Labelled tree sequents, Tree hypersequents and Nested (Deep) Sequents

Rajeev Goré [1]

Logic and Computation Group
Research School of Computer Science
The Australian National University
Canberra, ACT 0200, Australia

Revantha Ramanayake [2]

CNRS
LIX, École Polytechnique
91128 Palaiseau, France

Abstract

We identify a subclass of labelled sequents called "labelled tree sequents" and show that these are notational variants of tree-hypersequents in the sense that a sequent of one type can be represented naturally as a sequent of the other type. This relationship can be extended to nested (deep) sequents using the relationship between tree-hypersequents and nested (deep) sequents, which we also show.
We apply this result to transfer proof-theoretic results such as syntactic cut-admissibility between the tree-hypersequent calculus $CSGL$ and the labelled sequent calculus $G3GL$ for provability logic GL. This answers in full a question posed by Poggiolesi about the exact relationship between these calculi.
Our results pave the way to obtain cut-free tree-hypersequent and nested (deep) sequent calculi for large classes of logics using the known calculi for labelled sequents, and also to obtain a large class of labelled sequent calculi for bi-intuitionistic tense logics from the known nested (deep) sequent calculi for these logics. Importing proof-theoretic results between notational variant systems in this manner alleviates the need for independent proofs in each system. Identifying which labelled systems can be rewritten as labelled tree sequent systems may provide a method for determining the expressive limits of the nested sequent formalism.

Keywords: labelled tree sequents, notational variants, cut-elimination, proof theory.

[1] rajeev.gore@anu.edu.au

[2] revantha@lix.polytechnique.fr

1 Introduction

Gentzen [6] introduced the *sequent calculus* as a tool for studying proof systems for classical and intuitionistic logics. Gentzen sequent calculi are built from *traditional sequents* of the form $X \Rightarrow Y$ where X and Y are formula multisets. The main result is the cut-elimination theorem, which shows how to eliminate the cut-rule from these calculi. The resulting sequent calculi are said to be *cut-free*. A significant drawback of the Gentzen sequent calculus is the difficulty of adapting the calculus to new logics. For example, although there is a traditional cut-free Gentzen sequent calculus for $S4$, there is no known traditional cut-free Gentzen calculus for $S5$ despite the fact that the logic $S5$ can be directly obtained from a Hilbert calculus [3] for $S4$ by the addition of a single axiom corresponding to symmetry. This has lead to various generalisations of the Gentzen sequent calculus in an attempt to present many different (modal) logics in a single modular proof-theoretic framework with nice properties.

Hypersequent calculi [18,1] generalise Gentzen sequent calculi by using a /-separated list of traditional sequents (a *hypersequent*) rather than just a single one. Usually, the order of the sequents is not important so a multiset can be used instead of a list. In this case the hypersequent $X \Rightarrow Y/U \Rightarrow V$ is the same as $U \Rightarrow V/X \Rightarrow Y$, for example.

Tree-hypersequents [16] are defined from hypersequents through the addition of the symbols ; and ( ) to the syntax, and by attaching importance to the *order* of the traditional sequents. Furthermore, the placement of the / and ; symbols play a crucial role in the semantic meaning of a tree-hypersequent, enabling each tree-hypersequent to be associated with a tree-like frame. For example, the tree-hypersequents $-/(-/(-;-));-$ and $-/-/-/(-;-)$, where the dashes stand for sequents, correspond to the (tree) frame figures below left and below right respectively:

Nested (Deep) Sequents [11,2] generalise traditional sequents through the addition of the symbol [ ] to the syntax, giving us unordered but nested expressions of the form $\Rightarrow [\cdot, [\cdot], [\cdot]], [\cdot]$ and $\Rightarrow [\cdot, [\cdot, [\cdot], [\cdot]]]$ to capture the same frames as above. Further connections with display sequents are also known [7].

Labelled sequents [10,4,13] generalise the traditional sequent by the prefixing of indices or labels to formulae occurring in the sequent. A labelled sequent can be viewed as a directed graph with sequents at each node [20].

Negri [14] has presented a method for generating cut-free labelled sequent calculi for a large family of modal logics. These labelled sequent calculi incorporate the frame accessibility relation into the syntax of the calculi.

