PROJECTIVE MONOIDAL RESIDUATED ALGEBRAS

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Abstract

A characterization of finitely generated projective algebras with residuated monoid structure reduct is given. In particular it is shown that an m-generated subalgebra of m-generated free algebra is projective if and only if it is finitely presented by special kind of equation. Moreover, a characterization of finitely generated projective algebras is given in concrete varieties of MV-algebras and Heyting algebras.

Key words: Free algebra, projective algebra, finitely presented algebra, monoids, hoops, BL-algebra, MV-algebra, Heyting algebra.

1 Introduction and preliminaries

Residuated structures appears in many areas of mathematics, the main origin of which are monoidal operation multiplication $\odot$ that respects a partial order $\leq$ and a binary (left-) residuation operation $\rightarrow$ characterized by $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$. Such kind of structures are associated with logical systems. If the partial order is a semilattice order, and multiplication the semilattice operation, we get Browerian semilattices which are models of the conjuction-implication fragment of the intuitionistic propositional calculus. The well-known algebraic models of the conjucttion-implication fragment of Lukasiewicz many-valued logic are another example of special class of residuated structures. We are interested mainly with those monoidal structures which have in common the following basic properties : Integrality, commutativity of the monoidal operation $\odot$ and the existence of a binary operation $\rightarrow$ which is adjoint to the given operation $\odot$.

Bosbach ([4], [5]) undertook the investigation of a class of residuated structures (with *natural* partial order). He showed that the resulting class of structures can be viewed as an equational class, and the class is congruence distributive and congruence permutable.

J. R. Büchi and T. M. Owens [7] named the commutative members of this equational class by *hoops*.

The fundamental work is devoted by W. J. Blok and I. M. A. Ferreirim to investigation of hoops in [2]. We shall use the definitions and some results from this work, where hoop is defined as a naturally ordered *pocrim* (i. e., partially ordered commutative residuated integral monoid). If, in addition, pocrim is a lattice, compatible with the partially order, then it is called by integral, residuated, commutative l-monoid (see [24]).

We list now the algebraic structures which contain as a reduct a residuated operation : Wajsberg hoops [2], residuated lattices and BL-algebras [22], Wajsberg algebras [15], MV-algebras [8], Heyting algebras (alias pseudo-Boolean algebras) , Gödel algebras and product algebras [22]. We give to such kind of algebras the name *monoidal residuated algebras*, shortly MRA-algebras.

A structure $(A, \odot, 1, \leq)$ is *partially ordered monoid* if $(A, \odot, 1)$ is a monoid, $\leq$ is a partial order on A, and for all $x, y, z \in A$, if $x \leq y$, then $x \odot y \leq y \odot z$ and $z \odot x \leq z \odot y$. A is *integral* if, for all $x \in A$, $x \leq 1$. A is *residuated* if for all $x, y \in A$ the set $\{z : z \odot x \leq y\}$ contains the greatest element, called the *residual of* x *relative to* y, and denote by $x \rightarrow y$. A partially ordered, commutative, residuated and integral monoid (*pocrim*) $(A, \odot, 1, \leq)$ can be treated as an algebra $(A, \odot, \rightarrow, 1)$, since the partial order can be retrieved via $x \leq y$ iff $x \rightarrow y = 1$. The class $\mathcal{M}$ of all pocrims satisfies the following axioms [2]:

(M1) $x \odot 1 = x$,

(M2) $x \odot y = y \odot x$,

(M3) $x \rightarrow 1 = 1$,

(M4) $1 \rightarrow x = x$,

(M5) $(x \to y) \to ((z \to x) \to (z \to y)) = 1$,

(M6) $x \to (y \to z) = (x \odot y) \to z$,

(M7) $x \to y = 1 \,\&\, y \to x = 1 \Rightarrow x = y$.

Conversely, in every algebra $(A, \odot, \to, 1)$ satisfying (M1)-(M7) can be defined a partial order by setting $x \leq y$ iff $x \to y = 1$. This partial order makes $(A, \odot, 1, \leq)$ a commutative partially ordered monoid in which for all $x, y \in A$ $\;x \to y$ is the residual of x with respect to y. In addition to (M1)-(M7), pocrims also satisfies the following properties [2]:

(M8) $x \to x = 1$,

(M9) $x \to (y \to z) = (y \to (x \to z)$,

(M10) If $x \leq y$, then $y \to z \leq x \to z$ and $z \to x \leq z \to y$,

(M11) $x \leq (x \to y) \to y$,

(M12) $x \leq y \to x$.

Theorem 1. *Any pocrim $(A, \odot, 1, \leq)$ satisfies the following identities*

(M13) $(x \to y) \odot (y \to z) \leq (x \to z)$,

(M14) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$.

Proof. $(x \to y) \odot (y \to z) \odot x \leq$

according to the definition of residuation : $x \odot (x \to y) \leq y$,

$\leq y \odot (y \to z) \leq z$ (according to the definition of residuation). Therefore (according to the definition of residuation)

$$(x \to y) \odot (y \to z) \leq (x \to z).$$

$$(x \to y) \to ((y \to z) \to (x \to z)) = ((x \to y) \odot (y \to z)) \to (x \to z) \geq$$

according to **M13**

$$\geq (x \to z) \to (x \to z) = 1.$$

$\square$

The quasivariety $\mathcal{M}$ of all pocrims is the equivalent algebraic semantics - in the sense of [3] - of the algebraizable deductive system $\mathbf{S}_{\mathcal{M}}$ [31] :

(S1) $(p \to q) \to ((r \to p) \to (r \to q))$,

(S2) $(p \to (q \to r)) \to (q \to (p \to r))$,

(S3) $p \to (q \to p)$,

(S4) $p \to (q \to (p \odot q))$,

(S5) $(p \to (q \to r)) \to ((p \odot q) \to r)$.

The only inference rule of $\mathbf{S}_{\mathcal{M}}$ is *Modus Ponens* :

(MP) $p, p \to q \vdash q$.

Let us note that if $\mathbf{R}$ is a commutative ring with identity 1 and $Id(\mathbf{R})$ is the monoid of ideals of $\mathbf{R}$, with the usual ideal multiplication, ordered by inclusion, then for any two ideals I, J of $\mathbf{R}$, the residual of I relative to J exists and is given by $I \to J = \{x \in \mathbf{R} : xI \subseteq J\}$. Hence, $(Id(\mathbf{R}), \cdot, \to, R)$ is a pocrim.

Let us note, that the class $\mathcal{BCK}$ of BCK-algebras consists of all algebras $(A, \to, 1)$ satisfying (M3), (M4), (M5), (M7), (M8) and (M9). Since $\mathcal{M}$ satisfies all of these, the class of $\{\to, 1\}$-subreducts of algebras from $\mathcal{M}$ consists of BCK-algebras. Conversely, every BCK-algebra is a subreduct of a pocrim [30], [29], [14]. Wronski [33] and Higgs [23] showed that $\mathcal{BCK}$ is not a variety.

We say that a partially ordered commutative monoid $(A, \odot, 1, \leq)$ is *naturally ordered* if for all $x, y \in A$, $x \leq y$ iff $(\exists z \in A)(x = z \odot y)$ (divisibility condition).

An algebra $(A, \odot, \to, 1)$ is called a *hoop* if it is naturally ordered pocrim.

Theorem 2. *[4] An algebra $(A, \odot, \to, 1)$ is a hoop if and only if $(A, \odot, 1)$ is a commutative monoid that satisfies the following identities*

(M6) $x \to (y \to z) = (x \odot y) \to z$,

(M8) $x \to x = 1$,

(M15) $(x \to y) \odot x = (y \to x) \odot y$.

