Substructural Logics - Part 1

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Outline

- This is an introduction to the study of Substructural Logics, which is an attempt to understand various nonclassical logics in a uniform way.

N. Galatos, P. Jipsen, T. Kowalski, HO: Residuated Lattices: an algebraic glimpse at substructural logics, Studies in Logic, vol.151, Elsevier, April, 2007

A. Various nonclassical logics

Two main directions in nonclassical logics:

- Logics with additional operators
  modal logics, temporal logics, epistemic logics etc.
- Logics with nonclassical implications

(1) Constructive reasoning

Mathematical arguments are often infinitary and non-constructive. From intuitionists' viewpoint, mathematical arguments must be constructive.

- To infer $\alpha \to \beta$, it is required to have an algorithm for constructing a proof of β from any given proof of α,
- To infer $\alpha \vee \beta$, it is required to tell which of α and β holds, and also to have the justification.

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- To infer $\alpha \vee \beta$, it is required to tell which of α and β holds, and also to have the justification.

Thus, in constructive reasoning, both the law of double negation $\neg\neg\alpha \to \alpha$ and the law of excluded middle $\alpha \vee \neg\alpha$ are rejected.

(2) Relevant reasoning

The implication in classical logic is material implication, i.e. $\alpha \to \beta$ is identified with $\neg\alpha \vee \beta$.

Thus, both $(\alpha \wedge \neg\alpha) \to \beta$ and $\beta \to (\alpha \to \alpha)$ are classically valid (as both $\neg(\alpha \wedge \neg\alpha)$ and $\neg\alpha \vee \alpha$ are true), but their validity will be counterintuitive.

Relevant logicians try to formalize relevant implication, which expresses “implication” used in our daily reasoning.

For instance, relevant implication must satisfy:

Relevance principle: If α (relevantly) implies β, there must be some "connections" between α and β. (Without such a connection, why does β follow from α?)

(3) Many-valued logics

In 1920s, J. Łukasiewicz introduced both $n+1$-valued logic (for each $n>0$) with the set of truth values $\{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$, and also infinite-valued logic with the unit interval $[0,1]$ as the set of truth values.

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The truth table of each connective is defined as follows:

$$\begin{array}{rcl} a \wedge b & = & \min\{a, b\} \\ \neg a & = & 1 - a \end{array} \qquad \begin{array}{rcl} a \vee b & = & \max\{a, b\} \\ a \to b & = & \min\{1, 1 - a + b\} \\ & = & \begin{cases} 1 & a \le b \\ 1 - a + b & a > b \end{cases} \end{array}$$

(4) Fuzzy logics

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A binary operation on $[0, 1]$ is a t-norm if it is associative, commutative and monotone with the unit element 1.

A t-norm $\cdot$ is left-continuous if $x \cdot \sup Z = \sup(x \cdot Z)$ for each $x \in [0, 1]$ and each $Z \subseteq [0, 1]$. For each left-continuous t-norm $\cdot$, define an implication $\rightarrow$ by

$$a \rightarrow b = \sup\{z : a \cdot z \leq b\}$$

Examples of t-norms

(1) $a \cdot b = \min\{a, b\}$

$a \to b = b$ *if* $a > b$, *and* $= 1$ *otherwise. Implication of Gödel logic.*

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$a \rightarrow b = \min\{1, 1 - a + b\}$, *i.e. many-valued implication.*

(3) $a \cdot b = a \times b$

$a \rightarrow b = b/a$ *if* $a > b$, *and* $= 1$ *otherwise.*

- Are there something common among these logics?
- Is it possible to discuss them within a uniform framework?

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Substructural Logics

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Substructural Logics

We will explain what are substructural logics. Usually, they are introduced as sequent systems.

B. Sequent system LJ

A sequent is an expression of the following form with $m \geq 0$:

$$\alpha_1, \ldots, \alpha_m \Rightarrow \beta$$

Intuitively, it means "β *follows from assumptions* $\alpha_1, \ldots, \alpha_m$". (cf. sequents in classical logic)

Each sequent system consists of initial sequents (axioms) and rules that determine *correct* sequents in the system.

The sequent system **LJ** for intuitionistic logic introduced by Gentzen consists of initial sequents, i.e. sequents of the form $\alpha \Rightarrow \alpha$, and the following three kinds of rules.

- Structural rules
- Cut rule
- Rules for logical connectives

(a) Structural rules

Capital Greek letters denote finite sequences of formulas.

Structural rules control the meaning of commas in sequents. (i) together with (o) is called (w) (weakening rules).