All of these extended sequent formalism are modular, since a new logic can be presented by the inclusion of extra rules corresponding to the properties of its accessibility relation or to the appropriate modal axioms. The modal logic $S5$, for example, can be given a cut-free presentation by adding rules for reflexivity, transitivity and symmetry to the base calculus, or by adding rules that capture the axioms T, 4 and 5, as appropriate to the formalism.

A *labelled tree sequent* is a special instance of a labelled sequent where the underlying graph structure is restricted to a tree. Labelled tree sequents have appeared in various guises in the literature, where they have been used to construct calculi for non-classical logics (for example, see [9]). We observe that restricting the underlying graph structure of a labelled tree sequent to a tree (forcing irreflexivity, for example) does not limit the logics that we can handle to simply K-like or GL-like logics. This is because the formulae used to construct the inference rules may of course contain modalities, consequently enriching the expressiveness of the framework. As a trivial example, a standard Gentzen sequent calculus $\mathcal{C}_{S4}$ for the reflexive and transitive logic $S4$ induces a labelled tree sequent calculus C'_{S4} for $S4$, obtained by replacing each traditional sequent $\Gamma \Rightarrow \Delta$ in each inference rule in $\mathcal{C}_{S4}$ with the labelled tree sequent $x : \Gamma \Rightarrow x : \Delta$ where $x : \Gamma$ is obtained by prefixing each formula in Γ with the label x.

Here we establish mappings between tree-hypersequent calculi and labelled tree sequent calculi. This result shows that these systems are notational variants. Using this result it becomes possible to transfer proof-theoretic results between these systems, alleviating the need for independent proofs in each system. As an application of this work, we answer in full a question posed by Poggiolesi [17] regarding the relationship between tree-hypersequent and labelled sequent calculi for provability logic. We envisage that the existing results in the general labelled sequent framework may be coerced under suitable restrictions to provide new proof systems for tree-hypersequent and nested sequent calculi.

Related Work. This paper is based on work appearing in Ramanayake's [19] PhD thesis. The independent but contemporaneous work of Fitting [5] shows that there is a to-and-fro correspondence between prefixed tableaux and Brünnler's deep sequents [2], and coincidentally, uses exactly the same term "notational variants" as does Ramanayake.

Fitting does not prove syntactic cut-admissibility for his prefixed tableaux, but uses the standard semantic completeness proof to establish "cut-free completeness". On the other hand, in this paper, we obtain syntactic transformations between the various calculi that we study, which lead to syntactic proofs of cut-admissibility.

Fitting notes that prefixed tableaux are subsumed by Negri's [14] labelled systems, and states that clarifying the relationship between labelled systems and Bruennler's deep sequent systems is an interesting question. We answer this question as follows.

Since deep sequents are an independent re-invention of Kashima's much older notion of nested sequents [11], we use the term nested (deep) sequents uniformly. In Section 5, we present to-and-fro maps between labelled tree sequent and nested (deep) sequent systems, thus providing an answer to Fitting's question. Finally, since the labelled tree sequent is a proper subclass of the labelled sequent, by ascertaining which labelled sequents cannot be written as labelled tree sequents, it may be possible to determine the expressive limits of the nested sequent formalism. This is a primary motivation for studying labelled tree sequents, rather than just working with the more expressive labelled sequents of Negri.

2 Preliminaries

The *basic modal language* $\mathcal{ML}$ is defined using a countably infinite set $Atms = \{p_i\}_{i \in \mathbb{N}}$ of propositional variables, the usual propositional connectives $\neg$, $\vee$, $\wedge$ and $\supset$, the unary modal operators $\Box$ and $\Diamond$, and the parenthesis symbols ( ).

A *formula* is an expression generated by the following grammar

$$A \ ::= \ p \in Atms \ | \neg A \ | (A \vee B) \ | (A \wedge B) \ | (A \supset B) \ | \Box A \ | \Diamond A$$

In this paper we work exclusively with classical modal logics. In this context we have the freedom of working with certain proper subsets of $\{\neg, \vee, \wedge, \supset, \Box, \Diamond\}$, as the missing language elements can be defined in terms of the remaining ones.

A *traditional sequent* (denoted $X \Rightarrow Y$) is an ordered pair (X, Y) where X (the 'antecedent') and Y (the 'succedent') are finite multisets of formulae.

