September 2011 Tbilisi

Introduction to Gödel logics

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4 x 45 mins

slides available at
http://www.logic.at/people/fasching/tbilisi2011

contents:
- Gödel logics
  propositional, prop. quantified, first-order
- logic with many names, many characterisations
- logic with many variants
- also: introduction/repetition of many-valued logics

(A very incomplete) list of names and papers

- very short paper 1932/1933
  Gödel: A family of logics: G_n, $n \in \mathbb{N}$, and G_ω
  (propositional) (semantical characterisation)
- Dummett: Proof system for $G_\omega =: \underline{LC}$ (1959)
- Horn: Generalisation to first-order; proof system (1969)
- Takeuti, Titani: intuitionistic fuzzy logic (1984)
- Takano (1987)
- Viennese group: Baaz, Preining, ... (different truth value sets)
- Avron: hypersequent calculus (proof systems)
- links to other logics, e.g. relevant logic

Syntax 1/2

propositional: e.g. $(A \supset (B \wedge C)) \vee (B \supset \bot)$

- countable set of variables $A, B, C, \ldots$
  (formally: $V_1, V_2, V_3, \ldots$)
  however, often $A, B, C, \ldots$ also used for formulas
- connectives: often: $\supset/2$, $\wedge/2$, $\vee/2$, $\bot/0$
  sometimes: $\circ/1$ ("ring"), $\Delta/1$ ("Delta")
- negation is syntactic sugar "$\neg X$" stands for "$X \supset \bot$"
- truth "$\top$" stands for "$\bot \supset \bot$"

first-order: e.g. $\forall x.(P(x) \vee Q(a,b))$

- individuum variables (two types: bound and free)
- individuum constants
- connectives, as above; they connect atoms
- quantifiers $(\forall, \exists)$: bind an individuum variable

basic knowledge about languages is assumed

Syntax 2/2

quantified propositional, e.g. $\forall A. (\exists B. (A \supset B) \vee C)$

- propositional variables (bound and free)
- connectives, as above; they connect prop. vars.
- quantifiers ($\forall, \exists$): bind a prop. var.

Remark: In classical logic this language is only rarely considered in introductions to logic:

"$\forall A . G$" just means "$G(A/\top) \wedge G(A/\bot)$"

prop. var. formula substitute A by $\top$ (everywhere)

However useful in applications and used in complexity theory

Yields "new" logics in multiple-valued logic

if we have an infinite truth-value set (instead of $\{0,1\}$)

Repetition: Classical semantics for propositional logics

Definition: A <u>classical interpretation</u> I is a function
$I : \text{Formula}^{(prop)} \to \{0,1\}$ such that
$I(\bot) = 0$ and such that for all $A, B \in \text{Formula}^{prop}$:

$$I(A \wedge B) = \begin{cases} 1 : & \text{if } I(A) = 1 = I(B) \\ 0 : & \text{otherwise} \end{cases} \quad (= \min\{I(A), I(B)\})$$

$$I(A \vee B) = \begin{cases} 1 : & \text{if } I(A)=1 \text{ or } I(B)=1 \\ 0 : & \end{cases} \quad (= \max\{I(A), I(B)\})$$

$$I(A \supset B) = \begin{cases} 1 : & \text{if } I(A)=0 \text{ or } I(B)=1 \\ 0 : & \end{cases} \quad (\text{"material implication"})$$

Problem/Feature with this definition:
Yes, there are such functions; indeed, every $I : \text{Var}^{(prop)} \to \{0,1\}$ creates one:

Proposition: Every function $I : \text{Var}^{(prop)} \to \{0,1\}$ can be (uniquely) <u>extended</u> to an interpretation $I' : \text{Formula}^{(prop)} \to \{0,1\}$, i.e. such that $I(A) = I'(A)$ for all $A \in \text{Var}^{(prop)}$.

Essential data of an interpretation: values on $\text{Var}^{(prop)}$.

Subtleties...

Gödel semantics for propositional logics

We do not follow Gödel, who used $\mathbb{N}$ as a truth-value set, but:

t-norm-based interpretations, multiple-valued logic,

truth-value set is $[0,1]$ (i.e. <u>real</u> numbers, not only rationals)

Problems and explanations after the definition

A <u>Gödel interpretation I</u> is a function

$I: \mathrm{Formula}^{(prop)} \to [0,1]$ such that

$I(\bot)=0$ and such that for all $A, B \in \mathrm{Formula}^{(prop)}$:

$$I(A \wedge B) = \min\{I(A), I(B)\}$$

$$I(A \vee B) = \max\{I(A), I(B)\}$$

$$I(A \supset B) = \begin{cases} 1 & : \ I(A) \leq I(B) \\ I(B) & : \ I(B) < I(A) \end{cases}$$ (see next slide)

<u>Proposition</u>: Every classical interpretation is a Gödel interpretation

<u>Proposition</u>: Every function $I: \mathrm{Var}^{(prop)} \to [0,1]$ has an extension

$I': \mathrm{Formula}^{(prop)} \to [0,1]$, i.e. $\forall A \in \mathrm{Var}^{(prop)}\ I(A) = I'(A)$.

This extension is unique; thus the essential data are the values of I' on $\mathrm{Var}^{(prop)}$.

Why is the truth-value of $I(A \supset B)$ defined that way?

- Gödel saw that it fits his purposes in the 1932/1933 paper.
- Generalises classical implication (✓).
- "Nice" logics should have the property

$$\frac{\text{From } I(A)=1 \text{ and } I(A \supset B)=1}{\text{conclude } I(B)=1}$$

  for all interpretations in question (i.e. Gödel interpretations).
- The following proposition is a simplification of an idea of Takeuti (also used in t-norm-based logics):

- For every classical interpretation I, for all $A, B, C \in \text{Formula}^{(prop)}$:

  We have

$$I(A)=0 \text{ or } I(B)=1 \text{ or } I(C)=0$$

  if and only if

$$I(A \wedge C) \leq I(B)$$

  if and only if

$$I(C) \leq I(A \supset B).$$

  In particular, we always have $I(A \wedge (A \supset B)) \leq I(B)$.

Try to retain the equivalence (for all Gödel interpretations):

$$(*) \quad \underbrace{I(A \wedge C)}_{\substack{\text{known:} \\ \min\{I(A), I(C)\}}} \leq I(B) \quad \text{if and only if} \quad I(C) \leq \underbrace{I(A \supset B)}_{\text{imagine: to be defined}}$$

Indeed, the following does: "residuum"

$$I(A \supset B) := \sup\{ \underset{\substack{\vdots \\ \text{represents } I(C)}}{z} \in [0,1] ; \underbrace{\min\{I(A), z\} \leq I(B)}_{\text{stands for } I(A \wedge C)} \}$$

Check: If $I(A) \leq I(B)$, we have $\min\{I(A), 1\} \leq I(B)$,

thus $\sup\{\ldots\} = 1$

If $I(B) < I(A)$, we have $\min\{I(A), I(B)\} \leq I(B)$

but $\min\{I(A), I(B)+\varepsilon\} > I(B)$ for all $\varepsilon > 0$

thus $\sup\{\ldots\} = I(B)$.

