Fundamentals of Fuzzy Logics

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1 Introduction

Logics come in many guises. Classical logic, to take the most obvious example, may be presented *semantically* using truth tables or Boolean algebras to define the meaning of connectives like “and” or “implies”, or *syntactically* via proof methods such as axiomatizations, Gentzen systems, or Tableaux. Other logics may take one guise as primary; *substructural logics* are often defined using Gentzen systems, while *modal logics* originate via classes of Kripke frames. We may think of such guises as *frameworks*, within which logics arise naturally as a result of various “design choices”. For example, semantically we might select certain properties that we want from our logic, principles like the law of excluded middle “every proposition is either true or false” that we think should hold or not hold. From a syntactic point of view we might choose certain axioms or rules over others. Such choices might be made on philosophical grounds, or on a more practical level, based on mathematical or computational considerations.

Fuzzy logics, the subject matter of this course, are characterized as “logics based on the real numbers”. That is, logics where the truth degrees are taken from the real line $\mathbb{R}$, and connectives are interpreted as functions on $\mathbb{R}$. Such logics are usually designed with applications in mind as workhorses of the wider enterprise of *fuzzy logic*, originating with the formalisation of fuzzy sets by Zadeh [15]. Fuzzy logics provide the basis for logical systems dealing with *vagueness*, e.g. for formalising common natural language predicates such as “tall” or “fast”. Design choices in this framework are made as to which real numbers to take as truth values, and which properties connectives should have. In fact logics based on real numbers occur in a number of areas in logic.

Example 1 (T-norm based fuzzy logics). One widely used method for defining fuzzy logics is to take the real unit interval $[0, 1]$ as a set of truth values, and interpret connectives like conjunction “and” and implication “implies” as functions on $[0, 1]$ having certain intuitive properties, such as commutativity, associativity etc. For example:

– *Gödel logic* $\mathbf{G}$ where “and” is interpreted by the “minimum” t-norm $x * y = min(x, y)$, was introduced by Dummett [2] in 1959 as the infinite-valued version of a sequence of finite-valued logics defined by Gödel [6] in the 1930s.
– *Łukasiewicz logic* **Ł** where “and” is interpreted by the t-norm $x * y = max(0, x + y - 1)$, is the infinite-valued version of a famous family of many-valued logics introduced by Łukasiewicz [12] in the 1920s,
– *Product logic* $\mathbf{\Pi}$, where “and” is interpreted by the “product” t-norm $x.y$ (multiplication on [0,1]), is a more recent addition to the many-valued logic canon, introduced by Hájek et al. in 1996 [9].

As well as defining logics based on one particular t-norm, logics can also be considered based on *classes* of t-norms, such as Hájek's logic of continuous t-norms $\mathbf{BL}$ [8] and Godo and Esteva's logic of left-continuous t-norms $\mathbf{MTL}$ [3].

Example 2 (Resource Based Logics). In resource based logics *how often* a formula is used in a proof matters; in some logics like Anderson and Belnap's relevance logics [1], they must be used *at least once*, in others like Girard's Linear logic [5] *once exactly*. In some cases resources can be modelled by real numbers:

- In Meyer and Slaney's *Abelian logic* $\mathbf{A}$ [13], conjunction and implication are interpreted by ordinary addition and subtraction on $\mathbb{R}$, and true formulae are those having a truth value greater than 0.
- The logic $\mathbf{RM}$ [1] is a relevance logic with truth values in $\mathbb{R}$, where conjunction is interpreted by the function:

$$x * y = \begin{cases} min(x, y) & \text{if } x \leq -y \\ max(x, y) & \text{otherwise} \end{cases}$$

Example 3 (MYCIN-Like Expert Systems). MYCIN was one of the first expert systems capable of reasoning under uncertainty, its job being to diagnose certain blood infections using *certainty factors* taken from the interval $[-1, 1]$ and rules like:

IF the infection is primary-bacteremia
AND the site of the culture is one of the sterile sites
AND the suspected portal of entry is the gastrointestinal tract
THEN there is suggestive evidence (0.7) that infection is bacteroid.

To combine certainty factors MYCIN uses the function:

$$x * y = \begin{cases} x - y(1 - x) & \text{if } min(x, y) \geq 0 \\ \dfrac{x + y}{1 - min(|x|, |y|)} & \text{if } min(x, y) < 0 < max(x, y) \\ x - y(1 + x) & \text{if } max(x, y) \leq 0 \end{cases}$$

Generally, when a logic encountered in one framework shows up in another, this is celebrated; both as a proof of the "generality" of the logic, and also because different frameworks are good for different purposes: truth tables give us a good intuitive understanding of the logic, while algebraic semantics may be necessary to establish mathematical properties; for investigating proofs in a logic and defining automated reasoning methods, syntactic approaches like Gentzen systems or tableaux are needed. Remarkably enough, for logics based on the real numbers there exists (1) an algebraic perspective based on the class of commutative residuated lattices obeying *pre-linearity*, (2) an axiomatic perspective where fuzzy logics are substructural logics with the pre-linearity axiom $(A \to B) \vee (B \to A)$, and (3) a proof-theoretic perspective in the framework of *hypersequents*, a generalization of Gentzen sequents.

In this course we provide a general methodology for defining logics "based on the real numbers". What we mean by this last phrase will be made precise soon enough, but intuitively our interest lies with logics defined "truth table style" with truth values taken from the real line $\mathbb{R}$, and logical connectives such as "and", "or", and "implies" interpreted by functions on $\mathbb{R}$. For example, we might define a logic with truth values taken from the real unit interval $[0, 1]$ where the truth value of "A and B" is the minimum of the truth values of "A" and "B". We call such logics *fuzzy logics*, since such definitions characterize the central intuitions of Fuzzy Logic, and indeed cover the main formalisms investigated in the literature.

2 Formal Languages

A *formal language* is an essential ingredient of any logical framework: a formal language provides the basic materials for making *statements* in a logic, statements like: "Paris is the capital of France", "John is tall", etc. Such statements are called *propositions* and are built up using atomic statements called *propositional variables*, and *logical connectives* that put propositions together to form more complex ones, i.e.:

Definition 1 (Propositional Language). *A* propositional language $\mathcal{L}$ *consists of:*

1. *A denumerable (countably infinite) set of symbols called* propositional variables $VAR(\mathcal{L})$ *with typical members* $p, q, r, p_1, p_2, \ldots$.
2. *A set of connectives* $CON(\mathcal{L}) = \{\#_1, \ldots, \#_n\}$ *with arities (i.e. how many arguments a connective takes) given by a function* $AR(\mathcal{L}) : CON(\mathcal{L}) \to \mathbb{N}$.

Connectives with arity 0 *are called* (logical) constants, *those with arity* 1 *and* 2 *are called* unary *and* binary *connectives respectively.*

In this course we will need more connectives than those usually supplied for classical logic: these are displayed together with their arities and some clues as to their expected behaviour in Table 2.