Denote by $\mathcal{HO}$ the variety of all hoops.

1.1 Filters and congruences

Let $(A, \odot, \to, 1)$ be a hoop. We say that $F \subseteq A$ is *filter* of A if (i) $1 \in F$; (ii) if $x \in F$, $y \in A$ and $y \geq x$, then $y \in F$; (iii) if $x \in F$ and $y \in F$, then $x \odot y \in F$.

Lemma 3. *Any filter F of a hoop A satisfies the following condition : if $x \in F$ and $x \to y \in F$, then $y \in F$.*

Proof. According to the definition of residuation $x \odot (x \to y) \leq y$. But $x \odot (x \to y) \in F$. Hence $y \in F$. □

One can easily check that, given $X \subseteq A$, the least filter generated by X, denoted by $[X)$, is $\{b \in A : a_1 \odot a_2 \cdots \odot a_n \leq b \text{ for some } a_1, a_2, \ldots, a_n \in X, n \in \omega\}$. If, in particular, $X = \{a\}$, then $[a) = \{b \in A : a^n \leq b \text{ for some } n \in \omega\}$. It easy to check that if ϱ is a congruence of A then $1/\varrho$ is a filter of A. Moreover, the map $\varrho \to 1/\varrho$ establishes an order isomorphism between the lattice of congruences of A and its lattice of filters. The inverse of this map is $F \to \varrho_F$, where $\varrho_F = \{(x, y) : (x \to y) \odot (y \to x) \in F\}$ is a congruence of A. If F is the filter associated with the congruence ϱ we often write A/F for A/ϱ.

We express the considerations in

Theorem 4. *Let F be a filter of a hoop A. Then the binary relation ϱ_F on A defined by $x\varrho_F y$ iff $x \to y \in F$ and $y \to x \in F$ is a congruence relation. Moreover, $F = \{x \in A : x\varrho_F 1\}$.*

Conversely, if ϱ is a congruence on A, then $\{x \in A : x\varrho 1\}$ is a filter, and $x\varrho y$ iff $(x \to y)\varrho 1$ and $(y \to x)\varrho 1$. Therefore, the correspondence $F \mapsto \varrho_F$ is a bijection from the set of filters of A and the set of congruences on A.

Proof. It is obvious that ϱ_F is reflexive and simmetric. From **M13** it follows that ϱ_F is transitive. Hence ϱ_F is an equivalence relation.

Now, suppose that $x\varrho_F s$ and $y\varrho_F t$. Since

$$x \odot y \odot (x \to s) \odot (y \to t) = x \odot (x \to s) \odot y \odot (y \to t) \leq s \odot t$$

(since $x \odot (x \to s) \leq s$, $y \odot (y \to t) \leq t$),
$(x \to s) \odot (y \to t) \leq x \odot y \to s \odot t$. Hence, $x \odot y \to s \odot t \in F$. Interchanging x with s and y with t, we get $s \odot t \to x \odot y \in F$. Therefore $x \odot y\varrho_F s \odot t$. By **M5**, we have that

$$(y \to t) \to ((x \to y) \to (x \to t)) \in F,$$

and since $y \to t \in F$, we get that

$$(x \to y) \to (x \to t) \in F.$$

By **M14**, we have that

$$(s \to x) \to ((x \to t) \to (s \to t)) \in F,$$

and since $s \to x \in F$, we get that

$$(x \to t) \to (s \to t) \in F.$$

Hence

$$(x \to y) \to (x \to t) \odot (x \to t) \to (s \to t) \in F.$$

According to **M13**

$$(x \to y) \to (x \to t) \odot (x \to t) \to (s \to t) \leq (x \to y) \to (s \to t).$$

Hence, $(x \to y) \to (s \to t) \in F$. In a similar way we can see that also $(s \to t) \to x \to y) \in F$. Hence $(x \to y)\varrho_F(s \to t)$.

From **M2** and **M3** it follows that $F = \{x \in A : x\varrho_F 1\}$.

Conversely, suppose that ϱ is a congruence on A. If $x\varrho 1$ and $y\varrho 1$, then $(x \odot y)\varrho 1$. If $x\varrho 1$ and $x \leq y$, then $x \to y = 1$ and $1 = (x \odot (x \to y))\varrho 1 \to= y$ Hence $\{x \in A : x\varrho 1\}$ is a filter.

If $x\varrho y$, then $(x \to y)\varrho(y \to x)\varrho(x \to x) = 1$. Conversely, if $(x \to y)\varrho 1$ and $(y \to x)\varrho 1$, then $x\varrho x \odot (x \to y) = y \odot (y \to x)\varrho y$. Therefore $x\varrho y$. □

Now suppose that $(A, \odot, \to, 1)$ is a pocrim. As it is easy to see, if F is a filter of A, then ϱ_F is a congruence of A. Moreover, $(A/\varrho_F, \odot, \to, 1)$ is a pocrim, where $x/\varrho_F \odot y/\varrho_F = (x \odot y)/\varrho_F$, $x/\varrho_F \to y/\varrho_F = (x \to y)/\varrho_F$ and $1 = 1_{A/\varrho_F} = 1/\varrho_F$. Indeed, it is enough to check the quasi-identity **M7**. If $x/\varrho_F \to y/\varrho_F = 1/\varrho_F = (x \to y)/\varrho_F$, then $1 \to (x \to y) = (x \to y) \in F$. Similarly, if $y/\varrho_F \to x/\varrho_F = 1/\varrho_F = (y \to x)/\varrho_F$, then $1 \to (y \to x) = (y \to x) \in F$. Hence $(x \to y) \odot (y \to x) \in F$, i. e. $x\varrho_F y$. It means, that $x/\varrho_F = y/\varrho_F$. Therefore $(A/\varrho_F, \odot, \to, 1)$ is a pocrim. Since all pocrims form quasi-variety, but not a variety [23], the set $\{x \in A : x\varrho 1\}$ may be not a filter. In other words not every congruence ϱ converts a pocrim A into a pocrim A/ϱ.

From here we conclude that holds

Theorem 5. *If* $(A, \odot, \to, 1)$ *is a pocrim and* $F \subset A$ *is a filter, then* $(A/\varrho_F,\ \odot/\varrho_F,\ \to/\varrho_F,\ 1/\varrho_F)$ *is a pocrim.*

1.2 Wajsberg hoops and MV-algebras

A hoop is called *Wajsberg hoop* if it additionally satisfies the following condition

(Wh) $(x \to y) \to y = (y \to x) \to x$.

Any hoop which satisfies **(Wh)** is in fact a lattice, where the join operation is defined by $x \vee y =_{Def} (x \to y) \to y$. The identity **(Wh)** expresses the fact that $\vee$ is a commutative operation.

Wajsberg hoops are closely related to the Wajsberg algebras [15], which are algebraic models of Łukasiewicz's many-valued logic. Actually, Wajsberg hoops with least element are termwise equivalent to Wajsberg algebras [3]. Moreover, it is easy to see that Wajsberg hoops which have a least element are exactly the $\{\odot, \to, 1\}$-reducts of Wajsberg algebras.

Wajsberg algebras are definitionally equivalent to Chang's MV-algebras [8]. It is well known that MV-algebras are algebraic models of infinite valued Łukasiewicz logic. We assume the reader's familiarity with MV-algebras. For all needed notions in MV-algebras we refer to [8], [10].