(e) exchange rule (commutativity): $\dfrac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi}$

(c) contraction rule (square-increasing): $\dfrac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$

(i) left weakening rule (integrality): $\dfrac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$

(o) right weakening rule (minimality of 0): $\dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}$

(b) Cut rule

Cut rule

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \alpha, \Xi \Rightarrow \varphi}{\Sigma, \Gamma, \Xi \Rightarrow \varphi} \text{ (cut)}$$

(c) Rules for ∨ and ∧

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \varphi \quad \Gamma, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \varphi} \ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow \vee 2)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \varphi} \ (\wedge 1 \Rightarrow) \qquad \frac{\Gamma, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \varphi} \ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \ (\Rightarrow \wedge)$$

- weakening and contraction rules in a proof of distributive law in **LJ**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \beta \Rightarrow \alpha}\,(weak) \quad \dfrac{\beta \Rightarrow \beta}{\alpha, \beta \Rightarrow \beta}\,(weak)}{\alpha, \beta \Rightarrow \alpha \wedge \beta}}{\alpha, \beta \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)} \quad \dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \gamma \Rightarrow \alpha}\,(weak) \quad \dfrac{\gamma \Rightarrow \gamma}{\alpha, \gamma \Rightarrow \gamma}\,(weak)}{\alpha, \gamma \Rightarrow \alpha \wedge \gamma}}{\alpha, \gamma \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}}{\alpha, \beta \vee \gamma \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}}{\alpha \wedge (\beta \vee \gamma), \beta \vee \gamma \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}}{\alpha \wedge (\beta \vee \gamma), \alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}}{\alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\,(cont)
$$

(d) Rules for implication

Rules for implication

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \to \beta, \Delta \Rightarrow \varphi} \ (\to\Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \ (\Rightarrow\to)$$

- Find a proof of $\Rightarrow \alpha \rightarrow (\beta \rightarrow \alpha)$ in **LJ**

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$$
\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \beta \Rightarrow \alpha}\ (\mathit{weak})}{\alpha \Rightarrow \beta \rightarrow \alpha}\ (\Rightarrow\rightarrow)}{\Rightarrow \alpha \rightarrow (\beta \rightarrow \alpha)}\ (\Rightarrow\rightarrow)
$$

- Find a proof of $\alpha \to (\beta \to \gamma) \Rightarrow (\alpha \to \beta) \to (\alpha \to \gamma)$ in **LJ**.

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$$\dfrac{\dfrac{\alpha \Rightarrow \alpha \quad \dfrac{\alpha \Rightarrow \alpha \quad \dfrac{\beta \Rightarrow \beta \quad \gamma \Rightarrow \gamma}{\beta \to \gamma, \beta \Rightarrow \gamma}}{\beta, \alpha, \alpha \to (\beta \to \gamma) \Rightarrow \gamma}}{\alpha, \alpha, \alpha \to \beta, \alpha \to (\beta \to \gamma) \Rightarrow \gamma}}{\alpha, \alpha \to \beta, \alpha \to (\beta \to \gamma) \Rightarrow \gamma}\ (cont)$$

When exchange rule is missing ...

In the following, we will consider also sequent systems which lack some of structural rules. In particular when a system lacks exchange rule, it will be natural to introduce two kinds of "implication" (division), left-residuation $\backslash$ and right residuation $/$, with the following rules.

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$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \backslash \beta} \ (\Rightarrow \backslash) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \theta}{\Delta, \Gamma, \alpha \backslash \beta, \Sigma \Rightarrow \theta} \ (\backslash \Rightarrow)$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta / \alpha} \ (\Rightarrow /) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \theta}{\Delta, \beta / \alpha, \Gamma, \Sigma \Rightarrow \theta} \ (/ \Rightarrow)$$

$$\dfrac{\alpha \Rightarrow \alpha \qquad \dfrac{\beta \Rightarrow \beta \quad \gamma \Rightarrow \gamma}{\beta/\gamma, \gamma \Rightarrow \beta}}{\dfrac{\alpha, \alpha\backslash(\beta/\gamma), \gamma \Rightarrow \beta}{\dfrac{\alpha\backslash(\beta/\gamma), \gamma \Rightarrow \alpha\backslash\beta}{\alpha\backslash(\beta/\gamma) \Rightarrow (\alpha\backslash\beta)/\gamma}}}$$

Note

In each rule except Cut, every formula in upper sequents will appear also as a subformula of the lower sequent.

(e) Negation

The negation $\neg\alpha$ means that assuming α is led to a contradiction. Thus, by using a constant 0 (falsehood), the negation $\neg\alpha$ of a formula α is defined by

$$\neg\alpha = \alpha \to 0.$$

For 0, we assume the initial sequent $0 \Rightarrow$, and the following rule:

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0 \text{ weakening})$$

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0 means empty formula in the right-hand side.