Table 1: Common Connectives

Connectives	Arity	Behaviour
$\wedge, \odot$	2	Conjunction: "... and ..."
$\vee, \oplus$	2	Disjunction: "... or ..."
$\to$	2	Implication: "if... then..."
$\leftrightarrow$	2	Bi-implication: "... if and only if ..."
$\neg$	1	Negation: "not ...."
$t, \top$	0	Truth
$f, \bot$	0	Falsity

Formulae for a language are built up out of propositional variables and connectives:

Definition 2 (Well Formed Formulae). *The* well formed formulae $FOR(\mathcal{L})$ *for a propositional language* $\mathcal{L}$ *is the smallest set such that:*

1. *If* $p \in VAR(\mathcal{L})$ *then* $p \in FOR(\mathcal{L})$.
2. *If* $A_1, \ldots, A_m \in FOR(\mathcal{L})$ *then* $\#(A_1, \ldots, A_m) \in FOR(\mathcal{L})$ *for all* $\# \in CON(\mathcal{L})$ *where* $AR(\mathcal{L})(\#) = m$.

Connectives may also be defined as *abbreviations* of other connectives, e.g.

$$\neg A =_{def} A \rightarrow \bot \text{ for all formulae } A$$

Here the defined connective should be thought of simply as a syntactic convenience, not actually present in the language itself. Note finally that for binary connectives we will in this course freely swap between *prefix* notation $\#(x, y)$ and *infix* notation $(x\#y)$. We will also disregard brackets where readability is not at stake, and assume that $\neg$ binds more tightly than other connectives e.g. reading $\neg p \wedge q$ as $\neg(p) \wedge q$ rather than $\neg(p \wedge q)$.

3 Logical Matrices

In this section we explain a particular framework for introducing *truth-functional* logics i.e. logics obeying the "principle of extensionality" where the meaning of a compound formula is uniquely determined by the meanings of its constituents. In this framework, a logic for a language has a set of *truth values* with some "designated" as the true ones (sometimes also "anti-designated" values but we ignore this possibility here), and *meanings* for the connectives of the language that tell us how truth values are assigned to non-atomic formulae. All these elements taken together are called a *logical matrix*.

Definition 3 (Logical Matrix). *A* logical matrix $\mathcal{M}$ *for a language* $\mathcal{L}$ *consists of:*

– *A non-empty set of* truth-values $\mathcal{N}$.
– *A subset* $\mathcal{D} \subseteq \mathcal{N}$ *of* designated truth values, *denoting (partial) truth.*
– *A set of* truth-functions *(or* meanings*) for each connective of* $\mathcal{L}$*:*

$$\mathcal{C} = \{\#^i : \mathcal{N}^m \rightarrow \mathcal{N} \mid \# \in CON(\mathcal{L}) \textit{ where } AR(\mathcal{L})(\#) = m\}$$

and we write: $\mathcal{M} = [\mathcal{N},\mathcal{D},\mathcal{C}]$.

A language with a logical matrix is called a *propositional many-valued logic*:

Definition 4 (Propositional Many-Valued Logic). *Let* $\mathcal{M}$ *be a matrix for a propositional language* $\mathcal{L}$*, then* $\mathbf{L} = (\mathcal{L}, \mathcal{M})$ *is called a* (matrix-defined) propositional many-valued logic *and we say that* $\mathcal{M}$ *is* characteristic *for* $\mathbf{L}$.

Definition 5 (Finite-Valued and Infinite-Valued Logics). *If there exists an integer* n *such that* $\mathcal{N}$ *for a logic* $\mathbf{L}$ *contains exactly* n *elements then we say that* $\mathbf{L}$ *is a* finite-valued (weakly n-valued) logic*; otherwise we say that* $\mathbf{L}$ *is an* infinite-valued *logic.*

Logics defined by a (single) matrix are *truth-functional* in the sense that the truth value of a non-atomic formula is uniquely determined by the truth values of its constituents. Since not all logics are truth-functional, not all logics are definable via logical matrices. Such logics may be defined via particular sets of matrices, or algebras but although this is a more general perspective, the matrix method is preferred where possible.

We illustrate the logical matrix method with a series of examples:

Example 4 (Classical Logic). Classical logic $\mathbf{CL}$ obeys the *principle of bivalence* which states that every proposition is either true or false, i.e. there are exactly two truth values, which we call here 1 (true) and 0 (false). $\mathbf{CL}$ can be based on a language with binary connectives $\wedge$, $\vee$, $\rightarrow$, and (a constant) $\bot$, and matrix:

$$[\{0,1\},\{1\},\{\wedge^i,\vee^i,\rightarrow^i,\bot^i\}]$$

where $\bot^i = 0$ and truth-functions for other connectives are defined by *truth tables* as follows (reading vertically then horizontally):

$\wedge^i$	1	0
1	1	0
0	0	0

$\vee^i$	1	0
1	1	1
0	1	0

$\rightarrow^i$	1	0
1	1	0
0	1	1

There are $2^{2^2} = 16$ different truth functions possible for defining binary connectives in a 2-valued logic: 2 options for each position in the truth table. However, all such connectives are definable from the connectives given above, for example the usual "not" function can be defined as $\neg A =_{def} A \rightarrow \bot$. We say in this case that the set of connectives $\{\wedge,\vee,\rightarrow,\bot\}$ is *functionally complete*. In fact the subset $\{\rightarrow,\bot\}$ is itself functionally complete, and it is even possible to find just one connective, the so-called Scheffer stroke, from which all others may be defined.

Example 5 (Finite-Valued Logics). By abandoning the principle of bivalence we obtain so-called *many-valued logics*, e.g. .:

– As in classical logic all *finite-valued* logics can be described with the aid of truth-tables. One of the first many-valued logics was introduced by Łukasiewicz, and includes, in addition to the usual truth values 1 "true" and 0 "false", $\frac{1}{2}$, "the possible" supposed to model future contingents such as "Prince Charles will be the next King of England". This logic is known as $\mathbf{Ł_3}$ and can be based on a language with connectives $\neg$ and $\rightarrow$.

$$[\{0,\frac{1}{2},1\},\{1\},\{\neg^i,\rightarrow^i\}]$$

$\neg^i$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\rightarrow^i$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

Note that there are $(3*3)^3 = 729$ different truth tables possible for binary connectives in three-valued logics; more generally $(n^2)^n$ for an n-valued logic. However, in contrast with the situation for classical logic, the connectives given here are not functionally complete.

– In $\mathbf{Ł_3}$ the only designated value is 1. By including $\frac{1}{2}$ as designated we obtain a different logic i.e. for the same language with the same truth tables:

$$[\{0,\frac{1}{2},1\},\{\frac{1}{2},1\},\{\neg^i,\rightarrow^i\}]$$

This is the "paraconsistent" logic $\mathbf{J_3}$ i.e. a logic where a contradiction such as $\neg(A \rightarrow A)$ does not have every formula as a logical consequence.

– All the sets of truth values we have encountered so far have been real numbers. In fact any *linearly ordered* set (i.e. where $x \leq y$ or $y \leq x$) of truth values can be "normalised" to give a set of reals. However there are many logics where the truth values are *not* linearly ordered. For example we might allow that propositions can be *both* true and false b, or *neither* n. This idea gives rise to the following lattice with "four corners of truth".