We recall that an algebra $A = (A; 0, 1, \oplus, \odot, *)$ is said to be an MV -algebra iff it satisfies the following equations:

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $x \oplus y = y \oplus x$;
3. $x \oplus 0 = x$;
4. $x \oplus 1 = 1$;
5. $0^* = 1$;
6. $1^* = 0$;
7. $x \odot y = (x^* \oplus y^*)^*$;
8. $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Henceforth we shall write ab for $a \odot b$ and a^n for $\underbrace{a \odot \cdots \odot a}_{n \text{ times}}$, for given $a, b \in A$. Every MV-algebra has an underlying ordered structure defined by

$$x \leq y \text{ iff } x^* \oplus y = 1.$$

$(A; \leq, 0, 1)$ is a bounded distributive lattice. Moreover, the following property holds in any MV-algebra :

$$xy \leq x \wedge y \leq x \vee y \leq x \oplus y.$$

A subset J of an MV-algebra A is called *a filter* provided that (1) $1 \in J$, (2) $a \in J$ and $b \in J$ imply $ab \in J$, and (3) $a \leq b$ and $a \in J$ imply $b \in J$. The correspondence : $\theta \mapsto \varphi(\theta) = 1/\theta = \{a \in A : (a, 1) \in \theta\}$ establishes an isomorphism φ from the lattice of all congruences onto the lattice of all filters (ordered by inclusion) [8], [10], [19]. The unite interval of real numbers $[0, 1]$ endowed with the following operations : $x \oplus y = \min(1, x + y), x \odot y = \max(0, x + y - 1), x^* = 1 - x$, becomes an MV-algebra. For $(0 \neq) m \in \omega$ we set $\mathcal{L}_m = (S_m; \oplus, \odot, *, 0, 1)$, where

$$S_m = \left\{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1\right\},$$

which is a proper subalgebra of the unite interval MV-algebra.

Denote by $\mathcal{WHO}$, $\mathcal{WA}$, $\mathcal{MV}$ the varieties of all Wajsberg hoops, Wajsberg algebras and MV-algebras, respectively.

1.3 Dual hoops

A *dual hoop* is an algebra $(A, \oplus, \dot{-}, 0)$ such that $(A, \oplus, 0, \leq)$ is partially ordered commutative monoid , with identity 0, which is the least element of A, and for all $x, y \in A$, $x \dot{-} y$ is the smallest element of the set $\{z : x \leq z \oplus y\}$. Let us note that while in hoops the partial order satisfies $x \leq y$ iff $x = z \odot y$, for some $z \in A$ $(z = x \to y)$, the partial order in dual hoops satisfies $x \leq y$ iff $y = z \oplus x$, for some $z \in A$. Thus if $(A, \odot, \to, 1)$ is a hoop then $(A, \oplus, \dot{-}, 0)$ is a dual hoop, where $x \oplus y := x \odot y,\ x \dot{-} y := y \to x,\ 0 := 1$. Conversely, if $(A, \oplus, \dot{-}, 0)$ is a dual hoop then $(A, \odot, \to, 1)$ is a hoop, where $x \odot y := x \oplus y,\ x \to y := y \dot{-} x, 1 := 0$. The classes of hoops and dual hoops are therefore term equivalent.

EXAMPLE. Let $(G, +, -, 0, \vee, \wedge)$ be a lattice- ordered Abelian group, or Abelian ℓ-group, for short, with strong unit $u \in G,\ u > 0$. Define on the set $G[u] = \{x \in G : 0 \leq x \leq u\}$ the following operations : $a +_u b = (a + b) \wedge u,\ a \dot{-} b = (a - b) \vee 0$. Then $(G[u], +_u, \dot{-}, 0)$ is dual hoop. This dual hoop may be seen as an instance of a reduction of the action of the Gamma functor from Abelian ℓ-groups with strong unit to MV-algebras [27], [10].

1.4 PL-algebras

Product logic algebras, or PL-algebras, for short, were introduced by Hájek, Godo and Esteva [21]. The fundamental work on PL-algebras belong to R. Cignoli and A. Torrens [9].

A *PL-algebra* is an algebra $A, \odot, \rightarrow, 0)$ of type $(2, 2, 0)$ such that, upon derived operations : $1 = 0 \rightarrow 0$, $\neg x = x \rightarrow 0$, $x \wedge y = x \odot (x \rightarrow y)$, $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$, satisfies the following identities :

(PL1) $(A, \odot, 1)$ is a commutative monoid with identity,

(PL2) $(A, \vee, \wedge, 0, 1)$ is a lattice with smallest (0) and greatest (1) elements,

(PL3) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$, $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,

(PL4) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,

(PL5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$,

(PL6) $x \rightarrow x = 1$,

(PL7) $(\neg\neg z \odot ((x \odot z) \rightarrow (y \odot z))) \rightarrow (x \rightarrow y) = 1$,

(PL8) $x \wedge \neg x = 0$.

Let A be PL-algebra and F a filter of A. Then by familiar arguments one can show that the stipulation :

For $x, y \in A$, $\quad x\varrho_F y$ iff $(x \rightarrow y) \wedge (y \rightarrow x) \in F$

defines a congruence relation ϱ_F on A, and that $F = \{x \in A : x\varrho_F 1\}$. The correspondence $F \mapsto \varrho_F$ defines a one-one inclusion preserving mapping.

Denote by $\mathcal{PL}$ the variety of all PL-algebras.

1.5 BL-algebras

Let us consider the real unit interval with interval topology. A *continuous t-norm* is a continuous function $* : [0, 1]^2 \rightarrow [0, 1]$ that is associative, commutative, monotone in each argument, and satisfying $a * 1 = a$ and $a * 0 = 0$, for every $a \in [0, 1]$. Every continuous t-norm $*$ induces a *residuum* (i. e. *implication*) $\rightarrow$, defined by $a \rightarrow b =$sup$\{c : c* \leq b\}$.

The structure $([0,1],*,\to,0,1)$ is a hoop, and $([0,1],\vee,\wedge,*,\to,0,1)$ is a main example of a *residuated lattice.*

Every continuous t-norm is locally isomorphic to one of the following (see, for example, [22]) :

- Łukasiewicz t-norm $\odot$, defined by $a \odot b = max\{a+b-1,0\}$;
- Gödel t-norm $\wedge$, defined by $a \wedge b = min\{a,b\}$;
- product t-norm $\cdot$, i. e. , ordinary product of real numbers.

Given a continuous t-norm and its residuum , the algebra $([0,1],\ \vee,\ \wedge,\ *,\ \to,0,1)$ generates a variety of residuated lattices, and the set of propositional formulas in the language $(*,\to,0,1)$ that take value 1 under every interpretation is called the *logic* of $*$. Let us note that we do not need to have the lattice operations in the language , since they are definable by $a \wedge b = a*(a\to b)$ and $a \vee b = ((a\to b)\to b)\wedge((b\to a)\to a)$ [22]. Specializing to the tree basic cases above, we have Łukasiewicz logic, Gödel logic, and product logic. All the tree logics are interesting, and the most general fuzzy logic is the logic of all continuous t-norms, i. e. , Hájek's Basic Logic [22].

BL-algebras is introduced by P. Hájek in [22] as an algebraic counterpart of one of the extensions of fuzzy logic. BL-algebra [22]

$$(B,\vee,\wedge,\to,\odot,0,1)$$

is a universal algebra of type $(2,2,2,1,0,0)$ such that :

1) $(B,\vee,\wedge,0,1)$ is a bounded lattice ;
2) $(B,\odot,1)$ is a commutative monoid with identity :

$$x\odot q = q\odot p$$
$$p\odot(q\odot r) = (p\odot q)\odot r$$
$$p\odot 1 = 1\odot p$$

3) (1) $p\wedge(q\to(p\odot q)) = p$,
(2) $((p\to q)\odot p)\vee q = q$,
(3) $(p\to(p\vee q)) = 1$,
(4) $((p\to r)\to(r\to(p\vee q))) = 1$,
(5) $(p\wedge q)\odot r = (p\odot r)\wedge(q\odot r)$,
(6) $p\wedge q = p\odot(p\to q)$,
(7) $p\vee q = ((p\to q)\to q)\wedge((q\to p)\to p)$,
(8) $(p\to q)\vee(q\to p) = 1$.