When exchange rule is missing, it will be natural to introduce two kinds of "negation"

$$\sim \alpha = \alpha \backslash 0 \textit{ and } -\alpha = 0/\alpha.$$

For our algebraic understanding of sequents, we will introduce a constant 1 and assume the initial sequent $\Rightarrow 1$, and the following rule:

$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, 1, \Delta \Rightarrow \varphi} \ (1 \text{ weakening})$$

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$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, 1, \Delta \Rightarrow \varphi} \text{ (1 weakening)}$$

1 means empty formula in the left-hand side.

Intuitively, 1 denotes the weakest truth and 0 the strongest falsehood.

C. Structural rules and commas

a) Exchange rule allows us to use assumptions in an arbitrary order:

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi}$$

b) Without contraction rule, every (occurrence of each) assumption is used at most once in deriving a conclusion:

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$$

c) Without weakening rule (i), every assumption is used at least once in deriving a conclusion:

$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$$

(i) What are commas in sequents?

Commas of **LJ** can be understood as conjunctions.

In fact, using contraction and (left) weakening, we can show that :

a sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable in **LJ** iff
$\alpha_1 \wedge \ldots \wedge \alpha_m \Rightarrow \beta$ is provable in **LJ**.

On the other hand, commas are not expressed by conjunctions in general, when either contraction or (left) weakening is missing.

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To express commas in general situation, we will introduce a new logical connective $\cdot$, called the fusion or the multiplicative conjunction.

Rules for $\cdot$ are given as follows:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\Rightarrow \cdot) \qquad \frac{\alpha, \beta, \Gamma \Rightarrow \gamma}{\alpha \cdot \beta, \Gamma \Rightarrow \gamma}\ (\cdot \Rightarrow)$$

(ii) Fusions as Commas

Then we have the following.

- $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable iff $\alpha_1 \cdot \ldots \cdot \alpha_m \Rightarrow \beta$ is provable,

(iii) Implications as Residuals of fusion

Moreover, we can show the following equivalences which say that implications are residuals of fusion:

With exchange rule:

$\alpha \cdot \beta \Rightarrow \varphi$ *is provable iff* $\alpha \Rightarrow \beta \rightarrow \varphi$ *is provable.*

(iii) Implications as Residuals of fusion

Moreover, we can show the following equivalences which say that implications are residuals of fusion:

With exchange rule:

$\alpha \cdot \beta \Rightarrow \varphi$ *is provable iff* $\alpha \Rightarrow \beta \to \varphi$ *is provable.*

Without exchange rule:

$\alpha \cdot \beta \Rightarrow \varphi$ *is provable iff* $\beta \Rightarrow \alpha \backslash \varphi$ *is provable iff* $\alpha \Rightarrow \varphi / \beta$ *is provable.*

D. Summary of initial sequents and rules

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- Rules for implication(s)
- Rules for fusion
- Initial sequents for constants 1 and 0
- Rules for constants 1 and 0

E. What does fusion mean?

Let α: *one pays 1000 yen.*
β: *one can get a hardcover.*
γ: *one can have lunch.*

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β: *one can get a hardcover.*
γ: *one can have lunch.*

Assume that

1) one (fixed) hardcover costs 1000 yen,
2) lunch at a Japanese restaurant costs 1000 yen.

Thus, we can assume both $\alpha \Rightarrow \beta$ and $\alpha \Rightarrow \gamma$ are provable. Then

(1) $\alpha \cdot \alpha \Rightarrow \beta \cdot \gamma$ *is provable,*
(2) $\alpha \Rightarrow \beta \cdot \gamma$ *is not always provable,*
(3) $\alpha \Rightarrow \beta \wedge \gamma$ *is provable.*

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What are differences among them?

(1) Fusions are "consumed"

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(1) if one pays 1000 plus 1000 yen, i.e. 2000 yen, then one can have both a hardcover and a lunch.

(2) 1000 yen is not enough to have both of them.

(3) if one pays 1000 yen then one can get a hardcover and also can have lunch, "but not both".

(2) Conjunction = Disjunction?

(3) *if one pays 1000 yen then one can get a hardcover and also can have lunch, "but not both".*

Then, what is a difference between *conjunction* and *disjunction*?

F. Substructural logics

We introduce several sequent systems of basic substructural logics. They are obtained from **LJ** for intuitionistic logic by deleting some or all of structural rules (and then sometimes adding the law of double nagation)

- **FL** — deleting all structural rules from **LJ**
- $\mathbf{FL_e}$ — **FL**+ exchange (**IMALL**)
- $\mathbf{FL_c}$ — **FL**+ contraction
- $\mathbf{FL_{ew}}$ — **FL**+ exchange + weakening
- $\mathbf{CFL_e}$ — $\mathbf{FL_e} + \neg\neg\alpha \to \alpha$ (**MALL**)

Various substructural logics

Substructural logics are axiomatic extensions of **FL**.