Definition 2.1 [Gentzen sequent calculus] The *Gentzen sequent calculus* consists of some set of traditional sequents (the *initial sequents*) and some set of *inference rules*, each of the form

$$\frac{\mathcal{S}_1 \ldots \mathcal{S}_n}{\mathcal{S}}$$

where the traditional sequents $\mathcal{S}_1, \ldots, \mathcal{S}_n$ are called the *premises* of the rule, and $\mathcal{S}$ is called the *conclusion* sequent.

2.1 Tree-hypersequent calculi

A tree-hypersequent is built from traditional sequents using the symbols '/' and ';', and the parenthesis symbols ( ).

Definition 2.2 A *tree-hypersequent* is defined inductively as follows:

(i) if $\mathcal{S}$ is a traditional sequent, then $\mathcal{S}$ is a tree-hypersequent,

(ii) if $\mathcal{S}$ is a traditional sequent and $G_1, G_2, \ldots, G_n$ are tree-hypersequents, then $\mathcal{S}/(G_1; G_2; \ldots; G_n)$ is a tree-hypersequent.

We write THS as an abbreviation for the term 'tree-hypersequent(s)'.

As usual, we often introduce or delete parentheses for the sake of clarity. Following are some examples of THS:

$$p \Rightarrow p \vee q \qquad\qquad p \Rightarrow q/((q \Rightarrow r/\Box r \Rightarrow \neg s); l \Rightarrow \Box(p \supset s))$$

We write $G\{\overset{\pi}{H}\}$ to mean that π is an occurrence of the THS H in the THS G. Within the context of any discussion within this paper, we will be concerned with at most *one fixed occurrence* of a THS H in G. Thus, following standard practice, we will refer only implicitly to the specific occurrence, by dropping the occurrence name π and writing $G\{H\}$ to mean that H occurs at a 'distinguised position' in G. This will not cause ambiguity in practice. For a THS H', we write $G\{H'\}$ to mean the THS obtained from $G\{H\}$ by replacing *that* fixed occurrence of H with H'.

Throughout this paper, we will use an underlined capitalised letter of the form $\underline{X}$ (possibly with subscripts) to denote a ;-separated sequence $G_1; \ldots ; G_n$ of THS. We define the notion of *equivalent position* between occurrences of the traditional sequents $\mathcal{S}_1$ and $\mathcal{S}_2$ at distinguished positions in the THS $G_1\{\mathcal{S}_1\}$ and $G_2\{\mathcal{S}_2\}$ respectively (denoted $G_1\{\mathcal{S}_1\} \sim G_2\{\mathcal{S}_2\}$) as follows:

(i) if G is $\mathcal{S}_1$ and H is $\mathcal{S}_2$, then $G\{\mathcal{S}_1\} \sim H\{\mathcal{S}_2\}$;

(ii) if G is $\mathcal{S}_1/\underline{X}_1$ and H is $\mathcal{S}_2/\underline{X}_2$, where the distinguished positions of $\mathcal{S}_1$ and $\mathcal{S}_2$ are the pictured ones, then $G\{\mathcal{S}_1\} \sim H\{\mathcal{S}_2\}$;

(iii) if $H_1\{\mathcal{S}_1\} \sim H_2\{\mathcal{S}_2\}$ then $\mathcal{T}_1/(H_1\{\mathcal{S}_1\}; \underline{X}_1) \sim \mathcal{T}_2/(H_2\{\mathcal{S}_2\}; \underline{X}_2)$ where $\mathcal{T}_1$ and $\mathcal{T}_2$ are traditional sequents.

The intended interpretation $\mathcal{I}$ of a THS as a formula is:

$$(X \Rightarrow Y)^{\mathcal{I}} = \wedge X \supset \vee Y \qquad (\mathcal{S}/(G_1; \ldots G_n))^{\mathcal{I}} = \mathcal{S}^{\mathcal{I}} \vee \Box G_1^{\mathcal{I}} \vee \ldots \vee \Box G_n^{\mathcal{I}}$$

Definition 2.3 [tree-hypersequent calculus] Obtained from Definition 2.1 with the phrase 'traditional sequent' replaced with 'tree-hypersequent'.