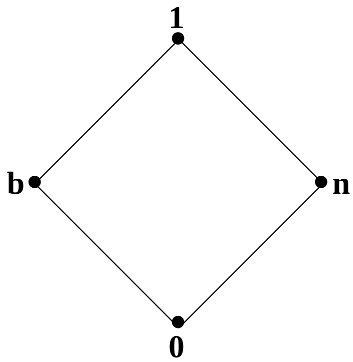

Fig. 1: Four corners of truth

The relevant logic $\mathbf{FDE}$ is based on a language with connectives $\wedge$, $\vee$ and $\neg$, and has the following matrix.

$$[\{0, n, b, 1\}, \{b, 1\}, \{\wedge^i, \vee^i, \neg^i\}]$$

$\neg$	
1	0
b	b
n	n
0	1

$\wedge^i$	1	b	n	0
1	1	b	n	0
b	b	b	0	0
n	n	0	n	0
0	0	0	0	0

$\vee^i$	1	b	n	0
1	1	1	1	1
b	1	b	1	b
n	1	1	n	n
0	1	b	n	0

Here $\neg$ can be viewed as mirroring the lattice along the horizontal axis, swapping 0 and 1 but leaving b and n in place. $\wedge$ and $\vee$ can be read as lattice meet and join respectively. Notice also that if we remove n then we get $\mathbf{Ł_3}$.

Example 6 (Infinite-Valued Logics). The truth table approach is fine for defining three or four valued logics but becomes tedious for more values, and impossible for an infinite number of values. Hence it is usual where possible to use *functions*. For example Łukasiewicz's infinite-valued logic **Ł** (one of the most important formalisations of fuzzy logic) can be based on a language with connectives $\rightarrow$ and $\perp$, and matrix:

$$[[0, 1], \{1\}, \{\rightarrow^i, \perp^i\}]$$

where truth functions are given by:

$$\perp^i = 0 \text{ and } x \rightarrow^i y = min(1, 1 - x + y)$$

If we swap $[0, 1]$ for $[0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1]$ in this matrix then we get the n-valued Łukasiewicz logic $\mathbf{Ł_n}$, where $\mathbf{Ł_3}$ has been introduced above, and $\mathbf{Ł_2}$ is classical logic.

By assigning particular truth values to the propositional variables for a logic, we get truth values for all formulae, looked up or calculated from the truth-functions of the connectives.

Definition 6 (Valuation). *A* valuation *(also known as an interpretation or assignment) for a logic with language $\mathcal{L}$ is a function $v : VAR(\mathcal{L}) \rightarrow \mathcal{N}$ extended to $FOR(\mathcal{L})$ by:*

$$v(\#(A_1, \ldots, A_m)) = \#^i(v(A_1), \ldots, v(A_m))$$

where $\# \in CON(\mathcal{L})$ and $AR(\mathcal{L})(\#) = m$.

A formula is then *valid* in a logic (or a tautology of the logic) if every valuation gives it a designated "true" truth value, and *satisfiable* if one valuation gives it such a value. Two formulae are equivalent if they always take the same value. More formally:

Definition 7 (Validity, Satisfiability, Logical Equivalence). *For $\mathbf{L} = (\mathcal{L}, \mathcal{M})$:*

- *$A \in FOR(\mathcal{L})$ is* valid *in $\mathbf{L}$ (also called a* tautology *of $\mathbf{L}$), written $\models_{\mathbf{L}} A$ iff $v(A) \in \mathcal{D}$ for all valuations v for $\mathbf{L}$.*
- *$A \in FOR(\mathcal{L})$ is* satisfiable *in $\mathbf{L}$ iff $v(A) \in \mathcal{D}$ for some valuation v for $\mathbf{L}$.*
- *$A, B \in FOR(\mathcal{L})$ are $\mathbf{L}$-*equivalent*, written $A \equiv_L B$, iff $v(A) = v(B)$ for all valuations v for $\mathbf{L}$.*

For example it is easy to see that $A \rightarrow (B \rightarrow A)$ is a tautology of both $\mathbf{CL}$ and $\mathbf{Ł_3}$, but not $\mathbf{FDE}$. In fact $\mathbf{FDE}$ is curious in that it has no tautologies whatsoever.

We can also define the notion of logical consequence between sets of formulae. Intuitively, a set of formulae is a logical consequence of another if whenever all the formulae in the first have designated values, then at least one formula in the second has a designated value.

Definition 8 (Logical Consequence). *Let $\mathbf{L} = (\mathcal{L}, \mathcal{M})$ be a logic, and $\Gamma, \Delta \subseteq FOR(\mathcal{L})$, Δ is a* logical consequence *of Γ, written $\Gamma \models_{\mathbf{L}} \Delta$ iff for all valuations v for $\mathbf{L}$, there exists $A \in \Gamma$ such that $v(A) \notin \mathcal{D}$ or $A \in \Delta$ such that $v(A) \in \mathcal{D}$.*

Notice that $\mathbf{FDE}$ *does* have logical consequences, for example $\{A\}$ is a consequence of $\{\neg\neg A\}$.

4 Truth Values

In this course we are interested mostly in logical matrices with truth values taken from the *real numbers* (or simply "the reals") $\mathbb{R}$. There are several motivations for investigating logics based on such matrices. Firstly, the real numbers are reasonably easy for humans to *understand*: we can easily compare magnitudes, order them, add them together, take differences etc. This makes the reals particularly suitable for use in AI applications where "non-black-box" behaviour i.e. giving explanations of reasoning, is often essential. Secondly, the reals are reasonably easy to *manipulate*: there exists a vast array of mathematical techniques and theorems to support representation, computation etc.; this is in contrast to the often complicated algebras of other non-classical logics. Finally, and perhaps most importantly, real numbers are *popular*, being by far the most common choice for degrees of truth in the fuzzy logic literature, and also used in other areas of logic e.g. to model resources in substructural logics or degrees of belief in expert systems.

So which real numbers should we be dealing with? That is, which subsets of the reals make suitable sets of truth values for fuzzy logics? We begin by introducing some well-known candidates from the literature.

- For $a, b \in \mathbb{R}$, we define: $[a, b] = \{x \mid x \in \mathbb{R}, a \leq x \leq b\}$, $\{x \mid x \in \mathbb{R}, a \leq x < b\}$ by $[a, b)$, $\{x \mid x \in \mathbb{R}, a < x \leq b\}$ by $(a, b]$ and $\{x \mid x \in \mathbb{R}, a < x < b\}$ by (a, b).

A key property of all subsets of real numbers is that they are *linearly (or totally) ordered*, i.e. for a set $S \subseteq \mathbb{R}$:

$$x \leq y \text{ or } y \leq x \text{ for all } x, y \in S.$$

Some subsets have the further property of being *dense* i.e. for $S \subseteq \mathbb{R}$:

$$\text{for all } x, y \in S, \text{ if } x < y, \text{ then there exists } z \in S \text{ such that } x < z < y.$$

The choice of truth values for a given logic is influenced by a number of factors:

1. Some truth values are more “intuitive” or “natural” than others, e.g. $[0, 1]$ rather than $[0.23, 1.47]$. In particular we might want to insist on generalizing classical logic, in which case our set of truth values should be a subset of $[0, 1]$.
2. Although we can consider many possibilities it makes sense to stick to a few which are *representative* in the sense that choices of truth values such as $[0, 1]$ and $[0.23, 1.47]$ are order isomorphic, and hence logics based on them have the same sets of theorems.
3. Certain properties of the truth value set may be particularly desirable; for example having an infinite stock of truth values, insisting that that the set of truth values is dense, or wanting a “top truth” and/or “bottom falsity” that are more true or more false than any other truth value respectively.