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- Lambek calculus — logic without structural rules, i.e. **FL**

  Calculus for categorial grammer introduced by Ajdukiewicz and Bar-Hillel (J. Lambek, 1958), which was rediscovered in early 80s (J. van Benthem and W. Buszkowski).

- Relevant logics — logics without weakening rules

  A. Anderson, N. Belnap Jr., R.K. Meyer, M. Dunn, A. Urquhart etc.

- Logics without contraction rule

  V. Grishin (middle of 1970), H.O. & Y. Komori (1985).

- Linear logic — logic only with exchange rule, **MALL** $=$ $\mathbf{FL_e}$ $+$ double negation

  J.-Y. Girard (1987)

- Relevant logic **R** is $\mathbf{FL_{ec}}$ + double negation + distributive law
- Both fuzzy logics and Łukasiewicz's many-valued logics are extensions of $\mathbf{FL_{ew}}$

Appendix: Hilbert-style system for $\mathbf{FL_{ew}}$

The system consists of *modus ponens* as a single rule and the following axiom schemata. You may observe that $\rightarrow$ takes multiple jobs of implications, commas and arrows.

- $\alpha \rightarrow (\beta \rightarrow \alpha)$ (wekening),
- $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma))$ (exchange),
- $0 \rightarrow \alpha$ and $(\alpha \rightarrow \beta) \rightarrow ((\gamma \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta))$,
- $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma)$,
- $\alpha \rightarrow (\alpha \vee \beta)$ and $\beta \rightarrow (\alpha \vee \beta)$,
- $((\gamma \rightarrow \alpha) \wedge (\gamma \rightarrow \beta)) \rightarrow (\gamma \rightarrow (\alpha \wedge \beta))$,
- $(\alpha \wedge \beta) \rightarrow \alpha$ and $(\alpha \wedge \beta) \rightarrow \beta$,
- $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$,
- $\alpha \rightarrow (\beta \rightarrow (\alpha \cdot \beta))$,
- $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \cdot \beta) \rightarrow \gamma)$.

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- $\alpha \to (\beta \to \alpha)$ (wekening),
- $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$ (exchange),
- $0 \to \alpha$ and $(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$,
- $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma)$,
- $\alpha \to (\alpha \vee \beta)$ and $\beta \to (\alpha \vee \beta)$,
- $((\gamma \to \alpha) \wedge (\gamma \to \beta)) \to (\gamma \to (\alpha \wedge \beta))$,
- $(\alpha \wedge \beta) \to \alpha$ and $(\alpha \wedge \beta) \to \beta$,
- $\alpha \to (\beta \to (\alpha \wedge \beta))$,
- $\alpha \to (\beta \to (\alpha \cdot \beta))$,
- $(\alpha \to (\beta \to \gamma)) \to ((\alpha \cdot \beta) \to \gamma)$.

- $(\alpha \to (\alpha \to \gamma)) \to (\alpha \to \gamma)$ (contraction).

Substructural Logics - Part 2

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1. Proof theory

Many important logical results are obtained from analyzing *structures of proofs*, in particular of cut-free proofs.

Proof Theory = analysis of structures of proofs

A. Cut elimination

Cut elimination is one of most important tools in proof-theoretic approach. Cut elimination for a sequent system **L** means:

> If a sequent is provable in **L** then it is also provable in **L** without using cut rule.

A. Cut elimination

Cut elimination is one of most important tools in proof-theoretic approach. Cut elimination for a sequent system **L** means:

> If a sequent is provable in **L** then it is also provable in **L** without using cut rule.

While cut-free proofs may be much longer than proofs with cut, they have many good properties.

(1) Cut elimination in substructural logics

Though cut elimination holds only for a limited number of sequent systems, it holds for most of sequent systems for basic substructural logics discussed so far.

Cut elimination holds for **FL**, $\mathbf{FL_e}$, $\mathbf{FL_w}$, $\mathbf{FL_{ew}}$, $\mathbf{FL_{ec}}$ and **LJ**.

For more details, see: H. O., "Proof-theoretic methods in nonclassical logics – an introduction", 1998

(2) Consequences of cut elimination

a) Subformula property

Any cut-free proof of a given sequent $\Gamma \Rightarrow \theta$ *contains only sequents that consist of subformulas of some formulas in* $\Gamma \Rightarrow \theta$*.*

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b) Decidability

All of these substructural logics are decidable. Moreover, all of these substructural predicate logics without contraction rule are decidable.

c) Disjunction property

Every basic logic without right contraction rule has the disjunction property, i.e. if $\alpha \vee \beta$ is provable then either α or β is provable.