Define $X \Rightarrow Y \otimes U \Rightarrow V$ as $X, U \Rightarrow Y, V$. Let $H\{\mathcal{S}\}$ and $H'\{\mathcal{S}'\}$ be THS such that $H\{\mathcal{S}\} \sim H'\{\mathcal{S}'\}$. Then define $H\{\mathcal{S}\} \star H'\{\mathcal{S}'\}$ inductively as follows:

(i) $\mathcal{S} \star \mathcal{S}' = \mathcal{S} \otimes \mathcal{S}'$

(ii) $(\mathcal{S}/\underline{X}) \star (\mathcal{S}'/\underline{Y}) = (\mathcal{S} \otimes \mathcal{S}'/\underline{X}; \underline{Y})$

(iii) $(\mathcal{T}/H\{\mathcal{S}\}; \underline{X}) \star (\mathcal{T}'/H'\{\mathcal{S}'\}; \underline{Y}) = \mathcal{T} \otimes \mathcal{T}'/(H\{\mathcal{S}\} \star H'\{\mathcal{S}'\}; \underline{X}; \underline{Y})$ where $\mathcal{T}$ and $\mathcal{T}'$ are traditional sequents:

We define the cut-rule as follows: for THS $G\{X \Rightarrow Y, A\}$ and $G'\{A, U \Rightarrow V\}$ such that $G\{X \Rightarrow Y, A\} \sim G'\{A, U \Rightarrow V\}$,

$$\frac{G\{X \Rightarrow Y, A\} \qquad G'\{A, U \Rightarrow V\}}{G\{X \Rightarrow Y\} \star G'\{U \Rightarrow V\}} \; cut$$

The $\star$ operation can be viewed as a merge operation on trees, and it ensures that the conclusion sequent of the cut-rule is indeed a THS.

2.2 Labelled sequent calculi

Fitting [4] has described the incorporation of frame semantics into tableau proof systems for the purpose of obtaining tableau systems for certain logics. Approaches to internalise the frame semantics into the Gentzen sequent calculus via the labelling of formulae appear in Mints [13], Vigano [21] and Kushida and

Okada [12]. Both approaches originate from Kanger's "spotted formulae" [10]. Here, we use the labelled systems for modal logic presented in Negri [14].

Assume that we have at our disposal an infinite set $\mathbb{SV}$ of ('state') variables disjoint from the set of propositional variables. We will use the letters $x, y, z \ldots$ to denote state variables. A *labelled formula* has the form $x : A$ where x is a state variable and A is a formula. If $X = \{A_1, \ldots A_n\}$ is a formula multiset, then $x : X$ denotes the multiset $\{x : A_1, \ldots, x : A_n\}$ of labelled formulae. Notice that if the formula multiset X is empty, then the labelled formula multiset $x : X$ is also empty. A *relation term* is a term of the form Rxy where x and y are variables. A (possibly empty) set of relations terms is called a *relation set.* A *labelled sequent* (denoted $\mathcal{R}, X \Rightarrow Y$) is the ordered triple $(\mathcal{R}, X, Y)$ where $\mathcal{R}$ is a relation set and X ('antecedent') and Y ('succedent') are multisets of labelled formulae.

Definition 2.4 [labelled sequent calculus] Obtained from Definition 2.1 with the phrase 'traditional sequent' replaced with 'labelled sequent'. Moreover, each inference rule may include a *standard variable restriction* of the form "z does not appear in the conclusion sequent of the rule" for some state variable z.

Observe that the standard variable restriction is a specific type of side condition on an inference rule.

2.3 Labelled tree sequent calculi

We begin by introducing some terminology and notation.

A *frame* is a pair $F = (W, R)$ where W is a non-empty set of states and R is a binary relation on W. For $x \in W$, define the subframe $x\uparrow$ in the usual way [3] as (W', R') where W' is the minimal upward closed set of $\{x\}$ wrt R, and R' is the restriction of R to W'. When $F = x\uparrow$ we say that F is *generated* by x and x is said to be a root of F. If a frame has a root it is said to be rooted. A rooted frame whose underlying *undirected* graph does not contain a path from any state back to itself (ie. no cycles) is called a *tree.* For example, a frame containing a reflexive state is not a tree. Although a rooted frame may have multiple roots, due to the prohibition of cycles, a tree has exactly one root.