Compare with original definition (✓)

Check (*). (✓)

Semantical strength of propositional logic

Given: Gödel interpretation I

Formula F, containing only variables $A, B, C, \ldots$

Consider $I(F)$ as a function in $I(A), I(B), I(C), \ldots$

Which properties have these functions?

$$I(\neg A) = I(A \supset \bot) = \begin{cases} 1 : & I(A) = 0 \\ 0 : & I(A) > 0 \end{cases} \qquad \text{(0-comparison)}$$

$$I(\top) = I(\neg \bot) = I(\bot \supset \bot) = 1 \qquad \text{(absolute truth)}$$

$$I((A \supset B) \wedge (B \supset A)) = \ldots = \begin{cases} 1 & : I(A) = I(B) \\ \min\{I(A), I(B)\} & : I(A) \neq I(B) \end{cases}$$

Definition: $A \leftrightarrow B$ will stand for $(A \supset B) \wedge (B \supset A)$

$$I((B \supset A) \supset B) = \ldots = \begin{cases} 1 & \text{if } I(A) < I(B) \\ I(B) & \text{if } I(A) \geqslant I(B) \end{cases}$$

Definition: $A \prec B$ will stand for $(B \supset A) \supset B$

Conditions expressible in Gödel logic

I… Gödel interpretation, $A, B, C, \ldots$ formulas

- $I(\neg A) = 1$ if and only if $I(A) = 0$
- $I(A \supset B) = 1$ if and only if $I(A) \leq I(B)$
- $I(A \prec B) = 1$ if and only if $I(A) < I(B)$ or $I(A) = 1 = I(B)$
- $I(A \leftrightarrow B) = 1$ if and only if $I(A) = I(B)$

Is it possible to find a formula F whose only variable is A such that:

$$\left.\begin{array}{l} I(F) = 1 \text{ if } I(A) = 1 \\ \text{and } I(F) \leq \frac{9}{10} \text{ if } I(A) < 1 \quad ? \end{array}\right\} (*)$$

("1-comparator", "crisp comparison to 1")

Answer: No. Write down all formulas with three binary connectives. For every such formula F', one can find a formula F'' with two connectives such that $I(F') = I(F'')$ for all Gödel interpretations. (tedious)

Suppose F had property $(*)$: Replace F' by F'' in F, repeat, …
The result is either A, $A \supset \bot$, $A \supset A$, $\bot \supset \bot$, $\bot$, $\bot \supset A$ but none of these fulfil $(*)$. $\square$

Remedy: later on

Very trivial things: For every Gödel interpretation I for all formulas

$$I(A \wedge B) = I(B \wedge A)$$
$$I(A \vee B) = I(B \vee A)$$
$$I(A \wedge A) = I(A) = I(A \vee A)$$
$$I((A \wedge B) \wedge C) = I(A \wedge (B \wedge C))$$
$$I((A \vee B) \vee C) = I(A \vee (B \vee C))$$
$$I(A \wedge (B \vee C)) = I((A \wedge B) \vee (A \wedge C))$$
$$I(A \vee (B \wedge C)) = I((A \vee B) \wedge (A \vee C))$$
$$I(A \supset A) = 1$$
$$I((A \supset B) \vee (B \supset A)) = 1 \qquad \text{(LIN)}$$
$$I((A \prec B) \vee (A \leftrightarrow B) \vee (B \prec A)) = 1$$
$$I(\bot \supset A) = 1$$

like in classical logic. But:

$$I(A \vee \neg A) = \begin{cases} 1 : & I(A) = 0 \\ I(A) : & I(A) > 0 \end{cases}$$

Exercise: For every Gödel interpretation I, for every formula A and B:

$$I(A \vee B) = I\left(((A \supset B) \supset B) \wedge ((B \supset A) \supset A) \right).$$

Semantically, connective $\vee$ is superfluous.

Proposition: Given a formula F in propositional variables $A_1, \ldots, A_n$ and a Gödel interpretation I, we have
$I(F) \in \{0, I(A_1), \ldots, I(A_n), 1\}$. ("projection")

Proof: Structural induction. □

Trivial for classical logics. Does not hold for Łukasiewicz logics.

Proposition: Given a propositional context $E[\cdot]$, formulas A, B: Then
$I\left((A \Leftrightarrow B) \supset (E[A] \Leftrightarrow E[B]) \right) = 1$. ("equivalence scheme")

Proof: Structural induction. □

Remark as before.

Helps comparison to BL $(\wedge, \supset, \perp)$

Important: Restrict $[0,1]$.

<u>Lifting lemma</u> for propositional Gödel semantics.

Let $0 \le d \le 1$ and let I be a Gödel interpretation.

Define $h(x) := \begin{cases} x & x \le d \\ 1 & x > d \end{cases}$ and

let I^h be the Gödel interpretation given by

$$I^h(A) := h(I(A)) \quad \text{for all } \underline{\text{propositional variables}}\ A.$$

Then

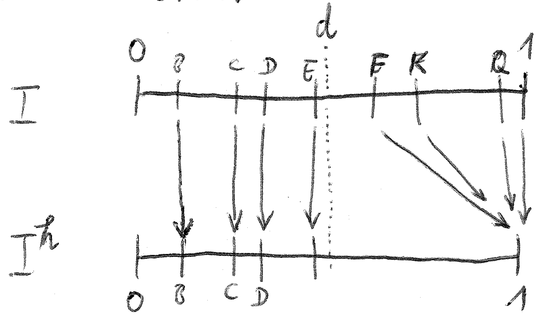
$$I^h(A) = h(I(A)) \quad \text{for all } \underline{\text{formulas}}\ A.$$

Proof: First, observe the arithmetical identities:

$$h(0) = 0,\quad h(1) = 1,\quad h(\underset{\max}{\min}\{x,y\}) = \underset{\max}{\min}\{h(x), h(y)\},$$

$$h\left(\begin{cases} 1 : x \le y \\ y : y < x \end{cases}\right) = \begin{cases} 1 & : h(x) \le h(y) \\ h(y) & : h(y) \le h(x) \end{cases}$$

Rest is just structural induction. $\square$



(propositional)

Order isomorphism lemma

For Gödel semantics, only order of the truth values is important

i.e. relative order of values assigned to variables but not exact position.

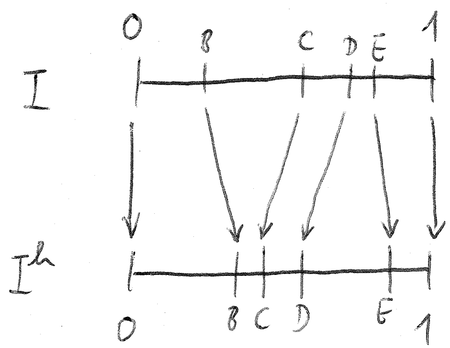
Let $h : [0,1] \to [0,1]$ such that

$$h(0) = 0, \; h(1) = 1 \text{ and } \forall x, y \in [0,1]: x < y \Leftrightarrow h(x) < h(y).$$

For any Gödel interpretation I, let I^h denote the Gödel interpretation given by $I^h(A) := h(I(A))$ for all prop. vars A.

Then $I^h(A) = h(I(A))$ for all prop. formulas A.

Proof: Same identities as before.



Valid/satisfiable formulas and entailment (propositional)

A formula A is valid [w.r.t. Gödel semantics]
if $I(A)=1$ for all Gödel interpretations I.

Observation: If this is the case, A is also classically valid.

A formula A is satisfiable [w.r.t. Gödel semantics]
if $I(A)=1$ for (at least) one Gödel interpretation I.

Proposition: A is satisfiable if and only if $I(A)>0$ for some Gödel int. I.
——— " ——— $I(A)=1$ for some classical int. I.

Proof: Lifting lemma. $\square$

Proposition: A is not satisfiable if and only if $\neg A$ is valid.

But: $\underbrace{\text{"}A \text{ is not valid"}}_{\Leftrightarrow\ \exists I:\ I(A)<1}$ differs from $\underbrace{\text{"}\neg A \text{ is satisfiable"}}_{\Leftrightarrow\ \exists I:\ I(A)=0}$

No classical duality! (Usually the case in many-valued logics)

Exercise: you have seen the formula before

$\{\forall I\ I(A)>0\} = CL$

The set of valid formulas is closed under <u>modus ponens</u> and under <u>substitution</u>, i.e. a "logic".

Write $\models A$ for "A is valid" [w.r.t. Gödel ~~logics~~ semantics]; also: $G \models A$ or $\models_G A$

Let Π be a <u>set</u> of formulas (can be infinite), and let I be a Gödel interpretation: Put $I(\Pi) := \inf\{I(B);\ B \in \Pi\}$

$$I(\emptyset) := 1$$

Compare: $I(B_1 \wedge \ldots \wedge B_n) = \min\{I(B_1), \ldots, I(B_n)\}$.