Intuitionistic logic has the disjunction property, but classical logic doesn't.

c) Disjunction property

Every basic logic without right contraction rule has the disjunction property, i.e. if $\alpha \vee \beta$ is provable then either α or β is provable.

Intuitionistic logic has the disjunction property, but classical logic doesn't.

d) Craig interpolation property

B. Deducibility

Let Σ be a set of formulas. A derivation of $\Gamma \Rightarrow \alpha$ from assumptions Σ in a sequent system **L** is a proof-figure to the sequent $\Gamma \Rightarrow \alpha$ which has also sequents $\Rightarrow \gamma$ (for each $\gamma \in \Sigma$) as extra initial sequents.

A formula α is deducible from Σ in **FL** ($\Sigma \vdash_{\mathbf{FL}} \alpha$) iff there exists a derivation of $\Rightarrow \alpha$ from Σ in **FL**.

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A formula α is deducible from Σ in **FL** ($\Sigma \vdash_{\mathbf{FL}} \alpha$) iff there exists a derivation of $\Rightarrow \alpha$ from Σ in **FL**.

Obviously, the provability of a formula α is equivalent to its deducibility from the empty assumption. For example,

$$\vdash_{\mathbf{Int}} \alpha \quad \text{iff} \quad \emptyset \vdash_{\mathbf{Int}} \alpha,$$

where $\vdash_{\mathbf{Int}} \alpha$ means that the sequent $\Rightarrow \alpha$ is provable in **LJ**.

The deducibility is different from the provability. For example, while $\alpha \Rightarrow \alpha^2$ is not provable in **FL**, $\alpha \vdash_{\mathbf{FL}} \alpha^2$ holds as the following shows.

$$\frac{\Rightarrow \alpha \quad \Rightarrow \alpha}{\Rightarrow \alpha \cdot \alpha} \ (\Rightarrow \cdot)$$

Can the deducibility relation be reduced to the provability?

(a) Deduction theorem

Yes, for both classical and intuitionistic logics. In fact, the following deduction theorem (DT) holds for them:

$$\Sigma \cup \{\alpha\} \vdash \beta \text{ iff } \Sigma \vdash (\alpha \to \beta).$$

By applying this repeatedly, the decidability of the deducibility in classical and intuitionistic logics follows from that of the provability.

Outline of the proof

The left-hand side follows immeadiately from the right-hand side.

Conversely, suppose that $\Sigma \cup \{\alpha\} \vdash \beta$. Take a derivation Π of $\Rightarrow \beta$ from assumptions $\Sigma \cup \{\alpha\}$. Replace every sequent $\Delta \Rightarrow \theta$ in Π by $\alpha, \Delta \Rightarrow \theta$.

- Then, $\Rightarrow \alpha$ is transformed into an initial sequent $\alpha \Rightarrow \alpha$, and $\Rightarrow \delta$ for $\delta \in \Sigma$ into $\alpha \Rightarrow \delta$, which follows from $\Rightarrow \delta$ by weakening.
- Using induction, we can show that $\alpha, \Delta \Rightarrow \theta$ is deducible from Σ for each sequent $\Delta \Rightarrow \theta$ in Π by the help of structural rules.
- In particular, $\alpha \Rightarrow \beta$ is deducible from Σ.

(b) Local deduction theorem

In a system with exchange rule, the following local deduction theorem holds.

$\Sigma \cup \{\alpha\} \vdash_{\mathbf{FL_e}} \beta$ iff $\Sigma \vdash_{\mathbf{FL_e}} (\alpha \wedge 1)^m \to \beta$ for some m.

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$$\Sigma \cup \{\alpha\} \vdash_{\mathbf{FL_e}} \beta \quad \text{iff} \quad \Sigma \vdash_{\mathbf{FL_e}} (\alpha \wedge 1)^m \rightarrow \beta \quad \text{for some } m.$$

As a corollary, we have:

$$\Sigma \cup \{\alpha\} \vdash_{\mathbf{FL_{ew}}} \beta \quad \text{iff} \quad \Sigma \vdash_{\mathbf{FL_{ew}}} \alpha^m \rightarrow \beta \quad \text{for some } m.$$

cf. many-valued logics

It is still local, as we cannot always determine such an *m* from given Σ, α, β. In fact,

- The provability problem of $\mathbf{FL_e}$ is decidable. (by cut elimination)
- The deducibility problem of $\mathbf{FL_e}$ is undecidable (essentially by Lincoln, Mitchell, Scedrov & Shankar).