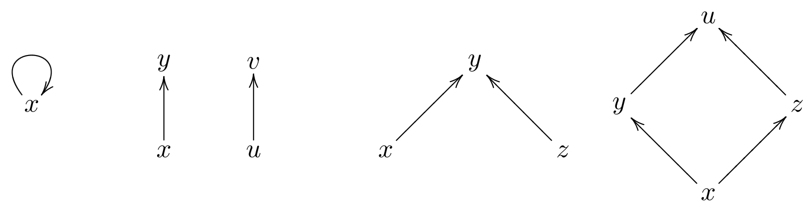
If S is a set of states and Γ is a multiset of labelled formulae, then Γ_S is the multiset $\{x : A \,|\, x \in S \text{ and } x : A \in \Gamma\}$. So $\Gamma_{\{x\}}$ is the multiset of labelled formulae in Γ that are labelled with the state x. With a slight abuse of notation, we write this multiset as Γ_x.

A (possibly empty) set of relations terms (ie. terms of the form Rxy) is called a *relation set.* For a relation set $\mathcal{R}$, the *frame $Fr(\mathcal{R})$ defined by* $\mathcal{R}$ is given by $(|\mathcal{R}|, \mathcal{R})$ where $|\mathcal{R}| = \{x \,|\, Rxv \in \mathcal{R} \text{ or } Rvx \in \mathcal{R} \text{ for some state } v\}$. In the reverse direction, given a frame $F = (W, R)$, write $\mathrm{Rel}(F)$ for the relation set corresponding to R, and let $|F| = W$.

Definition 2.5 [treelike] A relation set $\mathcal{R}$ is *treelike* if the frame defined by $\mathcal{R}$ is a tree or $\mathcal{R}$ is empty.

For a non-empty relation set $\mathcal{R}$ that is treelike, let $\mathrm{root}(\mathcal{R})$ denote the root of this tree.

To illustrate this definition, consider the relation sets $\{Rxx\}$, $\{Rxy, Ruv\}$, $\{Rxy, Rzy\}$, and $\{Rxy, Rxz, Ryu, Rzu\}$. The frames defined by these sets are, respectively,



None of the above relation sets are treelike because the frames defined by their relation sets are not trees. In the above frames from left-to-right, frame 1 contains a reflexive state (and hence a cycle); frame 2 and frame 3 are not rooted. Finally, frame 4 is not a tree because the underlying undirected graph contains a cycle.

Definition 2.6 [labelled tree sequent] A *labelled tree-sequent* is a labelled sequent of the form $\mathcal{R}, X \Rightarrow Y$ where

(i) $\mathcal{R}$ is treelike, and

(ii) if $\mathcal{R} = \emptyset$ then X has the form $\{x : A_1, \ldots, x : A_n\}$ and Y has the form $\{x : B_1, \ldots, x : B_m\}$ for some state variable x (ie. all labelled formulae in X and Y have the same label), and

(iii) if $\mathcal{R} \neq \emptyset$ then every state variable x that occurs in either X or Y (in some labelled formula $x : A$ for some formula A) also occurs in $\mathcal{R}$ (ie as a term Rxu or Rux for some state u).

We write LTS as an abbreviation for the term "labelled tree sequent(s)"

Each of the following is a LTS:

$$x : A \Rightarrow x : A \qquad \Rightarrow y : A \qquad Rxy, Rxz, x : A \Rightarrow y : A$$

Notice that it is possible for a state variable to occur in the relation set and not in the X, Y multisets (this is what happens with the state variable z in the example above far right). The following are *not* LTS:

$$x : A \Rightarrow x : A, z : A \qquad Rxy, x : A \Rightarrow z : A \qquad Rxy, Ryz, Rxz \Rightarrow$$

From left-to-right above, the first labelled sequent is not a LTS because the relation set is empty and yet two distinct state variables occur in the sequent, violating condition (ii). The next sequent violates condition (iii) because the state variable z appears in the succedent as $z : A$ but it does not appear in the relation set. The final sequent violates condition (i) because the relation set is not treelike.

Definition 2.7 [labelled tree sequent calculus] A *labelled tree sequent calculus* is a labelled sequent calculus whose initial sequents and inference rules are constructed from LTS.