Π can not be part of a formula if infinite

Π <u>entails</u> A $:\Longleftrightarrow$ $I(\Pi) \leq I(A)$ for all Gödel interpretations I.

Compare: $I(B) \leq I(A)$ if and only if $I(B \supset A) = 1$.

easy: <u>$\{B_1, \ldots, B_n\}$ entails A</u> if and only if <u>$(B_1 \wedge \ldots \wedge B_n) \supset A$ is valid.</u>

<u>Proposition</u>: Π entails A if and only if

For all Gödel int. I such that ($I(B) = 1$ for all $B \in \Pi$) holds: $I(A) = 1$

"Π <u>1-entails</u> A"

Proof: Lifting lemma

Write $\Pi \Vdash A$ for "Π entails A" [w.r.t. Gödel ~~logics~~ semantics]

⊙ Validity can be expressed in terms of entailment

Proposition: For every formula A:

$$\models A \quad \text{if and only if} \quad \emptyset \Vdash A.$$

⊙ Monotonicity: If $\Pi \Vdash A$ and $\Pi \subseteq \Sigma$, then $\Sigma \Vdash A$.

⊙ Correspondence between $\supset$ and $\Vdash$

syntax semantics

Proposition: Let Π be a set of formulas (maybe infinite). Let A, B be formulas. Then

$$\Pi \cup \{A\} \Vdash B \quad \text{if and only if} \quad \Pi \Vdash A \supset B.$$

Proof: trivial.

Note: In Łukasiewicz logic: $\{A \underset{Ł}{\supset} B, A\} \not\Vdash B$.

-) <u>Gödel logics</u> := set of all <u>valid</u> formulas w.r.t. Gödel semantics.

$\mathbb{G}_{[0,1]}$ or $\mathbb{G}_{\mathbb{R}}$ or $\mathbb{G}_\infty$ or LC (Care!)

-) For the variants discussed in this lecture, a better approach is entailment, i.e. Gödel logics is defined as all pairs (Π, A) such that $\Pi \Vdash A$. Leads to a different notion!

-) How to check if a formula A is valid?

Try all orderings of the variables involved in A and the number 1.

Suppose variables B, C, D occur. Consider e.g. interpretations

1) $I(B) < I(C) = I(D) = 1$
2) $I(B) = I(D) < I(C) = 1$
3) $I(B) = I(D) < I(C) < 1$

etc.

Only ordering type matters, not the exact values.

Evaluate A under these assumptions. If, for all, we have $I(A) = 1$, then A is valid; if $I(A) < 1$ for some I, choose real numbers for $I(B), I(C), I(D)$ that correspond to the order type.

In practise: "syntactical" evaluation with case distinctions is cheaper

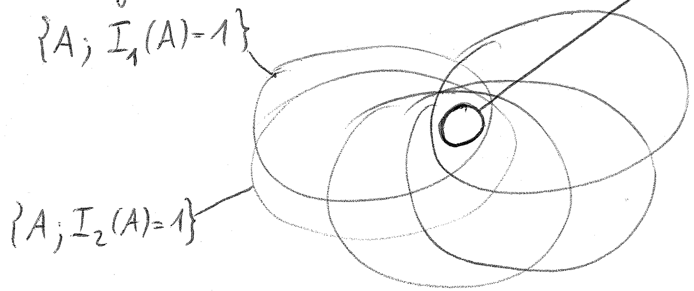
Possible, because min, max, $\trianglelefteq$ are decided by $<$-order.

Proof systems for Gödel logic (propositional)

Goal: Describe all valid formulas

Generate all valid formulas one-by-one systematically without a reference to definition of validity.

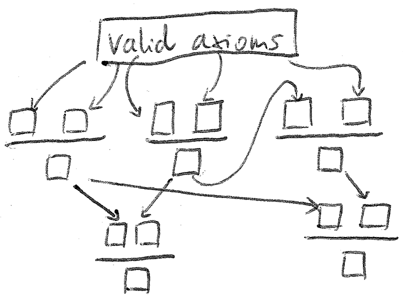
Semantics/algebraic approach:



Valid formulas are the intersection

interpretations rule out non-valid formulas

Syntactic/proof theoretic approach



union of all derived formulas

rules generate new formulas

Gödel logics are given by a semantic. Find proof system?

Definition: Let G denote the following Hilbert-style proof system:

One rule: modus ponens (MP)

$$\frac{A \qquad A \supset B}{B} \qquad (A, B : \text{propositional formulas})$$

Ten axioms (actually axiom schemes): A, B, C : propositional formulas

(1) $\bot \supset A$

(2) $(A \wedge B) \supset A$

(3) $(A \wedge B) \supset B$

(4) $A \supset (B \supset (A \wedge B))$

(5) $A \supset (A \vee B)$

(6) $B \supset (A \vee B)$

(7) $(A \supset C) \supset (B \supset C) \supset ((A \vee B) \supset C)$

(8) $A \supset (B \supset A)$

(9) $(A \supset (B \supset C)) \supset (A \supset B) \supset (A \supset C)$

(LIN) $(A \supset B) \vee (B \supset A)$

We write $G \vdash F$ if there is a G-proof of F.

<u>Example</u> (should be known from intuitionistic logic):

$$\text{(MP)}\ \dfrac{\overbrace{A\supset(A\supset A)}^{\text{instance of (8)}} \qquad \text{(MP)}\ \dfrac{\overbrace{A\supset((A\supset A)\supset A)}^{\text{instance of (8)}} \qquad [A\supset((A\supset A)\supset A)]\overbrace{\supset}^{\text{instance of (9)}}\big[[A\supset(A\supset A)]\supset[A\supset A]\big]}{[A\supset(A\supset A)]\supset[A\supset A]}}{A\supset A}$$

i.e. $G \vdash A \supset A$, in fact for any formula A.

<u>Important facts</u>: MP + (1) + ... + (9) is a well-known standard proof system for intuitionistic logic. Often called: $IL^{(prop)}$, IPL, ...

If $IL^{(prop)} \vdash A$, then $G \vdash A$.

e.g. $IL^{(prop)} \vdash (A \vee (B \vee C)) \leftrightarrow ((A \vee B) \vee C)$

$IL^{(prop)} \vdash (A \vee A) \supset A$

$IL^{(prop)} \vdash A \supset (A \wedge A)$

$IL^{(prop)} \vdash (A \leftrightarrow B) \supset (E[A] \leftrightarrow E[B])$

Use this to shorten proofs for G

We will use the following facts later on!

Recall $A \prec B$ stands for $(B \supset A) \supset B$

$A \leftrightarrow B$ stands for $(A \supset B) \wedge (B \supset A)$

We have:

$$G \vdash (A \prec B) \supset (E[A \wedge B] \leftrightarrow E[A])$$
$$G \vdash (A \leftrightarrow B) \supset (E[A \wedge B] \leftrightarrow E[A])$$
$$G \vdash (B \prec A) \supset (E[A \wedge B] \leftrightarrow E[B])$$

Guess:

$$G \vdash (A \prec B) \supset (E[A \vee B] \leftrightarrow \;?\;)$$

$$G \vdash (A \leftrightarrow B) \supset (E[A \supset B] \leftrightarrow \;?\;)$$

(propositional)

Dummett (1959):

(1) G is sound w.r.t. Gödel semantics.

(2) G is complete w.r.t. Gödel semantics.

(3) There is an algorithm that, for any propositional formula A, either

constructs a G-proof of A

or

constructs an interpretation I such that $I(A)<1$ (a countermodel of A)

In detail:

(1) means: If $G \vdash A$ then $\models A$ (w.r.t. Gödel semantics)

(2) means: If $\models A$ then $G \vdash A$

(3) means: • We can find a proof of valid A, not only know the mere existence of such a proof

• We can determine if A is valid or not.