Bearing these points in mind, and following fuzzy logic tradition, we consider here subsets of $[0, 1]$ as representative of other choices. We will also insist on subsets containing at least two elements (otherwise logic gets very boring indeed!), and to ensure that we are generalizing clasical logic, that if there are least and greatest elements of the set then these will be 0 and 1 respectively. Moreover we also consider only sets that are *dense* i.e. for all $x, y \in I$, if $x < y$, then there exists a z such that $x < z < y$. This last condition ensures that we have *infinite-valued* logics. We hence arrive at four possibilities: $[0, 1]$, $[0, 1)$, $(0, 1]$ and $(0, 1)$, representing the various situtations where an interval is open, closed or half-open. The *designated* truth degrees in such cases will be a set $[e, 1)$ or $[e, 1]$, often just $\{1\}$, the idea being that if $x < y$ and x denotes truth, then also y denotes truth.

5 Ands

Once decided on a set of truth values, the next step is to define truth functions for the various connectives in our chosen language. We begin here by considering interpretations for "ands" i.e. conjunction connectives, which together with interpretations for "ors", are taken from the vast panorama of *aggregation operators* used for combining values across a multitude of applications. Usually such interpretations are designed for applications "by hand"; however here we determine a set of basic properties that we wish conjunctions to have then investigate the effect of demanding further properties. We think of these as *design choices*, giving different classes of logics.

The first property which we consider basic for interpreting conjunctions is *associativity* which essentially means that bracketing is unimportant.

Definition 9 (Associativity). *$* : I^2 \to I$ is* associative *iff $(x * y) * z = x * (y * z)$ for all $x, y, z \in I$.*

Our second fundamental property is *commutativity* (also known as symmetry, neutrality and anonymity) which essentially means that the *order* of the arguments does not matter i.e. the criteria to be aggregated are of equal importance.

Definition 10 (Commutativity). *$* : I^2 \to I$ is* commutative *iff $x * y = y * x$ for all $x, y \in I$.*

A further property considered essential for "and-ness" is *monotonicity* i.e. increasing one of the arguments of the function only increases (does not decrease) the function.

Definition 11 (Monotonicity). *$* : I^2 \to I$ is* increasing *iff $x \leq y$ implies $x * z \leq y * z$ and $z * x \leq z * y$ for all $x, y, z \in I$.*

In certain cases we may require that the value *strictly increases*:

Definition 12 (Strict Monotonicity). *$* : I^2 \to I$ is* strictly increasing *iff $x < y$ implies $x * z < y * z$ and $z * x < z * y$ for all $x, y, z \in I$.*

A further property desirable for some applications is that A have the same truth value as A and A, i.e. that repeating something does not make it any more or less true.

Definition 13 (Idempotency). *$* : I^2 \to I$ is* idempotent *iff $x * x = x$ for all $x \in I$.*

We may also want some kind of "sensitivity" requirement to ensure that the value of the function is not too sensitive to changes in the values of its arguments. We can insist that our functions be *continuous* or *left-continuous*.

Definition 14 (Continuity). *$* : I^2 \to I$ is* continuous *iff for all $x, y \in I$, given a sequence $(x_i)_{i \geq 0}$, $x_i \in I$, such that $x = lim_{i \to \infty} x_i$, then also $lim_{i \to \infty}(x_i * y) = x * y$.*

Definition 15 (Left-Continuity). *$* : I^2 \to I$ is* left-continuous *iff for all $x, y \in I$, given a sequence $(x_i)_{i \geq 0}$, $x > x_i \in I$, such that $x = lim_{i \to \infty} x_i$, then also $lim_{i \to \infty}(x_i * y) = x * y$.*

Finally, we often require an element that plays the role of an *identity* for conjunction.

Definition 16 (Identity). *Given a function* $* : I^2 \to I$*, an element* $e \in I$ *is an* identity *for* $*$ *iff* $e * x = x * e = x$ *for all* $x \in I$*.*

Example 7 (Arithmetic Mean). One well known aggregation operator for real numbers is the *arithmetic mean*, which has a binary version $*_A : [0,1]^2 \to [0,1]$:

$$x *_A y = \frac{x + y}{2}$$

Note that $*_A$ is commutative, strictly increasing, idempotent and continuous, but is not associative and has no identity element.

In the rest of this section we will be concerned primarily with interpreting conjunction by functions on $[0,1]$ that behave classically on $\{0,1\}$, are commutative, associative and increasing, and have 1 as an identity. Such functions are known as *t-norms* (see [10] for a comprehensive survey).

Definition 17 (T-Norm). *A* t-norm *is a function* $* : [0,1]^2 \to [0,1]$ *such that for all* $x, y, z \in [0,1]$*:*

1. $x * y = y * x$ *(Commutativity)*
2. $(x * y) * z = x * (y * z)$ *(Associativity)*
3. $x \leq y$ *implies* $x * z \leq y * z$ *(Monotonicity)*
4. $1 * x = x$ *(Identity)*

As an immediate consequence of this definition we get that 0 is an "annihilating element" for each t-norm i.e.:

Lemma 1. *For each t-norm* $*$*,* $x * 0 = 0 * x = 0$ *for all* $x \in [0,1]$.

Proof. $1 * 0 = 0$ since 1 is an identity element, but $x \leq 1$ so by monotonicity $x * 0 = 0 * x = 0$. ☐

Note that in the literature the prefix notation $T(x, y)$ (instead of $x * y$) is often used for t-norms. However since we wish here to emphasize the interpretation of logical connectives via t-norms (and be consistent), we prefer the infix notation.

There are a lot (uncountably many in fact) of t-norms, often arranged into families bearing important or interesting properties, e.g. Frank t-norms, Hamacher t-norms etc. However here we concentrate on classifying classes of t-norms with some of the properties defined above, beginning with some fundamental examples of *continuous* t-norms:

Definition 18 (Fundamental T-Norms).

1. Łukasiewicz t-norm*:* $x *_{\mathbf{Ł}} y =_{def} max(0, x + y - 1)$
2. Gödel t-norm*:* $x *_{\mathbf{G}} y =_{def} min(x, y)$
3. Product t-norm*:* $x *_{\mathbf{\Pi}} y =_{def} x.y$ *(product of reals)*

Notice that the Gödel t-norm min is the only t-norm that is *idempotent*.

Proposition 1. $*_{\mathbf{G}}$ *is the only idempotent t-norm.*

Proof. Let $*$ be an idempotent t-norm. If $x \leq y$ then $x = x*x \leq x*y$ by monotonicity and $x*y \leq x*1 = x$ by monotonicity and identity, hence $x*y = x$ and by commutativity we have that $x*y = min(x,y)$. □

Continuity is often considered desirable for interpreting conjunctions in fuzzy logic since it means that the function is not over-sensitive to slight changes in its arguments. In fact the above t-norms play a special role with regard to continuous t-norms: it turns out that *any* continuous t-norm is locally isomorphic to one of these three (see e.g. [8] for details). Although continuous t-norms are most commonly used in fuzzy logics and have an elegant representation, there are also *non-continuous* t-norms of interest.