Here, the deducibility relation for a logic **L** is decidable iff

> there is an effective procedure of deciding whether or not $\Sigma \vdash_{\mathbf{L}} \alpha$ holds for each finite set of formulas Σ and each formula α.

C. Interpolation properties

A logic **L** has the Craig's interpolation property (CIP), if for all formulas α, β such that $\alpha \to \beta$ is provable in **L**, there exists a formula γ, called an interpolant, such that

- both $\alpha \to \gamma$ and $\gamma \to \beta$ are provable in **L**,
- $Var(\gamma) \subseteq Var(\alpha) \cap Var(\beta)$.

Note that when **L** is without exchange, $\to$ is replaced by $\backslash$.

(i) Maehara's method

S. Maehara gives a way of showing CIP as a consequence of cut elimination. Here is an outline of the method e.g. for $\mathbf{FL_{ew}}$. We show the CIP of the following form.

If $\Gamma \Rightarrow \varphi$ is provable in $\mathbf{FL_{ew}}$, then there exists a formula δ, such that

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- both $\Gamma \Rightarrow \delta$ and $\delta \Rightarrow \varphi$ are provable in $\mathbf{FL_{ew}}$,
- $Var(\delta) \subseteq Var(\Gamma) \cap Var(\varphi)$.

By cut elimination for $\mathbf{FL_{ew}}$, there is a cut-free proof Π of $\Gamma \Rightarrow \varphi$.

Take an arbitrary sequent $\Psi \Rightarrow \beta$ in Π, and let $\langle \Lambda, \Theta \rangle$ be an arbitrary partition of Ψ (i.e. the multiset union of Λ and Θ is equal to Ψ). Then, we show the following by induction on the length of a proof of $\Psi \Rightarrow \beta$ in Π.

There exists a formula γ such that

- both $\Lambda \Rightarrow \gamma$ and $\gamma, \Theta \Rightarrow \beta$ are provable in $\mathbf{FL_{ew}}$,
- $Var(\gamma) \subseteq Var(\Lambda) \cap (Var(\Theta \cup \{\beta\}))$.

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Then the CIP follows immediately. Using Maehara's method, an interpolant can be obtained in a constructive way as long as a cut-free proof is given. (CIP holds for $\mathbf{FL}$, $\mathbf{FL_e}$, $\mathbf{FL_{ew}}$ and $\mathbf{FL_{ec}}$).

(ii) Deductive interpolation property

A substructural logic **L** has the strong deductive interpolation property (strong DIP), if for every set of formulas $\Lambda \cup \Theta \cup \{\varphi\}$ such that $\Lambda \cup \Theta \vdash_L \varphi$, there exists a set of formulas Δ such that

- $\Lambda \vdash_L \delta$ for all $\delta \in \Delta$ and $\Delta \cup \Theta \vdash_L \varphi$,
- $Var(\Delta) \subseteq Var(\Lambda) \cap Var(\Theta \cup \{\varphi\})$.

When Θ is empty, it is called the DIP.

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When Θ is empty, it is called the DIP.

For each logic over $\mathbf{FL_e}$, CIP implies DIP, and DIP is equivalent to SDIP.

2. Algebraic approaches

While proof-theoretic methods provide us with fine and sharp information on particular logics, algebraic methods supply us with quite general results.

Algebraic logic = applying algebra & universal algebra to logic

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Algebraic traditions

- Boole, de Morgan, Schroeder
- Łukasiewicz, Tarski, Lindenbaum, Rasiowa, Sikorski: Polish school
- Birkhoff, Stone, Tarski, Jónsson, Mal'cev: universal algebra
- Blok, Pigozzi, Czelakowski: abstract algebraic logic

D. Algebraic interpretations

Let **A** be an algebra of a suitable type for substructural logics.

A sequent $\alpha_1, \alpha_2, \ldots, \alpha_m \Rightarrow \beta$ is valid in **A** iff $f(\alpha_1 \cdot \alpha_2 \cdots \alpha_m) \leq f(\beta)$ holds for every assignment f on **A**, in symbol

$$\mathbf{A} \models \alpha_1 \cdot \alpha_2 \cdots \alpha_m \leq \beta$$

In particular, a formula β is valid in **A** iff $\mathbf{A} \models 1 \leq \beta$.

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In particular, a formula β is valid in **A** iff $\mathbf{A} \models 1 \leq \beta$.

Then, what kind of algebras are suitable for substructural logics? They must be partially ordered monoids.