Negri [14] uses the following cut-rule for labelled sequent calculi:

$$\frac{\mathcal{R}_1, X \Rightarrow Y, x : A \qquad \mathcal{R}_2, x : A, U \Rightarrow V}{\mathcal{R}_1 \cup \mathcal{R}_2, X, U \Rightarrow Y, V}\ cut$$

We cannot use this rule directly in a labelled tree sequent calculus because $\mathcal{R}_1 \cup \mathcal{R}_2$ need not be treelike even if $\mathcal{R}_1$ and $\mathcal{R}_2$ are treelike. Instead of placing additional conditions on the cut-rule, we define an 'additive' cut-rule for labelled tree sequent calculi as follows:

$$\frac{\mathcal{R}, X \Rightarrow Y, x : A \qquad \mathcal{R}, x : A, X \Rightarrow Y}{\mathcal{R}, X \Rightarrow Y}\ cut_{LTS}$$

We close this section by revising some standard terminology. The terminology is applicable to each of the sequents and calculi we have defined in this section.

An *initial sequent instance* in the calculus $\mathcal{C}$ is a substitution instance of propositional variables/formulae (and state variables, if applicable) of an initial sequent from $\mathcal{C}$. A *rule instance* in the calculus $\mathcal{C}$ is a substitution instance of propositional variables/formulae (and state variables, if applicable) of one of the inference rules from $\mathcal{C}$. A *derivation* in the calculus $\mathcal{C}$ is defined in the usual way, as either an initial sequent instance, or an application of a rule instance to derivations of the premises of the rule. If there is a derivation of some sequent $\mathcal{S}$ in $\mathcal{C}$, then we say that $\mathcal{S}$ is derivable in $\mathcal{C}$. The *height* of a derivation is defined in the usual way as the maximum depth of the derivation tree. We write $\vdash^{\delta}_{\mathcal{C}} \mathcal{S}$ to mean that there is a derivation δ of the sequent $\mathcal{S}$ in $\mathcal{C}$. To avoid having to name the derivation we simply write $\vdash_{\mathcal{C}} \mathcal{S}$.

We say that an inference rule ρ is *admissible* in $\mathcal{C}$ if whenever premises of any rule instance of ρ is derivable in $\mathcal{C}$, then so is the conclusion of the rule instance. Following standard terminology, we say that a calculus $\mathcal{C}$ is *sound and complete* for the logic L if it derives exactly the theorems of L. Formally, for every formula A:

$\vdash_{\mathcal{C}} \ \Rightarrow A$ (or $\vdash_{\mathcal{C}} \ \Rightarrow x : A$ if $\mathcal{C}$ is a labelled sequent calculus) iff $A \in L$

The results we present in this paper are syntactic in the sense that there is an underlying algorithm witnessing each result.

3 Maps between THS and LTS

If $X \Rightarrow Y$ and $U \Rightarrow V$ are traditional sequents, recall that we defined $X \Rightarrow Y \otimes U \Rightarrow V$ to be the traditional sequent $X, U \Rightarrow Y, V$. Overloading the operator, if $\mathcal{R}_1, X \Rightarrow Y$ and $\mathcal{R}_2, U \Rightarrow V$ are two labelled sequents, then define $(\mathcal{R}_1, X \Rightarrow Y) \otimes (\mathcal{R}_2, U \Rightarrow V)$ to be the labelled sequent $\mathcal{R}_1 \cup \mathcal{R}_2, X, U \Rightarrow Y, V$. Because the order of elements in a multiset is irrelevant, in each case $\otimes$ is associative and commutative.

Definition 3.1 [THS to LTS] For a state variable x, define the mapping $\mathbb{TL}_x$ from a THS to a LTS as follows, where the variable $x_{\bar{s}}$ ($\bar{s} \in \mathbb{N}^*$) passed to a recursive call of $\mathbb{TL}$ is never reused:

$$\mathbb{TL}_x(X \Rightarrow Y) = x : X \Rightarrow x : Y$$
$$\mathbb{TL}_x(X \Rightarrow Y/G_1; \ldots; G_n) = \left(\otimes_{j=1}^{n} \mathbb{TL}_{x_j}(G_j)\right) \otimes (Rxx_1, \ldots, Rxx_n, x : X \Rightarrow x : Y)$$