Example 8 (Non-Continuous T-norms).

1. The *nilpotent minimum* t-norm defined by:
$$x *_N y =_{def} \begin{cases} min(x,y) & \text{if } x+y>1 \\ 0 & \text{otherwise} \end{cases}$$
is left-continuous but not continuous.
2. The *drastic product* t-norm defined by:
$$x *_D y =_{def} \begin{cases} 0 & \text{if } x,y \in [0,1) \\ min(x,y) & \text{otherwise} \end{cases}$$
is not even left-continuous.

We end this section with some results on the relative "strengths" of t-norms.

Definition 19. *For two functions $f : I^n \to J$ and $g : I^n \to J$:*

– *f is* weaker *than g or equivalently g is* stronger *than f, written $f \leq g$ iff $f(x_1,\ldots,x_n) \leq g(x_1,\ldots,x_n)$ for all $x_1,\ldots,x_n \in I$.*
– *If f is weaker than g and for some $x_1,\ldots,x_n \in I$, $f(x_1,\ldots,x_n) < g(x_1,\ldots,x_n)$ then we write $f < g$.*

Proposition 2. *We have the following orderings:*

1. $*_D \leq * \leq *_{\mathbf{G}}$ *for any t-norm* $*$.
2. $*_D < *_{\mathbf{Ł}} < *_{\mathbf{\Pi}} < *_{\mathbf{G}}$.

Proof. 1. For any t-norm $*$ we have that for all $x \in [0,1]$, $1*x = x*1 = 1$ by identity and commutativity. Moreover we have trivially that for all $x,y \in [0,1)$, $0 = x *_D y \leq x*y$, and for all $x,y \in [0,1]$, $x*y \leq x*1 = x$ by identity and monotonicity. Hence $*_D \leq * \leq *_{\mathbf{G}}$ as required.
2. Since $x.y = max(0, x+y-1)$ iff one of x and y is 0 or 1 and e.g. $\frac{1}{4} = \frac{1}{2}.\frac{1}{2} > max(0, \frac{1}{2}+\frac{1}{2}-1) = 0$, we have $*_{\mathbf{Ł}} < *_{\mathbf{\Pi}}$ and the result follows by 1. □

6 Ors

Many of the properties natural for interpreting "ands" are natural also for interpretations of "ors" i.e. disjunction connectives. Again it is reasonable to assume that such functions are commutative, associative and monotonic increasing; the main difference being that 0 instead of 1 is an identity. A suitable class of functions for interpreting disjunction are therefore a kind of "disjunctive dual" to t-norms called *t-conorms*.

Definition 20 (T-Conorm). *A* t-conorm *is a function* $\circ : [0,1]^2 \to [0,1]$ *such that for* $x, y, z \in [0,1]$*:*

1. $x \circ y = y \circ x$ *(Commutativity)*
2. $(x \circ y) \circ z = x \circ (y \circ z)$ *(Associativity)*
3. $x \leq y$ *implies* $x \circ z \leq y \circ z$ *(Monotonicity)*
4. $0 \circ x = x$ *(Identity)*

Just as for t-norms we get that 0 is an "annihilator", so 1 plays this role for t-conorms i.e. $1 * x = x * 1 = 1$ for all $x \in [0,1]$. In fact, as might be expected from the definitions, there is a very strong correspondence between t-norms and t-conorms: each t-conorm can be used to define a dual t-norm and vice versa.

Proposition 3. $\circ$ *is a t-conorm iff there exists a t-norm* $*$ *such that for all* $x, y \in [0,1]$*:*

$$x * y = 1 - ((1 - x) \circ (1 - y))$$

where $*$ *is called the* dual t-norm *of* $\circ$*, and* $\circ$ *the* dual t-conorm *of* $*$.

Example 9 (Fundamental T-conorms). The dual t-conorms of the fundamental t-norms are as follows:

1. *Bounded sum*: $x \circ_{Ł} y =_{def} min(1, x + y)$
2. *Maximum*: $x \circ_G y =_{def} max(x, y)$
3. *Probabilistic sum*: $x \circ_{\Pi} y =_{def} x + y - x.y$

All the representation theorems of the previous section on t-norms have dual versions for t-conorms.

7 Ands and Ors

As we have seen above, "ands" and "ors" generally have very similar properties, both conjunction and disjunction functions being usually commutative, associative and increasing in both arguments. The difference, at least when "ands" and "ors" are interpreted by t-norms and t-conorms respectively, lies solely with the location of the *identity element*: 1 for "and", 0 for "or". In this section we investigate a *generalization* of t-norms and t-conorms, introduced by Yager and Rybalov in [14], where the identity element can be *any* number taken from the real unit interval i.e.:

Definition 21 (Uninorm). *A* uninorm *is a function* $* : [0,1]^2 \to [0,1]$ *such that for some* $e \in [0,1]$*, for* $x, y, z \in [0,1]$*:*

1. $x * y = y * x$ *(Commutativity)*
2. $(x * y) * z = x * (y * z)$ *(Associativity)*
3. $x \leq y$ *implies* $x * z \leq y * z$ *(Monotonicity)*
4. $e * x = x$ *(Identity)*

Notice that if $e = 1$ or $e = 0$ then we just get the usual definition of a t-norm or a t-conorm respectively. However, uninorms with an identity in $(0, 1)$, unlike t-norms and t-conorms, allow for "compensatory behaviour": that is, whether or not the truth value is low or high, it may be possible to add suitable further evidence to either increase or decrease the truth value of the combined value. In this case the identity element e can be interpreted as the score or truth value given to an argument or proposition which does not have any influence either way. Hence we are naturally led to say that designated truth values should be those greater than or equal to e, i.e. the set $[e, 1]$.

Example 10 (Least and Greatest Uninorms). Consider the following uninorms:

$$x *_\bot y = \begin{cases} 0 & \text{if } x, y \in [0, e) \\ max(x, y) & \text{if } x, y \in [e, 1] \\ min(x, y) & \text{otherwise} \end{cases}$$

$$x *_\top y = \begin{cases} min(x, y) & \text{if } x, y \in [0, e] \\ 1 & \text{if } x, y \in (e, 1] \\ max(x, y) & \text{otherwise} \end{cases}$$

$*_\bot$ and $*_\top$ are the weakest and strongest uninorms with identity e respectively, i.e. for any uninorm $*$ with identity e, we have: $*_\bot \leq * \leq *_\top$.

Our aim now is to investigate the behaviour of uninorms on the unit square $[0, 1]^2$, beginning with the values taken at the extremal points 0 and 1.

Lemma 2. *If* $*$ *is a uninorm then* $0 * 0 = 0$ *and* $1 * 1 = 1$.

Proof. $e * 0 = 0$ so since $0 \leq e$, by monotonicity we get $0 * 0 = 0$. Similarly $e * 1 = 1$ and $e \leq 1$, so again by monotonicity $1 * 1 = 1$. □

Although $0 * 0 = 0$ and $1 * 1 = 1$ are the same for all uninorms, the final "classical" value $0 * 1 = 0 * 1$ is not fixed, as is obvious from the fact that for t-norms this value is 0 and for t-conorms 1. Nevertheless we can show instead that for a uninorm the value $0 * 1$ always takes one or ther other of the values 0 or 1, being either "and-like" *conjunctive* like a t-norm, or "or-like" *disjunctive* like a t-conorm.