Proof ideas:

Soundness "routine" matter, on the length of a proof by induction

- Check (1), ..., (9), (LIN) if they are indeed valid.

  e.g. for any Gödel interpretation I, we have
  $$I\big(A \supset (A \vee B)\big) = \begin{cases} 1 : & I(A) \leq I(A \vee B) \\ I(A \vee B) : & I(A) > I(A \vee B) \end{cases} = 1 \quad \text{since}$$
  $$I(A \vee B) = \max\{I(A), I(B)\} \geq I(A).$$ Thus (5) is valid.

- Check if (MP) is sound, i.e.

  if $I(A)=1$ and $I(A \supset B) = 1$, we should have $I(B) = 1$.

  (✓)

- Given any proof tree, work from the axioms to the end formula by applying the above points.

Completeness of G and the said algorithm are proved together.

Sketch:

- Recall the evaluation procedure that determines validity.
  (Take all orderings of $I(V_1), I(V_2), \ldots, I(V_n), 1$ and calculate the value $I(A)$ of the formula A in $V_1, \ldots, V_n$.)

  e.g. I is a Gödel-interpretation such that

  $$I(B) < I(C) = I(D) < I(E) = 1.$$

  Let A be $D \supset (B \wedge E)$. Evaluate A under I:

  $$I(A) = I(D \supset (B \wedge E)) = \begin{cases} 1 & : I(D) \leq I(B \wedge E) \\ I(B \wedge E) & : I(D) > I(B \wedge E) \end{cases} =$$

  $$= \begin{cases} 1 & : I(D) \leq I(B) \\ I(B) & : I(D) > I(B) \end{cases} = I(B) \qquad \text{(projection property!)}$$

- Proof system G can do something similar:

  $$G \vdash \underbrace{((B \prec C) \wedge (C \Leftrightarrow D) \wedge (D \prec E) \wedge (E \Leftrightarrow \top))}_{\text{“chain” in } B, C, D, E, \top} \supset (\underbrace{[D \supset (B \wedge E)]}_{\text{composite formula}} \leftrightarrow \underbrace{B}_{\in \{B, C, D, E, \top, \bot\} \text{ in the general case}})$$

Several tools:

semantical / evaluation procedure	syntactical / proof system G
order type of an interpretation	chain
$I(D) < I(B) = I(E) < I(C) < 1$ --- corresponds ---	$(D \prec B) \wedge (B \leftrightarrow E) \wedge (E \prec C) \wedge (C \prec \top)$

(for propositional variables B, C, D, E)

Let A be a formula in variables $V_1, \ldots, V_n$

projection property: For every Gödel interpretation I: $I(A) \in \{1, 0, I(V_1), \ldots, I(V_n)\}$	For every chain K in $V_1, \ldots, V_n$ there is $L \in \{\top, \bot, V_1, \ldots, V_n\}$ such that $G \vdash K \supset (A \leftrightarrow L)$ "K evaluates A to L" Proof idea: next slides

I - - - - K

$I(A) < 1$ if and only if $G \nvdash K \supset (A \leftrightarrow \top)$

i.e. $G \vdash K \supset (A \leftrightarrow \circledast)$

not equivalent with $\top$

(w/o proof)

Validity of A is tested by trying <u>all</u> order types.	List <u>all</u> chains $K_1, \ldots, K_q$. If $G \vdash K_1 \supset (A \leftrightarrow \top), \ldots, G \vdash K_q \supset (A \leftrightarrow \top)$ then $G \vdash A \leftrightarrow \top$, i.e. $G \vdash A$

./.

LIN axiom $(A \supset B) \vee (B \supset A)$ is used to prove in G:

$$(V_1 \prec V_2) \vee (V_1 \leftrightarrow V_2) \vee (V_2 \prec V_1).$$

Proposition: $G \vdash K_1 \vee K_2 \vee \ldots \vee K_q$

where $K_1, \ldots, K_q$ is a list of all chains for a given list of variables $V_1, V_2, \ldots, V_n$.

Proposition: If $G \vdash K_1 \supset F, \ldots, G \vdash K_q \supset F$, $G \vdash K_1 \vee \ldots \vee K_q$ then $G \vdash F$.

Proof: Use (7) several times. Actually holds in $IL^{(prop)}$.

Evaluation lemma: Proof tedious because actual proof trees are constructed.

$K \ldots$ chain in variables $V_1, \ldots, V_n$

$A \ldots$ formula in ——"——

Then there is $L \in \{\top, \bot, V_1, \ldots, V_n\}$ such that

$$G \vdash K \supset (A \leftrightarrow L).$$

Idea: Construct formulas $A_1, A_2, A_3, \ldots, A_p$ such that

$A_1 = A$ and $A_p = L$ and

$G \vdash K \supset (A_1 \leftrightarrow A_2)$ and $G \vdash K \supset (A_2 \leftrightarrow A_3)$ and ...

and ... and $G \vdash K \supset (A_{p-1} \leftrightarrow A_p)$. $\Rightarrow G \vdash K \supset (A \leftrightarrow L)$.

Continue example from before:

Let K abbreviate the chain $(B \prec C) \wedge (C \leftrightarrow D) \wedge (D \prec E) \wedge (E \leftrightarrow \top)$.

"Evaluate" $D \supset (B \wedge E)$ under K:

Start at an "innermost" formula, i.e. subformula $V_i \,\square\, V_j$

$V_i \in Var$, $\square$ binary connective, $V_j \in Var$

Here: $B \wedge E$ is the only choice.

Observe: $G \vdash K \supset (B \prec E)$

⚠ $G \vdash (B \prec E) \supset ((B \wedge E) \leftrightarrow B)$

more generally: $G \vdash (B \prec E) \supset (M[B \wedge E] \leftrightarrow M[B])$

in particular: $G \vdash (B \prec E) \supset ((D \supset (B \wedge E)) \leftrightarrow (D \supset B))$

Thus $G \vdash K \supset (\underbrace{D \supset (B \wedge E)}_{A_1} \leftrightarrow \underbrace{D \supset B}_{A_2})$

Now evaluate A_2 (as before):

$G \vdash K \supset (B \prec D)$

⚠ $G \vdash (B \prec D) \supset ((D \supset B) \leftrightarrow B)$

Thus $G \vdash K \supset (\underbrace{(D \supset B)}_{A_2} \leftrightarrow \underbrace{B}_{A_3})$.

It follows: $G \vdash K \supset ((D \supset (B \wedge E)) \leftrightarrow B)$ single letter as demanded

$\boxtimes$

⊙ Recall: Correspondence of $\supset$ and $\Vdash$

$$\Pi \cup \{A\} \underset{(\mathbb{G})}{\Vdash} B \quad \text{if and only if} \quad \Pi \underset{(\mathbb{G})}{\Vdash} A \supset B.$$

⊙ <u>Deduction theorem</u> Π: set of formulas. A, B formulas.

$$G + \Pi + A \vdash B \quad \text{if and only if} \quad G + \Pi \vdash A \supset B.$$

proof system G with extra axioms: A and all formulas of Π.

Proof: Induction of length of the $G+\Pi+A$-proof of B. □

Very typical for Gödel semantics.

[For Hájek's BL: $BL + A \vdash B$ if and only if $\exists n\ BL \vdash (\underbrace{A \wedge \ldots \wedge A}_{n \text{ times}}) \supset B$]

⊙ <u>Correspondence of $\Vdash$ and $\vdash$: "strong completeness"</u>

$$G + \Pi \vdash A \quad \text{if and only if} \quad \Pi \underset{(\mathbb{G})}{\Vdash} A$$

⊙ Recall: $G \vdash A$ if and only if $\underset{(\mathbb{G})}{\models} A$: <u>"weak completeness"</u>

We have seen: $IL^{(prop)}$ is part of G.