(a) Residuated structures

A p.o. monoid is a structure $\langle L; \cdot, 1; \leq \rangle$ such that

- $\langle L; \leq \rangle$ is a p.o. set,
- $\langle L; \cdot, 1 \rangle$ is a monoid such that

$$x \leq y \;\Rightarrow\; xz \leq yz \text{ and } zx \leq zy.$$

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A p.o. monoid is residuated if there exist division operations $\backslash$ and $/$ such that

$$xy \leq z \;\Leftrightarrow\; x \leq z/y \;\Leftrightarrow\; y \leq x\backslash z$$

(b) Residuated lattices

Moreover, when $\langle L; \leq \rangle$ forms a lattice in a given residuated p.o. monoid , the algebra $\langle L; \wedge, \vee, \cdot, 1, \backslash, / \rangle$ is called a residuated lattice. In *commutative* residuated lattices, $x \backslash y = y/x$ holds always. In this case, residuals are denoted as $x \to y$.

(b) Residuated lattices

Moreover, when $\langle L; \leq \rangle$ forms a lattice in a given residuated p.o. monoid , the algebra $\langle L; \wedge, \vee, \cdot, 1, \backslash, / \rangle$ is called a residuated lattice. In *commutative* residuated lattices, $x \backslash y = y / x$ holds always. In this case, residuals are denoted as $x \rightarrow y$.

Note that residuated lattices are equationally definable. In particular, the law of residuation is expressed by equations;

$$x(x \backslash z \wedge y) \leq z \text{ and } y \leq x \backslash (xy \vee z), \text{ etc.}$$

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$$x(x \backslash z \wedge y) \leq z \text{ and } y \leq x \backslash (xy \vee z), \text{ etc.}$$

An **FL**-algebra is a residuated lattice with a fixed element 0. Using 0, we can introduce two negations by defining $\sim x = x \backslash 0$ and $-x = 0/x$.

(c) Important RLs

- Lattice ordered groups: $x\backslash y = x^{-1}y, \; y/x = yx^{-1}$
- Heyting algebras: commutative residuated lattices with a least element 0 such that $x \cdot y = x \wedge y$ holds. 1 is the greatest element.
- Boolean algebras: *involutive* Heyting algebras, i.e. HAs with $x = - - x$, where $-x = x \to 0$.

- Łukasiewicz's many-valued models:

$$x \cdot y = \max\{0, x + y - 1\}, \textit{ and}$$
$$y \rightarrow z = \min\{1, 1 - y + z\}$$

- product algebras

$$x \cdot y = x \times y, \textit{ and}$$
$$y \rightarrow z = z/y \quad \textit{if } y > z, \textit{ and } \; = 1 \textit{ otherwise.}$$

- RLs determined by t-norms, in general: Each left-continuous t-norm over the unit interval [0,1] with the unit 1 is in particular a commutative residuated lattice. They are exactly models of fuzzy logics.

E. Varieties and equational classes

A class of algebras $\mathcal{K}$ is a variety iff it is closed under H (homomorphic images), S (subalgebras) and P (direct products).

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For a given set of equations Σ, let $\mathrm{Mod}(\Sigma)$ be the class of algebras $\mathbf{A}$ such that $\mathbf{A} \models s = t$ for all $s = t$ in Σ. A class of algebras $\mathcal{K}$ is an equational class iff $\mathcal{K} = \mathrm{Mod}(\Sigma)$ for some Σ.

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Birkhoff showed in 1935:

varieties $=$ equational classes

Important subvarieties

For instance, the class $\mathcal{RL}$ of all residuated lattices and the class $\mathcal{FL}$ of all **FL**-algebras are varieties.

The classes of all Boolean algebras and of all Heyting algebras are subvarieties of $\mathcal{FL}$.

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For instance, the class $\mathcal{RL}$ of all residuated lattices and the class $\mathcal{FL}$ of all **FL**-algebras are varieties.

The classes of all Boolean algebras and of all Heyting algebras are subvarieties of $\mathcal{FL}$.

Each of the following equations determine important subvarieties of the variety of $\mathcal{FL}$ (cf. structural rules).

- commutativity: $x \cdot y = y \cdot x$ (or equivalently, $x \backslash y = y / x$)
- square-increasingness: $x \leq x^2$,
- integrality: $x \leq 1$,
- minimality of 0: $0 \leq x$.

F. Logics vs Algebras

By a standard argument using Lindenbaum algebras, we can show

- a sequent is provable in **FL** iff it is valid in every **FL**-algebra.

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- a sequent is provable in **FL** iff it is valid in every **FL**-algebra.

This result can be easily generalized as follows (algebraic completeness).

For each substructural logic **L** there exists a class $\mathcal{K}$ of **FL**-algebras such that

- a sequent is provable in **L** iff it is valid in every **FL**-algebra in $\mathcal{K}$.