Lemma 3. *Given a uninorm* $*$, *if* $x \leq 0 * 1 \leq y$ *then* $x * y = 0 * 1$.

Proof. For $x \leq 0 * 1 \leq y$ we have by monotonicity that $0 * (0 * 1) \leq x * y \leq (0 * 1) * 1$; however by associativity we also have $0 * (0 * 1) = (0 * 0) * 1 = 0 * 1 = 0 * (1 * 1) = (0 * 1) * 1$, so we get $x * y = 0 * 1$ as required. □

Proposition 4. *Given a uninorm* $*$ *one of these conditions holds:*

1. $1 * 0 = 0 * 1 = 0$ *and* $*$ *is called* conjunctive.

2. $1 * 0 = 0 * 1 = 1$ *and* $*$ *is called* disjunctive.

Proof. If $0 * 1 \geq e$ then by Lemma 3 we have $x = e * x = 0 * 1$ for $0 * 1 \leq x \leq 1$, which gives that $0 * 1 = 1$. Similarly if $0 * 1 \leq e$ we get that $0 * 1 = 0$. □

Observe that this result means that conjunctive and disjunctive uninorms satisfy the intuitive requirements of generalizing classical conjunction and disjunction respectively. Moreover we also get that 0 is an annihilating element for conjunctive uninorms, while 1 plays the same role for disjunctive uninorms.

Lemma 4. *Given a uninorm* $*$*:*

1. *If* $*$ *is conjunctive then* $0 * x = 0$ *for all* $x \in [0, 1]$.
2. *If* $*$ *is disjunctive then* $1 * x = 1$ *for all* $x \in [0, 1]$.

Proof. If $*$ is conjunctive then we have $0 * 1 = 0$, but also $x \leq 1$ and hence by monotonicity we have $0 * x = 0$. The case where $*$ is disjunctive is very similar. □

We now show further that a uninorm with identity $e \in (0, 1)$ exhibits a "block-like" structure on the unit square, where the lower corner $[0, e]^2$ is isomorphic to a t-norm, and the upper corner $[e, 1]^2$ to a t-conorm.

Proposition 5. *If* $*$ *is a uninorm with identity* $e \in (0, 1)$*, then:*

1. $x *_T y =_{def} \dfrac{(ex) * (ey)}{e}$ *is a t-norm.*
2. $x *_S y =_{def} \dfrac{(e + (1 - e)x) * (e + (1 - e)y) - e}{1 - e}$ *is a t-conorm.*
3. $x * y = \begin{cases} e(\frac{x}{e} *_T \frac{y}{e}) & \text{if } x, y \in [0, e] \\ (e + (1 - e))(\frac{x - e}{1 - e} *_S \frac{y - e}{1 - e}) & \text{if } x, y \in [e, 1] \end{cases}$

Proof. We just check 1, leaving 2 and 3 as exercises. First notice that $*_T$ is clearly commutative and increasing by the corresponding conditions for $*$, so we just check identity and associativity as follows:

$$x *_T 1 = \frac{(ex) * (e1)}{e} = \frac{ex * e}{e} = \frac{ex}{e} = x$$

$$(x *_T y) *_T z = \frac{((ex) * (ey)) * ez}{e} = \frac{(ex) * ((ey) * (ez))}{e} = x *_T (y *_T z) \quad \square$$

On the rest of $[0, 1]^2$ such uninorms are bounded by min and max (see Figure 2).

Proposition 6. *If* $*$ *is a uninorm with identity* $e \in (0, 1)$*, then:*

$$min(x, y) \leq x * y \leq max(x, y) \text{ for all } (x, y) \in [0, 1]^2 \backslash ([0, e]^2 \cup [e, 1]^2)$$

Proof. WLOG take $x \in [0, e]$ and $y \in [e, 1]$; we have:

$$min(x, y) = x = x * e \leq x * y \leq e * y \leq y = max(x, y) \quad \square$$

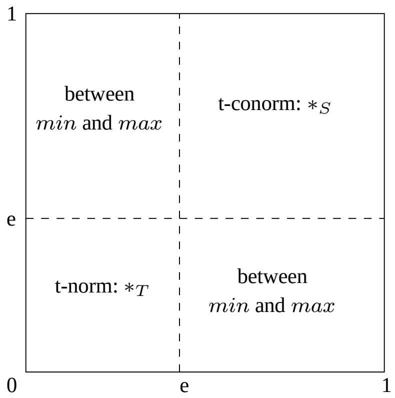

Fig. 2: The structure of a uninorm with identity $e \in (0, 1)$

We now turn our attention to various *classes* of uninorms. First observe that there are no *continuous* uninorms with an identity $e \in (0, 1)$.

Proposition 7. *If $*$ is a continuous conjunctive (disjunctive) uninorm then $e = 1$ ($e = 0$) and $*$ is a t-norm (t-conorm).*

Proof. We just prove the case where $*$ is a continuous conjunctive uninorm. Define $f(x) = x * 1$ for $x \in [0, 1]$. Since f is continuous with $f(0) = 0 * 1 = 0$ and $f(1) = 1 * 1 = 1$, f is a surjection i.e. onto $[0, 1]$. Hence for all $a \in [0, 1]$ there exists $b \in [0, 1]$ such that $a = f(b) = b * 1$, and $a * 1 = f(b) * 1 = (b * 1) * 1 = b * (1 * 1) = b * 1 = f(b) = a$. In particular $1 = e * 1 = e$ so $*$ is a t-norm. □

There are however uninorms continuous on the open square $(0, 1)^2$.

Example 11 (The Cross Product Uninorm). Consider the following continuous on $(0, 1)^2$ conjunctive uninorm:

$$x *_{CR} y = \begin{cases} \dfrac{xy}{xy + (1-x)(1-y)} & \text{if } \{x, y\} \neq \{0, 1\} \\ 0 & \text{otherwise} \end{cases}$$

This uninorm is interesting for many reasons; isomorphic versions of it are used to combine degrees of belied in the expert systems MYCIN and PROSPECTOR, also it has connections with Dempster-Schafer theory, and can be motivated geometrically using cross products.

In fact it is possible to classify *strictly increasing* almost-continuous uninorms, with the previous example being a prototype.

Theorem 1 ([4]). *Given a uninorm $*$ with identity $e \in (0, 1)$, the following are equivalent:*

1. *There exists a strictly increasing continuous mapping* $h : [0, 1] \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ *with* $h(0) = -\infty$*,* $h(e) = 0$ *and* $h(1) = +\infty$*, such that:*

$$x * y = h^{-1}(h(x) + h(y)) \text{ for all } (x, y \in [0, 1]^2 \setminus \{(0, 1), (1, 0)\}$$

2. $*$ *is strictly increasing on* $(0, 1)$ *and almost-continuous.*

A strictly increasing almost-continuous uninorm characterized by the previous theorem is called a *representable uninorm* with *additive generator* h.