We have seen: Any classical interpretation is also a Gödel interpretation

<u>Corollary:</u> For any propositional formula A:

A intuitionistically valid $\Rightarrow$ A valid w.r.t. Gödel semantics

A valid w.r.t. Gödel semantics $\Rightarrow$ A classically valid

Consider $(A \supset B) \vee (B \supset A)$, $A \vee \neg A$ to show proper inclusion.

Gödel logic is an "<u>intermediate logic</u>".

Gödel (1932) also shows (propositional logics):

$$VALID_{IL} \subsetneqq VALID_{G} \subsetneqq \dots \subsetneqq VALID_{G_4} \subsetneqq VALID_{G_3} \subsetneqq VALID_{CL}$$

for a family of logics $G_3, G_4, G_5, \dots$

Variants of Gödel logic

(propositional)

Projection property allows us to restrict truth value set.

Let a set V be given such that $\{0,1\} \subseteq V \subseteq [0,1]$.

A Gödel-V-interpretation is a Gödel-interpretation I
such that $I(A) \in V$ for all prop. vars. A.

Projection property $\Rightarrow$ $I(A) \in V$ for all formulas.

A formula A is V-valid if $I(A)=1$ for all Gödel-V-interpretation I.

Example. If I is a Gödel-V-interpretation, then

$$I(B \vee (B \supset C) \vee (C \supset \bot)) = \max\left\{ I(B), \begin{cases} 1 & I(B) \leq I(C) \\ I(C) & I(C) < I(B) \end{cases}, \begin{cases} 1 & I(C) \leq 0 \\ 0 & 0 < I(C) \end{cases} \right\}$$

$$= \begin{cases} 1: & I(B)=1 \text{ or } I(B) \leq I(C) \text{ or } I(C)=0 \\ I(B): & \text{otherwise, i.e. if } 0 < I(C) < I(B) < 1 \end{cases}$$

Thus, if $\#V \leq 3$, then $B \vee (B \supset C) \vee (C \supset \bot)$ is V-valid
if $\#V > 3$, then — " — is not V-valid

Proposition: If $\#V \leq n$, then $A_1 \vee (A_1 \supset A_2) \vee (A_2 \supset A_3) \vee \ldots \vee (A_{n-1} \supset \bot)$ is V-valid. If $\#V > n$, it is not V-valid. [Gödel]

Put $V_2 := \{0, 1\}$

$V_3 := \{0, \tfrac{1}{2}, 1\}$

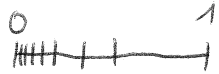
$V_4 := \{0, \tfrac{1}{3}, \tfrac{2}{3}, 1\}$

...

$V_n := \{0, 1\} \cup \{\tfrac{i}{n-1}\ ;\ i \in \mathbb{N},\ 1 \leq i < n-1\}$

$V_\uparrow := \{0\} \cup \{\tfrac{i}{i+1}\ ;\ i \in \mathbb{N},\ 1 \leq i\} \cup \{1\}$

$V_\downarrow := \{0\} \cup \{\tfrac{1}{i}\ ;\ i \in \mathbb{N},\ 1 \leq i\} \cup \{1\}$



<u>Examples</u>: V_2-valid

$[0,1]$-valid

<u>Recall</u>: Isomorphism lemma.

<u>Theorem</u>: $n \in \mathbb{N}$, $n \geq 2$. Then A is V_n-valid if and only if

$$G + FIN_n \vdash A;$$

here FIN_n is the axiom schema

$$A_1 \vee (A_1 \supset A_2) \vee (A_2 \supset A_3) \vee \ldots \vee (A_{n-1} \supset \bot).$$

<u>Theorem</u>: Let $\{0,1\} \subseteq V \subseteq [0,1]$, V infinite. Then A is V-valid iff $G \vdash A$.

In other words: Only <u>one</u> logic created by <u>all</u> infinite V.

Observe: ⊙ $G_{[0,1]}$ is the intersection of all $G_n := G + FIN_n$

⊙ $\mathbb{IL}^{(prop)} \subsetneq G_{[0,1]} \subsetneq \dots \subsetneq G_4 \subsetneq G_3 \subsetneq G_2 = CL$.

However, entailment relations for sets $V_\uparrow$, $V_\downarrow$, $[0,1]$ are all different.

Valid/satisfiable formulas and entailment (propositional)

A formula A is <u>valid</u> [w.r.t. Gödel semantics]
if $I(A)=1$ for all Gödel interpretations I.

<u>Observation</u>: If this is the case, A is also classically valid.

A formula A is <u>satisfiable</u> [w.r.t. Gödel semantics]
if $I(A)=1$ for (at least) one Gödel interpretation I.

<u>Proposition</u>: A is satisfiable if and only if $I(A)>0$ for some Gödel int. I.
——— " ——— $I(A)=1$ for some classical int. I.

Proof: Lifting lemma. $\square$

<u>Proposition</u>: A is not satisfiable if and only if $\neg A$ is valid.

<u>But</u>: $\underbrace{\text{"A is not valid"}}_{\Leftrightarrow\ \exists I:\ I(A)<1}$ differs from $\underbrace{\text{"}\neg A \text{ is satisfiable"}}_{\Leftrightarrow\ \exists I:\ I(A)=0}$

<u>No classical duality!</u> (Usually the case in many-valued logics)

Exercise: you have seen the formula before

$\{\forall I\ I(A)>0\} = CL$

The set of valid formulas is closed under <u>modus ponens</u> and under <u>substitution</u>, i.e. a "logic".

Write $\models A$ for "A is valid" [w.r.t. Gödel ~~logics~~ semantics]; also: $\mathfrak{G} \models A$ or $\models_{\mathfrak{G}} A$

Let Π be a <u>set</u> of formulas (can be infinite), and let I be a Gödel interpretation: Put $I(\Pi) := \inf\{I(B);\ B \in \Pi\}$

$I(\emptyset) := 1$

Compare: $I(B_1 \wedge \ldots \wedge B_n) = \min\{I(B_1), \ldots, I(B_n)\}$.

Π <u>entails</u> A $:\Longleftrightarrow$ $I(\Pi) \leq I(A)$ for all Gödel interpretations I.

Compare: $I(B) \leq I(A)$ if and only if $I(B \supset A) = 1$.

easy: <u>$\{B_1, \ldots, B_n\}$ entails A</u> if and only if <u>$(B_1 \wedge \ldots \wedge B_n) \supset A$ is valid.</u>

<u>Proposition</u>: Π entails A if and only if

For all Gödel int. I such that ($I(B) = 1$ for all $B \in \Pi$) holds: $I(A) = 1$

"Π <u>1-entails</u> A"

Proof: Lifting lemma

Write $\Pi \Vdash A$ for "Π entails A" [w.r.t. Gödel ~~logics~~ semantics]

Π can not be part of a formula if infinite

⊙ Validity can be expressed in terms of entailment

Proposition: For every formula A:

$$\models A \quad \text{if and only if} \quad \emptyset \Vdash A.$$

⊙ Monotonicity: If $\Pi \Vdash A$ and $\Pi \subseteq \Sigma$, then $\Sigma \Vdash A$.

⊙ Correspondence between $\supset$ and $\Vdash$

syntax semantics

Proposition: Let Π be a set of formulas (maybe infinite).
Let A, B be formulas. Then

$$\Pi \cup \{A\} \Vdash B \quad \text{if and only if} \quad \Pi \Vdash A \supset B.$$

Proof: trivial.

Note: In Łukasiewicz logic: $\{A \supset_{Ł} B, A\} \not\Vdash B$.

-) <u>Gödel logics</u> := set of all <u>valid</u> formulas w.r.t. Gödel semantics.

$\mathbb{G}_{[0,1]}$ or $\mathbb{G}_{\mathbb{R}}$ or $\mathbb{G}_\infty$ or LC (care!)

-) For the variants discussed in this lecture, a better approach is entailment, i.e. Gödel logics is defined as all pairs (Π, A) such that $\Pi \Vdash A$. Leads to a different notion!