Denote by $\mathcal{BL}$ the variety of all BL-algebras.

1.6 Heyting algebras

A *Heyting algebra* $(H, \wedge, \vee, \rightarrow, 0, 1)$ is a bounded distributive lattice $(H, \wedge, \vee, 0, 1)$ with an additional binary operation $\rightarrow: H \times H \rightarrow H$ such that for any $a, b \in H$

$$x \leq a \rightarrow b \text{ iff } a \wedge x \leq b.$$

(Here $x \leq y$ iff $x \wedge y = x$ iff $x \vee y = y$.)

It is well-known that the class of all Heyting algebras forms a variety, which will be subsequently denoted by $\mathcal{HA}$. Heyting algebras play an important role in different branches of mathematics: opens of a topological space, the lattice of congruences of a lattice, the object classificator of a topos, as well as algebraic models of Intuitionistic Logic all form Heyting algebras. These (and other) important features boosted a thorough investigation of Heyting algebras. A lot of results have been obtained. We will list only some of them: representation of Heyting algebras by means of Esakia spaces, which are "good" Priestley spaces (Esakia [13]); description of finitely generated free Heyting algebras (Urquhart [32], Grigolia [18], Bellissima [1], Ghilardi [16], Butz [6]).

Idempotent hoops, i. e. , hoops which satisfies $x \odot x = x$, are semilattices with respect to the operation of multiplication . They have been considered in the literature under the names *implicative semilattices* [28], and *Brouwerian semilattices* [25] and are the $\{\wedge, \rightarrow, 1\}$-subreducts of Heyting algebras.

All listed above classes of algebras are characterized with common property : filter defined congruences, all primitive operation of algebras are defined by monoidal operation and its adjoint - residuum. These algebras from the classes we name *monoidal residuated algebras.*

2 Free and projective algebras

Let $\mathbb{K}$ be an arbitrary variety. Recall that an algebra $A \in \mathbb{K}$ is said to be a *free algebra* in $\mathbb{K}$, if there exists a set $A_0 \subset A$ such that A_0 generate A and every mapping f from A_0 to any algebra $B \in \mathbb{K}$ is extended to a homomorphism h from A to B. In this case A_0 is said to be *the set of free generators* of A. If the set of free generators is finite then A is said to be a *free algebra of finitely many generators.*

The name "free algebra" came from the fact that free algebras of $\mathbb{K}$ are free from any additional identities on the free generators. In other words, for any identity $\mathbf{p}(x_1, \ldots, x_m) = \mathbf{q}(x_1, \ldots, x_m)$, $\mathbf{p}(x_1, \ldots, x_m) = \mathbf{q}(x_1, \ldots, x_m)$ is an identity of $\mathbb{K}$ iff the polynomials $P(g_1, \ldots, g_m)$ and $Q(g_1, \ldots, g_m)$, corresponding to $\mathbf{p}(x_1, \ldots, x_m)$ and $\mathbf{q}(x_1, \ldots, x_m)$ respectively, are equal to each other in the m-generated free algebra $F(m) \in \mathbb{K}$, where $g_1, \ldots, g_m$ denote the free generators of $F(m)$.

Also recall that an algebra $A \in \mathbb{K}$ is called *projective*, if for any $B, C \in \mathbf{K}$, any epimorphism (that is an onto homomorphism) $\gamma : B \to C$ and any homomorphism $\beta : A \to C$, there exists a homomorphism $\alpha : A \to B$ such that $\gamma\alpha = \beta$.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & B \arrow[d, "\gamma"] \\ & C \end{tikzcd}$$

An algebra $B \in \mathbb{K}$ is said to be a *retract* of an algebra $A \in \mathbb{K}$, if there exist a monomorphism (that is a one-to-one homomorphism) $\mu : B \to A$ and an epimorphism $\varepsilon : A \to B$ such that $\varepsilon\mu = Id_B$, where Id_B denotes the identity map on B.

$$\begin{tikzcd} B \arrow[r, shift right, "\mu"'] & A \arrow[l, shift right, "\varepsilon"'] \end{tikzcd}$$

It is well known that in every variety $\mathbb{K}$, projective algebras can be characterized as retracts of free algebras of $\mathbb{K}$. (In particular, every free algebra of $\mathbb{K}$ is projective.)

Let $\mathcal{K} \in \{\mathcal{HO}, \mathcal{WH}, \mathcal{WA}, \mathcal{MV}, \mathcal{PL}, \mathcal{BL}, \mathcal{HA}\}$.

Let $F(m, \Omega)$ be the free algebra of m generators in a variety of $\mathcal{K}$ defined by using the finite set Ω of extra axioms in m variables (for the definition see [17]). Let us note that if Ω is a finite set of m-ary identities, then it can be represented by just one identity $P = 1$. Indeed, if $P = Q$ is an identity, then the one is equivalent to $(P \to Q) \odot (Q \to P) = 1$ or , in abbreviated version, $P \leftrightarrow Q = 1$. In turn, if we have finite number of identities $\Omega = \{P_1 = Q_1, \ldots, P_n = Q_n\}$, then we can replace the one by the equivalent identity $\bigodot_{i=1}^{n}(P_i \leftrightarrow Q_i) = 1$.

Lemma 6. *Let P be an m-ary polynomial. Then there is a filter J such that*

$F(m, \{P = 1\}) \cong F(m)/J$.

Proof. Let $J = [P(g_1, \dots, g_m))$ be the least filter containing $P(g_1, \dots, g_m)$ where $g_1, \dots, g_m$ be free generators of $F(m)$. We need to prove that the "principal" filter J such that $F(m)/J \cong F(m, P = 1)$. Let $g_1, \dots, g_m$ be free generators of $F(m)$. Then $g_1/J, \dots, g_m/J$ are generators of $F(m)/J$. Let also A be an MV-algebra generated by $\{a_1, \dots, a_m\}$, $P(a_1, \dots, a_m) = 1$ and $f : F(m) \to A$ be a homomorphism such that $f(g_i) = a_i$, $i = 1, \dots, m$. Then $P^n(g_1, \dots, g_m) \in f^{-1}(\{1\})$, $n \in \omega$ and therefore $J \subseteq f^{-1}(\{1\})$. By the homomorphism theorem there is a homomorphism $f' : F(m)/J \to A$ such that the diagram

$$\begin{tikzcd}
F(m) \arrow[rr, "f"] \arrow[dr, "\pi_J"'] & & A \\
& F(m)/J \arrow[ur, "f'"'] &
\end{tikzcd}$$

commutes. It should be clear that f' is the needed homomorphism extending the map $g_i/J \to a_i$. □

Lemma 7. *Let $u \in F(m)$ be a generator of the proper filter $J = \{x : x \geq u^n, n \in \omega\}$. Then $F(m)/J \cong F(m, \{P = 1\})$, where P is some m-ary polynomial.*

Proof. Let J be a filter satisfying the condition of the Lemma. Then $u = P(g_1, \dots, g_m)$ for some polynomial P, where $g_1, \dots, g_m$ are free generators. We have that $F(m)/J$ is generated by $g_1/J, \dots, g_m/J$, and that

$$P(g_1/J, \dots, \dots, g_m/J) = P(g_1, \dots, g_m)/J = 1_{F(m)/J}.$$

The rest can be verified as in the proof of Lemma 1. □

Proposition 8. *If $A \in \mathbb{MV}$ is finite and generated by m elements, then there is a principal filter J such that $A \cong F(m)/J$.*