Finally we consider a representation theorem that has been given for *idempotent* uninorms. Here we restrict our attention to the case of left-continuous idempotent conjunctive uninorms:

Theorem 2. *Given a conjunctive uninorm* $*$ *with identity* $e \in (0, 1)$*, the following are equivalent:*

1. *There exists a left-continuous uniquely determined non-increasing function* $f : [0, 1] \rightarrow [0, 1]$ *with* $f(e) = e$ *and* $f(f(x)) \geq x$ *for* $x \in [0, 1]$*, such that for all* $x, y \in [0, 1]^2$ *we have:*

$$x * y = \begin{cases} min(x, y) & \text{if } y \leq f(x) \\ max(x, y) & \text{otherwise} \end{cases}$$

2. $*$ *is left-continuous and idempotent.*

Example 12 (An Idempotent Uninorm). If we take $f(x) = 1 - x$ in the previous theorem we obtain the following idempotent uninorm:

$$x * y = \begin{cases} min(x, y) & \text{if } x + y \leq 1 \\ max(x, y) & \text{otherwise} \end{cases}$$

8 Nots and Ifs

We turn our attention to other connectives, in particular "nots" (negations) and "ifs" (implications), beginning with some intuitive properties for the former:

Definition 22 (Negation). *A function* $n : [0, 1]^2 \rightarrow [0, 1]$ *is called a* negation *iff it is non-increasing,* $n(0) = 1$ *and* $n(1) = 0$*. A negation function is:*

– strict *iff it is strictly decreasing and continuous.*
– strong *iff it is strict and an involution i.e.:*

$$n(n(x)) = x \text{ for all } x \in [0, 1].$$

– weak *iff* n *is not strong.*

Example 13. The most widely used negation in fuzzy logic is the strong negation:

$$n_{Ł}(x) =_{def} 1 - x$$

There are also negations which are strict but not strong such as:

$$n_{S}(x) =_{def} 1 - x^2$$

Example 14. Consider the following two (weak) negations:

$$n_G(x) =_{def} \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ otherwise} \end{cases} \qquad n^*(x) =_{def} \begin{cases} 1 \text{ if } x < 1 \\ 0 \text{ if } x = 1 \end{cases}$$

n_G and n^* are the *weakest* and *strongest* negations respectively, i.e. for all negation functions n: $n_G \leq n \leq n^*$.

We now turn our attention to possible candidates for interpreting "ifs" i.e. *implication connectives*, beginning with some desirable properties for such functions.

Definition 23 (Properties for Implication). *For a function* $\Rightarrow: [0,1]^2 \rightarrow [0,1]$, *we define the following properties:*

- exchange principle*:* $x \Rightarrow (y \Rightarrow z) = y \Rightarrow (x \Rightarrow z)$ *for all* $x, y, z \in I$.
- left antinocity*:* $\Rightarrow$ *is decreasing in the first argument i.e. if* $x < y$ *then* $y \Rightarrow z \leq x \Rightarrow z$ *for all* $x, y, z \in [0,1]$.
- right isotonicity*:* $\Rightarrow$ *is increasing in the second argument i.e. if* $x < y$ *then* $z \Rightarrow x \leq z \Rightarrow y$ *for all* $x, y, z \in [0,1]$.
- left boundary condition*:* $0 \Rightarrow x = 1$ *for all* $x \in [0,1]$.
- right boundary condition*:* $x \Rightarrow 1 = 1$ *for all* $x \in [0,1]$.
- normality condition*:* $1 \Rightarrow 0 = 0$
- e-degree ranking property*:* $x \Rightarrow y \geq e$ *iff* $x \leq y$ *for all* $x, y \in [0,1]$ *where* $e \in [0,1]$.
- left neutrality*:* $1 \Rightarrow x = x$ *for all* $x \in [0,1]$.
- law of contraposition*:* $x \Rightarrow y = n(y) \Rightarrow n(x)$ *for all* $x, y \in [0,1]$ *w.r.t some strict negation function* $\neg$.

Example 15. Consider the following function $\Rightarrow: [0,1]^2 \rightarrow [0,1]$ where:

$$x \Rightarrow y =_{def} n_Ł(x) \circ_G y = max(1-x, y)$$

This satisfies the antonicity, boundary, normality, left-neutrality and law of contraposition conditions, but not the degree ranking property,

However, if we take conjunction as primary then it seems reasonable to seek an implication which "ties in" with our conjunction. At an intuitive level we insist that for a conjunction $*$ and implication $\Rightarrow$, $x * (x \Rightarrow y)$ be no more true than y, and that $x \Rightarrow y$ should be maximal subject to this restriction. This gives the following definition.

Definition 24 (Residuum). *A binary function* $* : [0,1]^2 \rightarrow [0,1]$ *is said to be* residuated *iff there exists a binary function* $\Rightarrow: [0,1]^2 \rightarrow [0,1]$ *called the* residuum *of* $*$, *such that* $*$ *and* $\Rightarrow$ *form an* adjoint pair, *i.e:*

$$x * y \leq z \textit{ iff } x \leq y \Rightarrow z \textit{ for all } x, y, z \in [0,1]$$

This is a natural but not the only way of obtaining a truth function $\Rightarrow$ for this connective (called an R-implication) from a t-norm $*$. Other implications can be defined e.g. using an involutive negation $\neg x =_{def} 1 - x$ and taking $x \Rightarrow y =_{def} \neg(x * \neg y)$ (called an S-implication).

Notice that if a conjunctive uninorm has a residuum, then we know what it is.

Lemma 5. *If a conjunctive uninorm $*$ has a residuum $\Rightarrow$ then:*

$$x \Rightarrow y = max\{z \mid z * x \leq y\} \textit{ for all } x, y \in [0,1]$$

Proof. By definition $(x \Rightarrow y) * x \leq y$ so $x \Rightarrow y \leq max\{z \mid z * x \leq y\}$. Moreover if $w * x \leq y$ then $w \leq x \Rightarrow y$, hence in fact $max\{z \mid z * x \leq y\} = x \Rightarrow y$ as required. □

One way of understanding this approach is as a generalization of *modus ponens*.

Lemma 6. *A function $* : [0,1]^2 \rightarrow [0,1]$ with residuum $\Rightarrow: [0,1]^2 \rightarrow [0,1]$ satisfies the* generalized modus ponens principle*, i.e.:*

$$x * (x \Rightarrow y) \leq y \textit{ for all } x, y \in [0,1]$$

Proof. Follows directly from adjointness. □

Residua for the fundamental t-norms can be calculated as follows:

Proposition 8.