-) How to check if a formula A is valid?

Try all orderings of the variables involved in A and the number 1.

Suppose variables B, C, D occur. Consider e.g. interpretations

1) $I(B) < I(C) = I(D) = 1$
2) $I(B) = I(D) < I(C) = 1$
3) $I(B) = I(D) < I(C) < 1$

etc.

Only ordering type matters, not the exact values.

Evaluate A under these assumptions. If, for all, we have $I(A) = 1$, then A is valid; if $I(A) < 1$ for some I, choose real numbers for $I(B), I(C), I(D)$ that correspond to the order type.

In practise: "syntactical" evaluation with case distinctions is cheaper

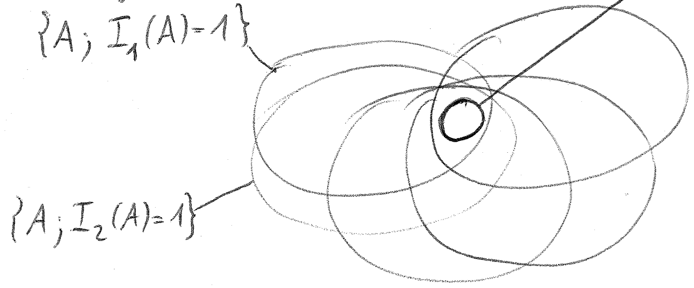
Possible, because min, max, $\trianglelefteq$ are decided by $<$-order.

Proof systems for Gödel logic (propositional)

Goal: Describe all valid formulas

Generate all valid formulas one-by-one systematically without a reference to definition of validity.

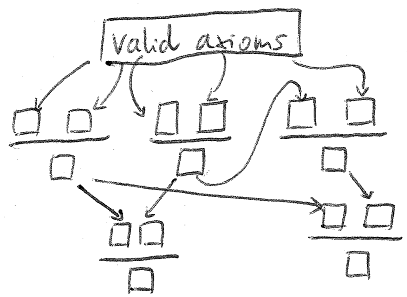
Semantics/algebraic approach:



Valid formulas are the intersection

interpretations rule out non-valid formulas

Syntactic/proof theoretic approach



union of all derived formulas

rules generate new formulas

Gödel logics are given by a semantic. Find proof system?

Definition: Let G denote the following Hilbert-Style proof system:

One rule: modus ponens (MP)

$$\frac{A \quad A \supset B}{B} \qquad (A, B: \text{propositional formulas})$$

Ten axioms (actually axiom schemes): A, B, C: propositional formulas

(1) $\perp \supset A$

(2) $(A \wedge B) \supset A$

(3) $(A \wedge B) \supset B$

(4) $A \supset (B \supset (A \wedge B))$

(5) $A \supset (A \vee B)$

(6) $B \supset (A \vee B)$

(7) $(A \supset C) \supset (B \supset C) \supset ((A \vee B) \supset C)$

(8) $A \supset (B \supset A)$

(9) $(A \supset (B \supset C)) \supset (A \supset B) \supset (A \supset C)$

(LIN) $(A \supset B) \vee (B \supset A)$

We write $G \vdash F$ if there is a G-proof of F.

(propositional)

Dummett (1959):

(1) G is <u>sound</u> w.r.t. Gödel semantics.

(2) G is <u>complete</u> w.r.t. Gödel semantics.

(3) There is an algorithm that, for any propositional formula A, either

constructs a G-proof of A

or

constructs an interpretation I such that $I(A) < 1$ (a <u>countermodel</u> of A)

In detail:

(1) means: If $G \vdash A$ then $\models A$ (w.r.t. Gödel semantics)

(2) means: If $\models A$ then $G \vdash A$

(3) means:
- We can <u>find</u> a proof of valid A, not only know the mere <u>existence</u> of such a proof
- We can determine <u>if</u> A is valid or not.

Example (should be known from intuitionistic logic):

$$(MP)\ \dfrac{\overset{\text{instance of (8)}}{A\supset(A\supset A)} \qquad (MP)\ \dfrac{\overset{\text{instance of (8)}}{A\supset((A\supset A)\supset A)} \qquad \overset{\text{instance of (9)}}{[A\supset((A\supset A)\supset A)]\supset\big[[A\supset(A\supset A)]\supset[A\supset A]\big]}}{[A\supset(A\supset A)]\supset[A\supset A]}}{A\supset A}$$

i.e. $G \vdash A\supset A$, in fact for any formula A.

Important facts: $MP + (1) + \ldots + (9)$ is a well-known standard proof system for intuitionistic logic. Often called: $IL^{(prop)}$, IPL, …

If $IL^{(prop)} \vdash A$, then $G \vdash A$.

e.g. $IL^{(prop)} \vdash (A\vee(B\vee C))\leftrightarrow((A\vee B)\vee C)$

$IL^{(prop)} \vdash (A\vee A)\supset A$

$IL^{(prop)} \vdash A\supset(A\wedge A)$

$IL^{(prop)} \vdash (A\leftrightarrow B)\supset(E[A]\leftrightarrow E[B])$

Use this to shorten proofs for G

We will use the following facts later on!

Recall $A \prec B$ stands for $(B \supset A) \supset B$

$A \leftrightarrow B$ stands for $(A \supset B) \wedge (B \supset A)$

We have:

$$G \vdash (A \prec B) \supset (E[A \wedge B] \leftrightarrow E[A])$$
$$G \vdash (A \leftrightarrow B) \supset (E[A \wedge B] \leftrightarrow E[A])$$
$$G \vdash (B \prec A) \supset (E[A \wedge B] \leftrightarrow E[B])$$

Guess:

$$G \vdash (A \prec B) \supset (E[A \vee B] \leftrightarrow \; ? \;)$$

$$G \vdash (A \leftrightarrow B) \supset (E[A \supset B] \leftrightarrow \; ? \;)$$

Hájek's Basic Logic (BL)

(propositional proof system in Hilbert style)

Language: $\supset/2$, $\wedge/2$, $\bot/0$ (original $\to$ & $\bar{0}$)

Rule: MP $\dfrac{A \quad A \supset B}{B}$

Axiom schemes

(1) $\bot \supset A$

(2) $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$

(3) $(A \wedge B) \supset A$

(4) $(A \wedge B) \supset (B \wedge A)$

(5) $(A \wedge (A \supset B)) \supset (B \wedge (B \supset A))$

(6) $((A \wedge B) \supset C) \supset (A \supset (B \supset C))$

(7) $(A \supset (B \supset C)) \supset ((A \wedge B) \supset C)$

(8) $((A \supset B) \supset C) \supset (((B \supset A) \supset C) \supset C)$

Hypersequent calculus (Avron)

$\Gamma, \Delta, \ldots$ denote finite (possibly empty) comma-separated lists of formulas: "multiformula"

$\Gamma \Rightarrow \Delta$ sequent where Γ, Δ multiformulas with $\# \Delta \leq 1$.