Proof. Let $A \in \mathbb{MV}$ be finite and generated by $a_1, \dots, a_m$, i. e. $A =< \{a_1, \dots, a_m\} >$. Let P_{a_i} be the m-ary polynomial x_i, and in general let P_x be a polynomial such that $P_x(a_1, \dots, a_m) = x$ for each $x \in A$. Let Ω be the collection of equations of the type $P_x \oplus P_y = P_{x \oplus y}$, $P_x \odot P_y = P_{x \odot y}$, $P_x^* = P_{x^*}$ for x, $y \in A$ and $P_0 = 0$, $P_1 = 1$. Then $A \cong F(m, \Omega)$. For if $A_1 =< \{b_1, \dots, b_m\} >$ and $b_1, \dots, b_m$ satisfy E, then $\{P_x(b_1, \dots, b_m) : x \in$

$A\} = A_1$ and the map $h : A \to A_1$ defined by $h(x) = P_x(b_1, \ldots, b_m)$ is a homomorphism extending the map $a_i \to b_i$ $i = 1, \ldots, m$. Since Ω is finite, the proposition follows. $\square$

Corollary 9. *If $0 < k \leq m$, then there is a principal filter J such that $F(k) \cong F(m)/J$.*

Proof. $F(k) \cong F(m, x_k = x_{k+1}, x_k = x_{k+2}, \ldots, x_k = x_m)$. $\square$

The assertions 1-4, formulated above, is given in [12].

From the above mentioned we arrive to

Theorem 10. *An MV-algebra A is finitely presented iff $A \cong F(n)/J$ for some principal filter J of $F(n)$.*

Now we give exact definition of finitely presented algebra [26]. An algebra A is called *finitely presented* if A is finitely generated, with the generators $a_1, \ldots, a_m \in A$, and there exists a finite number of equations

$$\begin{gathered} P_1(x_1, \ldots, x_m) = Q_1(x_1, \ldots, x_m) \\ \ldots \\ P_m(x_1, \ldots, x_m) = Q_m(x_1, \ldots, x_m) \end{gathered}$$

holding in A on the generators $a_1, \ldots, a_m \in A$ such that if there exists m-generated algebra B, with generators $b_1, \ldots, b_m \in B$, then there exists a homomorphism $h : A \to B$ sending a_i to b_i.

3 n-generated projective MV-algebras

Theorem 11. *[20] .Let $F(m)$ be the m-generated free algebra of a variety $\mathbb{K}$, and let $g_1, \ldots, g_m$ be its free generators. Then an m-generated subalgebra A of $F(m)$ with the generators $a_1, \ldots, a_m \in A$ is projective if and only if there exist polynomials $P_1(x_1, \ldots, x_m), \ldots, P_m(x_1, \ldots, x_m)$ such that*

$$P_i(g_1, \ldots, g_m) = a_i$$

and

$$P_i(P_1(x_1, \ldots, x_m), \ldots, P_m(x_1, \ldots, x_m)) = P_i(x_1, \ldots, x_m)$$

holds in the variety $\mathbb{K}$, $i = 1, \ldots, m$.

Proof. Let A be an m-generated projective subalgebra of $F(m)$. Then, there exists an epimorphism $h : F(m) \to A$ such that $h(g_i) = a_i$, $i = 1, \dots, m$, and $h(x) = x$ for every $x \in A$. Since $a_i \in F(m)$, $i = 1, \dots, m$, and $F(m)$ is m-generated, there exist polynomials $P_1(x_1, \dots, x_m), \dots, P_m(x_1, \dots, x_m)$ such that

$$P_1(g_1, \dots, g_m) = a_1, \dots, P_m(g_1, \dots, g_m) = a_m.$$

But then

$$h(a_i) = h(P_i(g_1, \dots, g_m)) = P_i(h(g_1), \dots, h(g_m)) = P_i(a_1, \dots, a_m) =$$

$$P_i(P_1(g_1, \dots, g_m), \dots, P_m(g_1, \dots, g_m)).$$

On the other hand, since $h(a_i) = a_i$, we have $h(a_i) = P_i(g_1, \dots, g_m)$. Combining these two identities we obtain

$$P_i(P_1(x_1, \dots, x_m), \dots, P_m(x_1, \dots, x_m)) = P_i(x_1, \dots, x_m),$$

$i = 1, \dots, m$.

Conversely, let A be an m-generated subalgebra of $F(m)$ and there exist polynomials $P_i(x_1, \dots, x_m)$ such that

$$P_i(g_1, \dots, g_m) = a_i \tag{1}$$

and

$$P_i(P_1(x_1, \dots, x_m), \dots, P_m(x_1, \dots, x_m)) = P_i(x_1, \dots, x_m), \tag{2}$$

$i = 1, \dots, m$. Since A is m-generated, there exists a homomorphism $h : F(m) \to A$ such that $h(g_i) = a_i$ $(i = 1, \dots, m)$. Let x be any element of $A \subseteq F(m)$. Then there exists a polynomial $Q(x_1, \dots, x_m)$ such that $Q(a_1, \dots, a_m) = x$. But then

$$h(x) = h(Q(a_1, \dots, a_m)) = h(Q(P_1(g_1, \dots, g_m), \dots, P_m(g_1, \dots, g_m))) =$$

$$Q(P_1(h(g_1), \dots, h(g_m)), \dots, P_m(h(g_1), \dots, h(g_m))) =$$

$$Q(P_1(a_1, \dots, a_m), \dots, P_m(a_1, \dots, a_m)) = \text{ (using (1))}$$

$$Q(P_1(P_1(g_1, \dots, g_m), \dots, P_m(g_1, \dots, g_m)), \dots,$$

$$P_m(P_1(g_1,\dots,g_m),\dots,P_m(g_1,\dots,g_m))) = \text{ (using (2))}$$
$$Q(P_1(g_1,\dots,g_m),\dots,P_m(g_1,\dots,g_m)) = \text{ (using (1))}$$
$$Q(a_1,\dots,a_m) = x.$$

Therefore $hId_A = Id_A$ and A is a retract of $F(m)$, which means that A is projective. □

Theorem 12. *Any finitely generated projective MV-algebra is finitely presented.*

Proof. Let A be n-generated projective MV-algebra. Then A is a retract of $F(n)$: there exist a monomorphism $\varepsilon : A \to F(n)$ and onto homomorphism $h : F(n) \to A$ such that $h\varepsilon = Id_A$ and there exist polynomials $P_1(x_1,\dots,x_n),\dots,P_n(x_1,\dots,x_n)$ such that $P_i(g_1,\dots,g_n)) = \varepsilon h(g_i)$ and $P_i(P_1(x_1,\dots,x_n),\dots,P_n(x_1,\dots,x_n)) = P_i(x_1,\dots,x_n)$ holds, $i = 1,\dots,n$, where $g_1,\dots,g_n$ are free generators of $F(n)$ (Theorem 6). Observe that $h(g_1),\dots,h(g_n)$ are generators of A which we denote by $a_1,\dots,a_n$ respectively. Let e be the endomorphism $\varepsilon h : F(n) \to F(n)$. This endomorphism has properties : $ee = e$ and $e(x) = x$ for every $x \in \varepsilon(A)$. Let