1. *Łukasiewicz implication:* $x \Rightarrow_{\mathbf{Ł}} y = min(1, 1 - x + y)$
2. *Gödel implication:* $x \Rightarrow_{\mathbf{G}} y = \begin{cases} 1 \textit{ if } x \leq y \\ y \textit{ otherwise} \end{cases}$
3. *Product implication:* $x \Rightarrow_{\mathbf{\Pi}} y = \begin{cases} 1 & \textit{if } x \leq y \\ y/x & \textit{otherwise} \end{cases}$

Proof. We consider the case of product implication as an example, leaving the others as exercises. Suppose that $x \leq y$ then $1.x = x \leq y$ so $x \Rightarrow y = 1$. If $x > y$ then $x.(y/x) = y$ and $x.z > y$ for $z > (y/x)$, so $x \Rightarrow y = y/x$. □

Example 16. We can also consider the more complicated case of the conjunctive uninorm of Example 11:

$$x \Rightarrow_{CR} y = \begin{cases} \dfrac{(1-x)y}{x(1-y) - (1-x)y} & \text{if } x, y \in [0,1]^2/\{(0,0),(1,1)\} \\ 1 & \text{otherwise} \end{cases}$$

In the case of a *conjunctive uninorm* $*$ (which recall includes t-norms) a necessary and sufficient condition for $\Rightarrow$ to exist, is that $*$ be *left-continuous*. We now consider which of the intuitive properties of implication listed above are satisfied by the residua of conjunctive uninorms:

Lemma 7. *Let $*$ be a conjunctive uninorm with identity e and residuum $\Rightarrow$, we have that $\Rightarrow$ satisfies all the properties of Definition 23 except the law of contraposition.*

Proof. As an example consider the right boundary condition. If $*$ is a conjunctive uninorm with residuum $\Rightarrow$ then for all $x \in [0,1]$ we have $1 * x \leq 1$ and hence by adjointness $1 \leq x \Rightarrow 1$ i.e. $x \Rightarrow 1 = 1$. □

One widely used method for obtaining suitable truth functions for *negation* is to take either an arbitary constant f or the constant 0 and define negations of x as $x \Rightarrow f$ and $x \Rightarrow 0$, i.e.:

Definition 25. *Let $*$ be a conjunctive uninorm with residuum $\Rightarrow$, we define:*

$$\neg x =_{def} x \Rightarrow f \quad -x =_{def} x \Rightarrow 0$$

Lemma 8. *If $\Rightarrow$ is the residuum of a conjunctive uninorm then the functions $\neg : [0,1] \rightarrow [0,1]$ and $- : [0,1] \rightarrow [0,1]$ defined by $\neg x = x \Rightarrow f$ and $-x = x \Rightarrow 0$ are negations.*

Proof. Both $\neg$ and $-$ are decreasing since $\Rightarrow$ is decreasing in its first argument. Moreover $\neg 0 = 0 \Rightarrow f = 1$ and $-0 = 0 \Rightarrow 0 = 1$, $\neg 1 = 1$ as required. □

In the case of a t-norm with an involutive negation $\neg$, we get that these two negations are the same.

Lemma 9. *For a t-norm where $\neg\neg x = x$, $f = 0$.*

Proof. We have $(0 \Rightarrow f) \Rightarrow f = 0$ but since $0 \Rightarrow f = 1$ and $1 \Rightarrow f = f$, we get $f = 0$ as required. □

For the fundamental t-norms we get the following negations:

Proposition 9.

1. *Łukasiewicz negation: $-x = 1 - x$*
2. *Gödel (Product) negation: $-0 = 1$, $-x = 0$ for $x > 0$*

Proof. By calculation. □

For *continuous* t-norms we also obtain the (perhaps unexpected) bonus of being able to define the functions min and max using just the t-norm and its residuum.

Proposition 10. *For all continuous t-norms $*$ with residuum $\Rightarrow$:*

1. *$x \leq y$ iff $x \Rightarrow y = 1$*
2. *If $x \leq y$ then $x = y * (y \Rightarrow x)$*
3. *$x * (x \Rightarrow y) = min(x, y)$*
4. *$min((x \Rightarrow y) \Rightarrow y, (y \Rightarrow x) \Rightarrow x) = max(x, y)$*

Proof.

1. If $x \leq y$ then $x = x * 1 \leq y$ so $x \Rightarrow y = 1$, and if $x \Rightarrow y = 1$ then $x = x * 1 = x * (x \Rightarrow y) \leq y$.
2. By the continuity of $*$ we have that $f(z) = z * y$ is a continuous function on $[0,1]$ where $f(0) = 0$ and $f(1) = 1$. Hence f is surjective and for some z, $0 \leq z \leq 1$, $f(z) = x$. Moreover for the maximal z such that $z = x * y$ we get that $z = x \Rightarrow y$.
3. If $x \leq y$ then $x \Rightarrow y = 1$ and $x * (x \Rightarrow y) = x$; if $x > y$ then $x * (x \Rightarrow y) = y$ by 2.
4. If $x \leq y$ then $x \Rightarrow y = 1$ and, since also $y \leq (y \Rightarrow x) \Rightarrow x$, $min((x \Rightarrow y) \Rightarrow y, ((y \Rightarrow x) \Rightarrow x) = y$. The case of $y \leq x$ is symmetrical. □

9 Putting Things Together

We now have all the ingredients required to define a fuzzy logic; our next step is to provide the recipe. Although it would be perfectly legitimate to take as many different implications, conjunctions etc. as we want for a logic, our main desire here is to define a set of connectives that is *coherent*. To this end we take a conjunctive uninorm to interpret conjunction, and, since this gives us a generalized modus ponens principle, its residuum as implication. What of the other connectives? Well for continuous t-norms we can automatically get the min and max functions, and indeed since these are used to define order, we take these for all our logics. For negation and disjunction we use definitions in terms of other connectives. Of course this might not be entirely satisfactory. What if for example we want an involutive negation to go along with the Gödel t-norm? Or we want both a product conjunction *and* a Łukasiewicz conjunction? Such needs are both plausible and indeed have been investigated in the literature. Nevertheless the logics we define here can be seen as a basis for adding further connectives.

Definition 26. *The language $\mathcal{L}_{\mathcal{F}^{\mathcal{B}}}$ has the following connectives:*

$$CON(\mathcal{L}_{\mathcal{F}^{\mathcal{B}}}) = \{\odot, \rightarrow, \wedge, \vee, t, f, \bot, \top\}$$

We also define $\neg A =_{def} A \rightarrow f$ and $A \oplus B =_{def} \neg(\neg A \odot \neg B)$.

Next we show how to interpret connectives.

Definition 27 (Uninorm Based Logics). *For a conjunctive uninorm $* : [0,1]^2 \rightarrow [0,1]$ with identity e and residuum $\Rightarrow$, we define the* fuzzy propositional logic $PC(*)$*:*

$$[[0,1],[e,1],\{*, \Rightarrow, min, max, e, 0\}$$

A valuation for $PC(*)$ is a function $v : VAR(\mathcal{L}_{\mathcal{F}^{\mathcal{B}}}) \rightarrow [0,1]$ extended to $FOR(\mathcal{L}_{\mathcal{F}^{\mathcal{B}}})$ as follows:

$$\begin{array}{rlrl} v(A \odot B) = v(A) * v(B) & & v(A \rightarrow B) = v(A) \Rightarrow v(B) \\ v(A \wedge B) = min(v(A), v(B)) & & v(A \vee B) = max(v(A), v(B)) \\ v(t) = e & & v(f) \in [0,1] \\ v(\bot) = 0 & & v(\top) = 1 \end{array}$$

$A \in FOR(\mathcal{L}_{\mathcal{F}^{\mathcal{B}}})$ is valid in PC(*) iff $v(A) \geq e$ for all valuations v.

We note also that if $*$ is a residuated t-norm then it is sufficient to base $PC(*)$ on a language with connectives $\odot$, $\rightarrow$, $\wedge$ and $\bot$, defining $A \vee B =_{def} ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A)$ and $t =_{def} \bot \rightarrow \bot$. Moreover if $*$ is continuous we can drop $\vee$ and have the definition $A \vee B =_{def} A \odot (A \rightarrow B)$.

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