(1) Correspondences

Correspondences between equations and formulas

terms: $s, t, u, \ldots \quad \longmapsto \quad$ formulas: $\alpha, \beta, \ldots$

- $s = t \qquad \Longrightarrow \qquad s \leftrightarrow t$, i.e. $(s \backslash t) \wedge (t \backslash s)$
- $1 \leq \alpha$, i.e. $\alpha \wedge 1 = 1 \qquad \Longleftarrow \qquad \alpha$

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- the variety of Boolean algebras $\longrightarrow$ classical logic
- the variety of Heyting algebras $\longrightarrow$ intuitionistic logic
- subvarieties of $\mathcal{RL}$ ($\mathcal{FL}$) $\longrightarrow$?

(2) Algebraization I

Algebraization a la Lindenbaum

1. For each subvariety $\mathcal{V}$ of $\mathcal{FL}$, the set $\mathbf{L}(\mathcal{V}) = \{\alpha; \mathcal{V} \models 1 \leq \alpha\}$ forms a substructural logic.
2. Conversely, for each substructural logic $\mathbf{L}$, the set of equations $\{s \approx t; (s\backslash t) \wedge (t\backslash s) \in \mathbf{L}\}$ determines a subvariety $\mathcal{V}(\mathbf{L})$ of $\mathcal{FL}$ (completeness).
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3. Moreover, these two maps $\mathbf{L}$ and $\mathcal{V}$ are dual lattice-isomorphisms.

Thus,

substructural logics are exactly logics of residuated lattices (more precisely, **FL**-algebras).

(3) Equational consequences

The equational consequence $\{u_i = v_i; i \in I\} \models_{\mathcal{V}} s = t$ of a subvariety $\mathcal{V}$ of $\mathcal{FL}$ is defined for a set of equations $\{u_i = v_i; i \in I\} \cup \{s = t\}$ by

> for each algebra $\mathbf{A}$ in $\mathcal{V}$ and each assignment f on A, $f(s) = f(t)$ holds whenever $f(u_i) = f(v_i)$ holds for all $i \in I$.

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> for each algebra **A** in $\mathcal{V}$ and each assignment f on A, $f(s) = f(t)$ holds whenever $f(u_i) = f(v_i)$ holds for all $i \in I$.

In particular, $\{u_i = v_i; 1 \leq i \leq m\} \models_V s = t$ is equivalent to the validity of the following quasi-equation in every **A** in $\mathcal{V}$.

- $(u_1 = v_1 \textit{ and } \ldots \textit{ and } u_m = v_m) \textit{ implies } s = t$.

(4) Algebraization II

Algebraization a la Blok-Pigozzi

The deducibility relation corresponds exactly to the equational consequence.

1. For each subvariety $\mathcal{V}$ of $\mathcal{FL}$, $\{u_i = v_i; i \in I\} \models_V s = t$ iff $\{u_i \backslash v_i \wedge v_i \backslash u_i; i \in I\} \vdash_{L(V)} s \backslash t \wedge t \backslash s$,
2. Conversely, for each substructural logic **L**, $\{\beta_j; j \in J\} \vdash_L \alpha$ iff $\{1 \leq \beta_j; j \in J\} \models_{V(L)} 1 \leq \alpha$,
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3. Moreover, they are mutually inverse transformations.

In abstract algebraic logic, we say this as:
for each substructural logic **L**, $\vdash_L$ *is algebraizable and* $\mathcal{V}(\mathbf{L})$ *is an equivalent algebraic semantics for it*.

F. Some important consequences

The deducibility relation for a logic **L** is decidable iff

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Algebraization theorem implies that the decision problem of the deducibility relation for a logic **L** is equivalent to the decision problem of quasi-equational theory of the corresponding variety $V(\mathbf{L})$.

The equational theory of residuated lattices is decidable.

On the other hand, since the deducibility relation for $\mathbf{FL_e}$ (without 0) is undecidable, we have:

The quasi-equational theory of commutative RLs is undecidable.

3. Why sequent systems?

Why sequent systems and their structural rules play critical roles in substructural logics?

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- Implication is admittedly the most important logical connective.
- In sequent formulation, a monoid operation is always introduced explicitly introduced as comma, and moreover
- implication(s) behaves exactly as its residual(s).
- Thus, different behaviors of commas, expressed usually by structural rules, will affect directly those of implications, and vice versa.

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- Thus, different behaviors of commas, expressed usually by structural rules, will affect directly those of implications, and vice versa.

In this way, the theory of implications (divisions) can be transferred faithfully into the theory of monoids (multiplications).