$$u = \bigodot_{i=1}^{n} g_i \leftrightarrow P_i(g_1,\dots,g_n)$$

and $[u) = \{x \in F(n) : x \geq u^n, n \in \omega\}$ be the principal filter generated by u, where $x \leftrightarrow y = (x^\star \oplus y) \odot (x \oplus y^\star)$. So $F(n,\Omega) \cong F(n)/[u)$, where $\Omega = \{x_i \leftrightarrow P_i(P_1(x_1,\dots,x_n) = 1 : i = 1,\dots,n\}$ [12]. Observe that the equations from Ω are true in A on the elements $\varepsilon(a_i) = e(g_i)$, $i = 1,\dots,n$. Indeed, since e is an endomorphism

$$e(u) = \bigodot_{i=1}^{n} e(g_i) \leftrightarrow P_i(e(g_1),\dots,e(g_n)).$$

But $P_i(e(g_1),\dots,e(g_n)) = P_i(P_1(g_1,\dots,g_n),\dots,P_n(g_1,\dots,g_n)) = P_i(g_1,\dots,g_n) = \varepsilon h(g_i) = e(g_i), i = 1,\dots,n$. Hence $e(u) = 1$ and $u \in e^{-1}(1)$, i. e. $[u) \subseteq e^{-1}(1)$. Therefore there exists homomorphism $f : F(n)/[u) \to \varepsilon(A)$ such that the diagram

$$\begin{tikzcd} F(n) \arrow[r, "e"] \arrow[dr, "r"'] & \varepsilon(A) \\ & F(n)/[u) \arrow[u, "f"'] \end{tikzcd}$$

commutes, i. e. $rf = e$, where r is a natural homomorphism sending x to $x/[u)$. Now consider the restrictions e' and r' on $\varepsilon(A) \subseteq F(n)$ of e and r respectively Then $fr' = e'$. But $e' = Id_{\varepsilon(A)}$. Therefore $fr' = Id_{\varepsilon(A)}$. From here we conclude that r' is an injection. Moreover r' is a surjection, since $r(\varepsilon(a_i)) = r(g_i)$. Indeed $e(g_i) = P_i(g_1, \ldots, g_n)$ and $g_i \leftrightarrow P_i(g_1, \ldots, g_n) = g_i \leftrightarrow e(g_i)$, where $e(g_i) = \varepsilon h(g_i)$. So $g_i \leftrightarrow P_i(g_1, \ldots, g_n) \geq \bigodot_{i=1}^n g_i \leftrightarrow P_i(g_1, \ldots, g_n)$, i. e. $g_i \leftrightarrow P_i(g_1, \ldots, g_n) \in [u)$. Hence r' is an isomorphism between $\varepsilon(A)$ and $F(n)/[u)$, Consequently $A(\cong \varepsilon(A))$ is finitely presented. $\square$

Theorem 13. *An n-generated subalgebra A, with generators $a_1, \ldots, a_n$, of n-generated free MV-algebra $F(n)$ is projective if and only if it is finitely presented by an equation*

$$\bigodot_{i=1}^{n} x_i \leftrightarrow P_i(x_1, \ldots, x_n) = 1,$$

where $P_i(x_1, \ldots, x_n)$ is some polynomial such that $P_i(g_1, \ldots, g_n) = a_i$, $i = 1, \ldots, n$.

Proof. $\Longrightarrow$ is given in Theorem 7.
$\Longleftarrow$. Let $u = \bigodot_{i=1}^n g_i \leftrightarrow P_i(g_1, \ldots, g_n)$. Then $A \cong F(n)/[u)$. Denote the isomorphism by $\varphi : F(n) \to A$ and the natural homomorphism by $h : F(n) \to F(n)/[u)$. Then $\varphi h : F(n) \to F(n)$ is an endomorphism such that $\varphi h(F(n)) = A$. Moreover $h(g_i) = h(a_i)$. Indeed, $a_i \leftrightarrow g_i = P_i(g_1, \ldots, g_n \leftrightarrow g_i \geq u \Rightarrow a_i \leftrightarrow g_i \in [u) \Rightarrow h(g_i) = h(a_i)$. Therefore $\varphi(h(a_i)) = \varphi(g_i/[u)) = a_i$. This means that $\varphi(h(x)) = x$ for every $x \in A$. From here we conclude that A is projective. $\square$

Let $\mathbb{V}$ be any subvariety of the variety from the following list : *hoops*, $\mathbb{H}, \mathbb{BL}, \mathbb{MV}, \mathbb{G}, \mathbb{PL}$, where $\mathbb{H}, \mathbb{BL}, \mathbb{G}, \mathbb{PL}$ are the varieties of all Heyting , BL-, Gõdel and $PL-$ (product) algebras respectively. In all these varieties we have monoidal operation $\odot$, which has left-adjoint operation - implication $\to$.

Theorem 14. *Any m-generated projective subalgebra A of the m-generated free algebra $F_{\mathbb{V}}(m)$ is finitely presented.*

Proof. Let $a_1, \ldots, a_m \in A$ be generators of A and $g_1, \ldots, g_m \in F_{\mathbb{V}}(m)$ be free generators. There exist homomorphisms $h : F_{\mathbb{V}}(m) \to A$ and $\varepsilon : A \to F_{\mathbb{V}}(m)$

such that $h\varepsilon = Id_A$, and $h(g_i) = a_i$, $i = 1, \ldots, m$. There exist polynomials $P_i(x_1, \ldots, x_m)$ such that $P_i(g_1, \ldots, g_m) = a_i$, $i = 1, \ldots, m$. Let $u = \bigodot_{i=1}^m (P_i(g_1, \ldots, g_m) \leftrightarrow g_i)$ and $[u)$ be the filter generated by u. Denote the natural homomorphism from $F_{\mathbb{V}}(m)$ onto $F_{\mathbb{V}}(m)/[u)$ by $\varphi : F_{\mathbb{V}}(m) \to F_{\mathbb{V}}(m)/[u)$. Denote $\varphi\varepsilon$ by f. Since $\varphi(u) = 1$, $\varphi(P_i(g_1, \ldots, g_m)) = \varphi(g_i)$, $i = 1, \ldots, m$, i. e. $\varphi(a_i) = \varphi(g_i)$, and , hence, $f(a_i) = \varphi(g_i)$ (we identify $\varepsilon(x)$ with x). It is obvious that f is onto homomorphism and $fh = \varphi$, since $h(g_i) = h(a_i)$. Since $h(g_i) = h(a_i)$, $h(u) = \bigodot_{i=1}^m (h(P_i(g_1, \ldots, g_m)) \leftrightarrow h(g_i)) = \bigodot_{i=1}^m (h(a_i) \leftrightarrow h(g_i)) = 1$. Therefore $f^{-1}(1) = 1$, i. e. f is injective, and ,hence, $A \cong F_{\mathbb{V}}(m)/[u)$. From here we deduce that A is finitely presented. $\square$

Theorem 15. *Let A be an m-generated , with generators $a_1, \ldots, a_m \in A$, finitely presented , with identity $P(x_1, \ldots, x_m) = 1$, subalgebra of the free m-generated algebra $F_{\mathbb{V}}(m)$ over the variety $\mathbb{V}$ with free generators $g_1, \ldots, g_m$. Then A is projective if $P(g_1, \ldots, g_m) \le \bigodot_{i=1}^m (a_i \leftrightarrow g_i)$.*

Proof. Let φ be natural homomorphism from $F_{\mathbb{V}}(m)$ onto $F_{\mathbb{V}}(m)/[u)$, where $u = P(g_1, \ldots, g_m)$ and $[u)$ is the principal filter generated by u. Then, as we know, A is isomorphic to $F_{\mathbb{V}}(m)/[u)$. Denote the isomorphism by $f : A \to F_{\mathbb{V}}(m)/[u)$. Then we have the following diagram :

where $\varepsilon(x) = x$ for every $x \in A$. Consider the homomorphism $h = f^{-1}\varphi$. Then $h(a_i) = f^{-1}(\varphi(a_i))$. But $\varphi(a_i) = \varphi(g_i)$, since $\varphi(g_i \leftrightarrow a_i) = \varphi(g_i) \leftrightarrow \varphi(a_i) \ge \varphi(u) = 1$. Therefore $h(a_i) = f^{-1}(\varphi(a_i)) = f^{-1}(\varphi(g_i)) = a_i$. It means that $h(x) = x$ for every $x \in A$, i. e. A is projective. $\square$