$G, G', \ldots$ denote hypersequents, i.e. bar-separated sequents:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

(finite, possibly empty)

$\mathbb{HC}$ has 2 axiom(schemes) and many rules:

Axioms $\quad A \Rightarrow A \qquad \bot \Rightarrow A$

Rules:

$$\frac{G}{G \mid \Gamma \Rightarrow \Delta} \ (ew) \qquad \frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta} \ (ec)$$

$$\frac{G \mid \Gamma \Rightarrow \Delta}{G \mid A, \Gamma \Rightarrow \Delta} \ (iw_l) \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow A} \ (iw_r) \qquad \frac{G \mid A, A, \Gamma \Rightarrow \Delta}{G \mid A, \Gamma \Rightarrow \Delta} \ (ic)$$

./.

hypersequent calculus ctd.

$$\frac{G \mid \Gamma \Rightarrow A \qquad G' \mid B, \Gamma \Rightarrow C}{G \mid G' \mid \Gamma, A \supset B \Rightarrow C} (\supset_l) \qquad \frac{G \mid \Gamma, A \Rightarrow B}{G \mid \Gamma \Rightarrow A \supset B} (\supset_r)$$

$$\frac{G \mid \Gamma, A \Rightarrow C}{G \mid \Gamma, A \wedge B \Rightarrow C} (\wedge_{l1}) \quad \frac{G \mid \Gamma, A \Rightarrow C}{G \mid \Gamma, B \wedge A \Rightarrow C} (\wedge_{l2}) \quad \frac{G \mid \Gamma \Rightarrow A \qquad G' \mid \Gamma \Rightarrow B}{G \mid G' \mid \Gamma \Rightarrow A \wedge B} (\wedge_r)$$

$$\frac{G \mid \Gamma, A \Rightarrow C \qquad G' \mid \Gamma, B \Rightarrow C}{G \mid G' \mid \Gamma, A \vee B \Rightarrow C} (\vee_l) \qquad \frac{G \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow A \vee B} (\vee_{r1}) \quad \frac{G \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow B \vee A} (\vee_{r2})$$

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow A \qquad G' \mid \Sigma, \Sigma' \Rightarrow B}{G \mid G' \mid \Gamma, \Sigma' \Rightarrow A \mid \Gamma', \Sigma \Rightarrow B} (Com)$$ "communication"

$$\frac{G \mid \Gamma \Rightarrow A \qquad G' \mid A, \Gamma' \Rightarrow B}{G \mid G' \mid \Gamma, \Gamma' \Rightarrow B} (cut)$$

We have seen: $IL^{(prop)}$ is part of G.

We have seen: Any classical interpretation is also a Gödel interpretation

<u>Corollary:</u> For any propositional formula A:

A intuitionistically valid $\Rightarrow$ A valid w.r.t. Gödel semantics

A valid w.r.t. Gödel semantics $\Rightarrow$ A classically valid

Consider $(A \supset B) \vee (B \supset A)$, $A \vee \neg A$ to show proper inclusion.

Gödel logic is an "<u>intermediate logic</u>".

Gödel (1932) also shows (propositional logics):

$$VALID_{IL} \subsetneqq VALID_{G} \subsetneqq \dots \subsetneqq VALID_{G_4} \subsetneqq VALID_{G_3} \subsetneqq VALID_{CL}$$

for a family of logics $G_3, G_4, G_5, \dots$

Variants of Gödel logic (propositional)

Projection property allows us to restrict truth value set.

Let a set V be given such that $\{0,1\} \subseteq V \subseteq [0,1]$.

A Gödel-V-interpretation is a Gödel-interpretation I such that $I(A) \in V$ for all prop. vars. A.

Projection property $\Rightarrow$ $I(A) \in V$ for all formulas.

A formula A is V-valid if $I(A) = 1$ for all Gödel-V-interpretation I.

Example. If I is a Gödel-V-interpretation, then

$$I(B \vee (B \supset C) \vee (C \supset \bot)) = \max \left\{ I(B), \begin{cases} 1 & I(B) \leq I(C) \\ I(C) & I(C) < I(B) \end{cases}, \begin{cases} 1 & I(C) \leq 0 \\ 0 & 0 < I(C) \end{cases} \right\}$$

$$= \begin{cases} 1 : & I(B) = 1 \text{ or } I(B) \leq I(C) \text{ or } I(C) = 0 \\ I(B) : & \text{otherwise, i.e. if } 0 < I(C) < I(B) < 1 \end{cases}$$

Thus, if $\#V \leq 3$, then $B \vee (B \supset C) \vee (C \supset \bot)$ is V-valid
if $\#V > 3$, then — " — is not V-valid

Proposition: If $\#V \leq n$, then $A_1 \vee (A_1 \supset A_2) \vee (A_2 \supset A_3) \vee \ldots \vee (A_{n-1} \supset \bot)$ is V-valid. If $\#V > n$, it is not V-valid. [Gödel]

Put $V_2 := \{0, 1\}$

$V_3 := \{0, \frac{1}{2}, 1\}$

$V_4 := \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

...

$V_n := \{0, 1\} \cup \{\frac{i}{n-1} ;\ i \in \mathbb{N},\ 1 \leq i < n-1\}$

$V_\uparrow := \{0\} \cup \{\frac{i}{i+1} ;\ i \in \mathbb{N},\ 1 \leq i\} \cup \{1\}$

$V_\downarrow := \{0\} \cup \{\frac{1}{i} ;\ i \in \mathbb{N},\ 1 \leq i\} \cup \{1\}$

Examples: V_2-valid

$[0,1]$-valid

Recall: Isomorphism lemma.

Theorem: $n \in \mathbb{N}$, $n \geq 2$. Then A is V_n-valid if and only if

$$G + FIN_n \vdash A ;$$

here FIN_n is the axiom schema

$$A_1 \vee (A_1 \supset A_2) \vee (A_2 \supset A_3) \vee \ldots \vee (A_{n-1} \supset \bot).$$

Theorem: Let $\{0,1\} \subseteq V \subseteq [0,1]$, V infinite. Then A is V-valid iff $G \vdash A$.

In other words: Only <u>one</u> logic created by <u>all</u> infinite V.

Observe: ⊙ $G_{[0,1]}$ is the intersection of all $G_n := G + FIN_n$

⊙ $\mathbb{IL}^{(prop)} \subsetneq G_{[0,1]} \subsetneq \dots \subsetneq G_4 \subsetneq G_3 \subsetneq G_2 = CL$.

However, entailment relations for sets $V_\uparrow$, $V_\downarrow$, $[0,1]$ are all different.

First-order Gödel logics

("predicate logics" in Hájek's book)

<u>Syntax</u>: like classical/intuitionistic first-order languages:

quantifiers $\forall, \exists$

connectives $\bot, \vee, \wedge, \supset$ to connect atoms

constants; function symbols (of any arity)

predicate symbols (of any arity)

free/bound variables

"typical" formula: $Q(c) \supset \forall x\, (P(x) \vee R(a,x))$

<u>monadic fragment</u>: no function symbols and only <u>unary</u> predicate symbols (arity = 1)

<u>semantics</u> (next page): like classical logic but for $[0,1]$ instead of $\{0,1\}$:

$\wedge \ldots \min$ $\quad \forall \ldots \inf$

$\vee \ldots \max$ $\quad \exists \ldots \sup$

First-order Gödel semantics

("usual" semantics, also "unwitnessed")

Recursive definition of an interpretation I (also "model")

Comprises universe (= domain) $|I| \neq \emptyset$

Connectives ... like in propositional languages

e.g. $I(X \wedge Y) = \min\{I(X), I(Y)\}$

$P^I : |I|^n \to [0,1]$ for every n-ary predicate symbol P

$f^I : |I|^n \to |I|$ " " " function " f

$c^I \in |I|$ for every constant c and every free variable

Quantifiers:

$I(\forall x\, F(x)) = \inf\{I(F(\underline{u})) : u \in |I|\}$

$I(\exists x\, F(x)) = \sup\{I(F(\underline{u})) : u \in |I|\}$

F ... "formula"

$\underline{u}$ ... fresh constant and interpretation I extended to $\underline{u}$ by $I(\underline{u}) = \underline{u}^I := u$

First-order Gödel semantics

Example: $c \ldots$ constant, $a \ldots$ free variable, Q, P, R pred. symbols

$$I\Big(Q(c) \wedge \forall x\, (P(x) \vee R(a,x)) \Big) =$$

$$= \min\Big\{ Q^I(c^I),\ \inf_{u \in |I|} \max\{ P^I(u),\ R^I(a^I, u)\} \Big\}$$

Example: $I(\exists x (Q(x) \supset \forall y\, Q(y))) =$

$$= \sup_{x \in |I|} \begin{cases} 1 : & \text{if } Q^I(x) \leq \inf_{y \in |I|} Q^I(y) \\ \inf_{y \in |I|} Q^I(y) : & \text{if } Q^I(x) > \inf_{y \in |I|} Q^I(y) \end{cases} =$$

$$= \begin{cases} 1 : & \text{if there is } x \in |I| \text{ such that } Q^I(x) = \inf_{y \in |I|} Q^I(y) \\ \inf_{y \in |I|} Q^I(y) : & \text{otherwise} \end{cases}$$

Example: $I(\exists x ((\exists y\, Q(y)) \supset Q(x))) = \ldots =$

$$\begin{cases} 1 : & \text{if there is } x \in |I| \text{ such that } Q^I(x) = \sup_{y \in |I|} Q^I(y) \\ \sup_{y \in |I|} Q^I(y) : & \text{otherwise} \end{cases}$$

Variants of first-order Gödel logic

Restricted truth-value set V:

Usual assumption: V is closed.