Let B be finite MV-algebra and A a subalgebra which satisfies the following condition :

(P) *for every nonzero smallest join-irreducible element $a \in A$ there exist incomparable nonzero join-irreducible elements $b_1, \ldots, b_k \in B$ such that b_j is the smallest join-irreducible in B for some $j \in \{1, \ldots, k\}$ and $b_1 \vee \cdots \vee b_k = a$*

Lemma 16. *Let A be a subalgebra of finite MV-algebra B which satisfies the condition (P). Then there exists a homomorphism $h : B \to A$ such that $h(x) = x$ for every $x \in A$.*

Proof. Let A and B be finite MV-algebras and A a subalgebra of B satisfying the condition (P). Since A and B are finite, each of them can be represented as finite product of simple finite totally ordered MV-algebras as follows : $A \cong \mathcal{L}_{m_1} \times \cdots \times \mathcal{L}_{m_k}$ and $B \cong \mathcal{L}_{n_1} \times \cdots \times \mathcal{L}_{n_r}$. Since A is a subalgebra of B, for every m_j $(j \in \{1, \ldots, k\})$ there exist $n_{i_1}, \ldots, n_{i_l}$ such that $\mathcal{L}_{m_j}$ is a subalgebra of $\mathcal{L}_{n_{i_1}} \times \cdots \times \mathcal{L}_{n_{i_l}}$, i. e. $m_j | n_{i_t}$, $n_{i_t} \in \{n_{i_1}, \ldots, n_{i_l}\}$, where $m_j | n_{i_t}$ means that m_j divides n_{i_t}. In other words there exist positive integers $c_1^{(j)}, \ldots, c_{\mu_j}^{(j)}$, $j = 1, \ldots, k$, such that $n_{i_t} = m_j c_{\mu_j}^{(j)}$. Observe that $\mathcal{L}_{m_j}$ is a retract of $\mathcal{L}_{n_{i_t}}$ if and only if $m_j = n_{i_t}$. Let us note also that nontrivial homomorphisms of a finite direct product are canonical projections. So, to be $\mathcal{L}_{m_j}$ a retract of $\mathcal{L}_{n_{i_1}} \times \cdots \times \mathcal{L}_{n_{i_l}}$ it is necessary and sufficient that in the integers $c_1^{(j)}, \ldots, c_{\mu_j}^{(j)}$ one of them should be coincided with 1, i. e. $c_q^{(j)} = 1$ for some $q \in \{1, \ldots, \mu_j\}$.

Condition (P) says that if $\mathcal{L}_{m_j}$ is a subalgebra of $\mathcal{L}_{n_1} \times \cdots \times \mathcal{L}_{n_l}$, then $\mathcal{L}_{m_j} \cong \mathcal{L}_{n_t}$ for some $t \in \{1, \ldots, l\}$. Let us note that the smallest nonzero join irreducible element $b_j (= (0, \ldots, 0, b'_j, 0, \ldots, 0))$ of B in the condition (P) has the component b'_j belonging to $\mathcal{L}_{m_j}$, which is at the same time the atom of $\mathcal{L}_{m_j}$. It is clear that $\mathcal{L}_{m_j}$ is a retract of $\mathcal{L}_{n_{i_1}} \times \cdots \times \mathcal{L}_{n_{i_l}}$. Indeed, the embedding $\varepsilon : \mathcal{L}_{m_j} \to \mathcal{L}_{n_{i_1}} \times \cdots \times \mathcal{L}_{n_{i_l}}$, since $\mathcal{L}_{m_j} \cong \mathcal{L}_{n_t}$, every element $a \in \mathcal{L}_{m_j}$ sends to $(\underbrace{a, \ldots, a}_{l \ times}) \in \mathcal{L}_{n_{i_1}} \times \cdots \times \mathcal{L}_{n_{i_l}}$. Then $\pi_t \varepsilon = Id$, where π_t is t-th projection. Making the same procedure for every $m_j, j = 1, \ldots, k$, we conclude that A is a retract of B. $\square$

By a *inverse system* in a category $\mathcal{C}$ we mean a family $\{B_i, \pi_{ij}\}_{i \in I}$ of objects, indexed by a directed poset I, together with a family of morphisms $\pi_{ij} : B_j \to B_i)$ satisfying the following conditions for each $i \leq j$:

(i) $\pi_{kj} = \pi_{ki} \circ \pi_{ij}$ for all $k \leq i \leq j$;

(ii) $\pi_{ii} = 1_{B_i}$ for every $i \in I$.

For brevity we say that $\{B_i, \pi_{ij}\}_I$ is an inverse system in $\mathcal{C}$. We shall omit to specify in which category we take an inverse system when this is evident from the context.

The *inverse limit* of an inverse system is an object B of $\mathcal{C}$ together with a family $\pi_i : B \to B_i$ of morphisms (which is often denoted by $\{B, \pi_i\}$) satisfying the condition: $\pi_{ij} \circ \pi_j = \pi_i$ for $i \leq j$, and having the following universal property: for any object D of $\mathcal{C}$ together with a family of morphisms $\lambda_i : D \to B_i$, if $\pi_{ij} \circ \lambda_j = \lambda_i$ for $i \leq j$, then there exists a morphism $\lambda : D \to B$ such that $\pi_i \circ \lambda = \lambda_i$ for any $i \in I$.

The *inverse limit* of the above system is denoted by $\varprojlim \{B_i, \pi_{ij}\}_I$, and its elements by $(b_i)_{i \in I}$, with $b_i \in B_i$. If π_{ij} is understood, we may simply write $\varprojlim \{B_i\}_I$.

Recall from Grätzer [17] that the inverse limits of families of algebras are constructed in the following way:

Suppose $\{B_i\}_{i \in I}$ is an inverse family of algebras. Consider their product $\prod_{i \in I} B_i$. Call $(b_i)_{i \in I} \in \prod_{i \in I} B_i$ a *thread*, if $\pi_{ij}(b_j) = b_i$ for $j \geq i$. Let B be the subset of $\prod_{i \in I} B_i$ consisting of all threads. Hence

$$B = \{(b_i)_{i \in I} \in \prod_{i \in I} B_i : \pi_{ij}(b_j) = b_i,\ j \geq i\}.$$

Then it is well known that B is a subalgebra of $\prod_{i \in I} B_i$, and that B is isomorphic to $\varprojlim \{B_i\}_I$.

We denote by $\mathbb{K}_n$ the variety of MV-algebras generated by $\{\mathcal{L}_1, \ldots, \mathcal{L}_n\}$, i.e. $\mathbb{K}_n = V(\{\mathcal{L}_1, \ldots, \mathcal{L}_n\})$. Let $F_n(m)$ be the m-generated free MV-algebra in the variety $\mathbb{K}_n$ and $F(m)$ be the m-generated free MV-algebra in the variety $\mathbb{MV}$.

Note that $\mathbb{K}_n$ is a locally finite variety. It is obvious that $\mathbb{MV} = V\left(\cup_{n \in \omega} \mathbb{K}_n\right)$.

On $\mathbb{Z}^+$ we define the function $v_m(x)$ as follows: $v_m(1) = 2^m$, $v_m(2) = 3^m - 2^m$, …, $v_m(n) = (n+1)^m - (v_m(n_1) + \ldots + v_m(n_{k-1}))$, where $n_1 (= 1), \ldots, n_{k-1}$ are all the divisors of n distinct from $n (= n_k)$.

By [11] (Lemma 2.2) we have:

Theorem 17. $F_n(m) \cong \mathcal{L}_1^{v_m(1)} \times \ldots \times \mathcal{L}_n^{v_m(n)}$.

Theorem 18. *[11]. $F(m)$ is isomorphic to a subalgebra of an inverse limit of an inverse family $\{F_n(m)\}_{n \in \omega}$ with free generators $G_i = (g_i^{(1)}, g_i^{(2)}, \ldots)$, where $i = 1, \ldots, m$ and $g_1^{(n)}, \ldots, g_m^{(n)}$ are free generators of the free m-generated MV-algebra $F_n(m) \in \mathbb{K}_n$.*

Let $\mathbb{V}$ be a variety and $\mathbb{V}_0$ its subvariety. Let A be an algebra from $\mathbb{V}$. A homomorphism h_0 from the algebra A onto an algebra $A_0 \in \mathbb{V}_0$ is called $\mathbb{V}_0$-*morphism (or universal morphism into* $\mathbb{V}_0$*)* if for any homomorphism $f : A \to B \in \mathbb{V}_0$ there exists a homomorphism $h : A_0 \to B$ such that $hh_0 = f$ (for detail information see [26]).

It is well known (see [26]) that if $F_{\mathbb{V}}(n)$ is n-generated free algebra of a variety $\mathbb{V}_0$, then its $\mathbb{V}_0$-morphism image into $\mathbb{V}_0$ is n-generated free algebra $F_{\mathbb{V}_0}(n)$ in the subvariety $\mathbb{V}_0$.

Theorem 19. *A subalgebra A of m-generated free algebra $F(m)$ is projective if and only if $\pi_n(A)$ $(\hookrightarrow \pi_n(F(m)))$ is a subalgebra of $\pi_n(F(m))$ satisfying the condition (P), where π_n is $\mathbb{K}_n$-morphism from $F(m)$ onto $F_n(m) \in \mathbb{K}_n$, $n \in \omega$.*

Proof. Let us suppose that A is an m-generated subalgebra of m-generated free algebra $F(m)$ such that $\pi_n(A) \hookrightarrow \pi_n(F(m))$ is a subalgebra of $\pi_n(F(m))$ satisfying the condition (P), where π_n is $\mathbb{K}_n$-morphism from $F(m)$ onto $F_n(m) \in \mathbb{K}_n$. Denote the embedding by ε, actually $\varepsilon(a) = a$ for every $a \in A$. According to Theorem 11 the m-generated free MV-algebra $F(m)$ is isomorphic to a subalgebra of the inverse limit $\varprojlim\{F_i(m)\}_\omega = \{F(m), \pi_i\}$, where π_i, being a projection, is at the same time $\mathbb{K}_i$-morphism. The embedding ε induces the embedding $\varepsilon_i : \pi_i(A) \to \pi_i(F(m))$, where $\pi_i(F(m)) = F_i(m)$. Denote $\pi_i(A)$ by A_i.

$$
\begin{tikzcd}
F_1(m) & F_2(m) \arrow[l] & \cdots \arrow[l] & \varprojlim\{F_i(m)\}_\omega & F(m) \arrow[l] \\
A_1 \arrow[u, "\varepsilon_1"] \arrow[ur, phantom, "\#"] & A_2 \arrow[l] \arrow[u, "\varepsilon_2"] & \cdots \arrow[l] & \varprojlim\{A_i\}_\omega \arrow[u] \arrow[ur, phantom, "\#"] & A \arrow[l] \arrow[u, "\varepsilon"]
\end{tikzcd}
$$

Then, since $A_i \hookrightarrow F_i(m)$ satisfies the condition P, there exists a homomorphism $\varphi_i : F_i(m) \to A_i$ such that $\varphi_i\varepsilon_i = Id_{A_i}$. Consequently, the mapping $\varphi = (\varphi_1, \varphi_2, \dots) : F(m) \to A$ will be a homomorphism such that $\varphi\varepsilon = ID_A$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$. It means that A is projective. The converse is trivial. $\square$

Theorem 20. *An m-generated subalgebra A of m-generated free MV-algebra $F(m)$, with generating set $\{a_1, \dots, a_m\} \subseteq A$, which is finitely presented by an equation $P(x_1, \dots, x_m) = 1$, is projective if $P(g_1, \dots, g_m) \odot g_i = a_i$ for $i = 1, \dots, m$, where $g_1, \dots, g_m$ are free generators of $F(m)$.*

Proof. Let A be m-generated subalgebra of m-generated free MV-algebra $F(m)$, with generating set $\{a_1, \dots, a_m\} \subseteq A$, which is finitely presented by an equation $P(x_1, \dots, x_m) = 1$ and satisfies the condition $P(g_1, \dots, g_m) \odot g_i =$

a_i for $i = 1, \ldots, m$. Then , according to Lemma 1, $A \cong F(m)/[u)$, where $u = P(g_1, \ldots, g_m)$ and $[u) = \{x \in F(m) : x \geq u^n, n \in \omega\}$ is a filter generated by u. Let us define new polynomial $P_i(x_1, \ldots, x_m) = P(x_1, \ldots, x_m)x_i$. It is obvious that $P_i(a_1, \ldots, a_m) = P(a_1, \ldots, a_m)a_i = a_i$. Consider new equation $\bigodot_{i=1}^m P_i(x_1, \ldots, x_m) \leftrightarrow x_i = 1$ which presents the algebra $F(m)/[u')$, where $u' = \bigodot_{i=1}^m P_i(g_1, \ldots, g_m) \leftrightarrow g_i$ and $[u')$ is a principal filter generated by u'. Let us observe that $\bigodot_{i=1}^m P_i(x_1, \ldots, x_m) \leftrightarrow x_i = 1$ holds in A on the elements $a_1, \ldots, a_m$. Therefore there exists onto homomorphism $h : F(m)/[u') \to A$ sending $g_i/[u')$ to a_i. On the other hand we have homomorphism $f : F(m) \to F(m)/[u')$, where $f^{-1}(1_{F(m)/[u')}) = [u')$. Then $f' : A \to F(m)/[u')$ is a homomorphism , which is restriction of f on the subalgebra $A \subset F(m)$, such that $a_i/[u') = g_i/[u')$, $i = 1, \ldots, m$. Indeed, since $a_i = P(g_i, \ldots, g_m)g_i$, $a_i \leftrightarrow g_i = P(g_i, \ldots, g_m)g_i \leftrightarrow g_i = P_i(g_i, \ldots, g_m) \leftrightarrow g_i \geq u'$. Hence $a_i/[u') = g_i/[u')$, $i = 1, \ldots, m$. It means that $f'(a_i) = g_i/[u')$. Therefore $hf' = Id_A$ which means that $A \cong F(m)/[u')$. Consequently A is finitely presented by $\bigodot_{i=1}^m P_i(x_1, \ldots, x_m) \leftrightarrow x_i = 1$ either. Moreover $P_i(a_1, \ldots, a_m) = a_i$ and, since $P_i(g_1, \ldots, g_m) = P(g_1, \ldots, g_m)g_i$, $P_i(P_1(g_1, \ldots, g_m), \ldots, P_m(g_1, \ldots, g_m)) = P_i(g_1, \ldots, g_m)$ for $i = 1, \ldots, m$. From here, according to Theorem 6, we conclude that A is projective. □

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