Other variant: V arbitrary but set of interpretations is restricted to safe ones.

(Evaluation of formula is not changed!)

Third variant: Witnessed interpretations ($[0,1]$ or restricted V)

Set of interpretations is restricted to those interpretations $\mathcal{I}$ such that for every "parametric" formula $A(\cdot)$ there are $u^A_{min}, u^A_{max} \in |\mathcal{I}|$ with

$$\mathcal{I}(A(\underline{a})) \leq \mathcal{I}(A(\underline{u^A_{max}})) \quad \text{for all } a \in |\mathcal{I}| \quad \text{and}$$
$$\mathcal{I}(A(\underline{a})) \geq \mathcal{I}(A(\underline{u^A_{min}})) \quad \text{—"—}$$

This yields

$$\mathcal{I}(A(\underline{u^A_{max}})) = \sup_{a \in |\mathcal{I}|} \mathcal{I}(A(\underline{a}))$$
$$\mathcal{I}(A(\underline{u^A_{min}})) = \inf_{a \in |\mathcal{I}|} \mathcal{I}(A(\underline{a}))$$

(Evaluation of formulas is not changed!)

For witnessed interpretations we thus have:

$$\exists u \in |\mathcal{I}| \quad \mathcal{I}(\forall x\, A(x)) = \mathcal{I}(A(\underline{u}))$$

and $\exists u \in |\mathcal{I}| \quad \mathcal{I}(\exists x\, A(x)) = \mathcal{I}(A(\underline{u}))$,

like in classical logic.

- Consequences: Lifting lemma holds only for witnessed interpretations, not for "usual" ones.

- For "usual" semantics, the $\forall$-quantifier is problematic.

- In all of the above variants:

$$\mathcal{I}(\underset{\exists}{\forall} x\, A(x)) = \underset{\sup}{\inf}_{a \in |\mathcal{I}|} A(\underline{a})$$

but validity/satisfiability changes because the set of interpretations is restricted.

- Projection property (Closure)

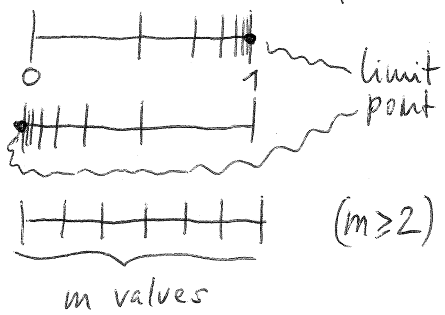
- Choice of truth-value set leads to different set of valid sentences. (see examples)

Important special cases of truth-value sets

$V_\uparrow :=$

$V_\downarrow :=$

$V_m :=$ $(m \geq 2)$



Order-isomorphism lemma!

V_2 ... classical logic

$\exists x\,(A(x) \supset \forall y\, A(y))$ is valid e.g. for all V_m, for $V_\uparrow$, not for $V_\downarrow$, not for $[0,1]$

$\exists x\,(\exists y\, A(y) \supset A(x))$ is valid e.g. for all V_m, for $V_\uparrow$, for $V_\downarrow$, not for $[0,1]$

We have: (G_V ... valid formulas for truth-value set V)

$$IL \subsetneq G_{[0,1]} \subsetneq G_\downarrow \subsetneq G_\uparrow = \bigcap_m G_m \subsetneq \dots \subsetneq G_4 \subsetneq G_3 \subsetneq G_2$$

intuitionistically valid (IL) — classically valid (G_2)

Proof systems for first-order Gödel logic

Horn: Weak completeness: A formula A is valid w.r.t. Gödel-semantics over the truth-value set $[0,1]$ if and only if $\mathcal{H}$ proves A:

$$A \in G_{[0,1]} \iff \mathcal{H} \vdash A$$

Here $\mathcal{H}$ is the proof system:

$$IL^{(\text{first-order})} + LIN + QS$$

$IL^{(\text{prop})}$

$(\forall x A(x)) \supset A(t)$

$A(t) \supset \exists x A(x)$

$$\frac{B \supset A(a)}{B \supset \forall x A(x)} \qquad \frac{A(a) \supset B}{\exists x A(x) \supset B}$$

(a not in B)

$(A \supset B) \vee (B \supset A)$

quantifier shift

$[\forall x (B \vee A(x))] \supset [B \vee \forall x A(x)]$

Horn: <u>weak completeness for</u> $[0,1]$

Takeuti, Titani (systems TT and TT^-); Takano:

<u>strong completeness for</u> $[0,1]$:

$$\Sigma \Vdash A \iff \underbrace{\mathcal{H}+\Sigma}_{\mathcal{H} \text{ plus new axioms } \Sigma} \vdash A$$

<u>Deduction theorem.</u> Let A, B be formulas <u>without</u> free variables. Then

$$\Sigma, A \Vdash B \iff \mathcal{H}+\Sigma \vdash A \supset B$$

Preining: Classification for other truth-value sets
(V closed, "usual" semantics)

- V finite, $\#V =: n$

  $$A \text{ valid} \iff \mathcal{H} + \mathrm{FIN}_n \vdash A$$

- V countable: No proof system (... with "good" properties)

  Set of valid formulas <u>not</u> recursively enumerable

- V uncountable:
  - 0 in the perfect kernel of V

    $$A \text{ valid} \iff \mathcal{H} \vdash A$$

  - 0 isolated in V

    $$A \text{ valid} \iff \mathcal{H} + \forall x \neg\neg B(x) \supset \neg\neg \forall x B(x) \vdash A$$

  - other cases: Set of valid formulas <u>not</u> r.e.

Examples of quantified propositional logic

Formulas: e.g. $\forall P(P \vee Q)$, $Q \vee \forall R((S \wedge (T \vee R)) \supset Q)$

Semantics: $\{0,1\} \subseteq V \subseteq [0,1]$, V closed, given.

$$I(\forall A\ F(A)) = \inf_{v \in V} I(F(\underline{v}))$$
$$\exists \qquad\qquad \sup$$

$\underline{v}$ ... fresh propositional constant

essential data of an interpretation:
$I: Var \longrightarrow V$

In contrast to first-order Gödel logic,
qp logic for $[0,1]$, $V_\uparrow$, $V_\downarrow$ are decidable.

E.g. for $[0,1]$ a hypersequent calculus is available.

Hilbert-style proof system for $G^{qp}_{[0,1]}$:

$$IL^{(prop)} + A(Q) \supset \exists P\, A(P)$$

$$+ \ (\forall P\, A(P)) \supset A(Q)$$

$$+ \ \frac{A(Q) \supset B}{(\exists P\, A(P)) \supset B} + \frac{B \supset A(Q)}{B \supset \forall P\, A(P)}$$

$$+ \left(\forall P\, (A(P) \vee B)\right) \supset \left((\forall P\, A(P)) \vee B\right) \qquad (QS)$$

$$+ \ \forall P\left((A \supset P) \vee (P \supset B)\right) \supset (A \supset B) \qquad (DENSE)$$

do not contain P

Moreover, interesting model-theoretic features, e.g. quantifier elimination