Of course, we need to verify for arbitrary THS H that $\mathbb{TL}_x(H)$ is indeed a LTS. This is a straightforward induction on the structure of H. In the base case (H is $X \Rightarrow Y$) the image $x : X \Rightarrow x : Y$ is a LTS. In the inductive case (H is $X \Rightarrow Y/G_1; \ldots; G_n$), by the induction hypothesis we have that $\mathbb{TL}_{x_i}(G_i)$ ($1 \leq i \leq n$) is a LTS. Then it is easy to see that the relation set $\mathcal{R}$ of $\mathbb{TL}_x(H)$ is non-empty and treelike — in particular, $Rxx_1, \ldots, Rxx_n \in \mathcal{R}$ and the frame defined by $\mathcal{R}$ has root x. It remains to check condition (iii) of Definition 2.6. The state variable of a labelled formula in $\mathbb{TL}_x(H)$ either is x or occurs in the relation set $\mathcal{R}_j$ of the LTS $\mathbb{TL}_{x_j}(G_j)$ for some j. Since x certainly occurs in $\mathcal{R}$ and $\mathcal{R}_j \subseteq \mathcal{R}$, the condition is satisfied in each case.

Example 3.2 If H is the THS $p \Rightarrow q/((q \Rightarrow r/\Box r \Rightarrow \neg s); l \Rightarrow \Box(p \supset s))$,

$$\begin{aligned}\mathbb{TL}_x(H) = \; & \mathbb{TL}_{x_1}(q \Rightarrow r/\Box r \Rightarrow \neg s) \otimes \mathbb{TL}_{x_2}(l \Rightarrow \Box(p \supset s)) \\ & \otimes (Rxx_1, Rxx_2, x : p \Rightarrow x : q) \\ = \; & \mathbb{TL}_{x_{11}}(\Box r \Rightarrow \neg s) \otimes (Rx_1x_{11}, x_1 : q \Rightarrow x_1 : r) \\ & \otimes (x_2 : l \Rightarrow x_2 : \Box(p \supset s)) \otimes (Rxx_1, Rxx_2, x : p \Rightarrow x : q) \\ = \; & (x_{11} : \Box r \Rightarrow x_{11} : \neg s) \otimes (Rx_1x_{11}, x_1 : q \Rightarrow x_1 : r) \\ & \otimes (x_2 : l \Rightarrow x_2 : \Box(p \supset s)) \otimes (Rxx_1, Rxx_2, x : p \Rightarrow x : q)\end{aligned}$$

The last equality simplifies to the LTS

$$Rx_1x_{11}, Rxx_1, Rxx_2, x_{11} : \Box r, x_1 : q, x_2 : l, x : p \Rightarrow x_{11} : \neg s, x_1 : r, x_2 : \Box(p \supset s), x : q$$

We sometimes suppress the subscript, writing $\mathbb{TL}$ instead of $\mathbb{TL}_x$ for the sake of clarity when the state variable that is used is not important. Observe that $\mathbb{TL}G$ assigns a unique state variable to each traditional sequent $\mathcal{S}$ appearing in the THS G. Moreover, given $G_1\{\mathcal{S}_1\} \sim G_2\{\mathcal{S}_2\}$, without loss of generality we may assume that the state variable assigned to $\mathcal{S}_1$ in $\mathbb{TL}(G_1\{\mathcal{S}_1\})$ and $\mathcal{S}_2$ in $\mathbb{TL}(G_2\{\mathcal{S}_2\})$ is identical.

Definition 3.3 Define the function $\mathbb{LT}$ from an LTS $\mathcal{R}, X \Rightarrow Y$ to a THS as:

$\mathcal{R} = \emptyset$: then $\mathcal{R}, X \Rightarrow Y$ must have the form $x : U \Rightarrow x : V$ for some state variable x, so let $\mathbb{LT}(x : U \Rightarrow x : V) = (U \Rightarrow V)$

$\mathcal{R} \neq \emptyset$: then suppose that $x = \text{root}(\mathcal{R})$ and $\mathcal{R}_x = \{Rxy_1, \ldots, Rxy_n\}$, and let $\Delta_i = |y_i\!\uparrow|$, and let

$$\mathbb{LT}(\mathcal{R}, X \Rightarrow Y) =$$
$$X_x \Rightarrow Y_x/(\mathbb{LT}(\text{Rel}(y_1\!\uparrow), X_{\Delta_1} \Rightarrow Y_{\Delta_1}); \ldots; \mathbb{LT}(\text{Rel}(y_n\!\uparrow), X_{\Delta_n} \Rightarrow Y_{\Delta_n})).$$

Recall that $\mathbb{SV}$ denotes the set of state variables. Let $\mathrm{Var}(\mathcal{S}) \subset \mathbb{SV}$ denote the finite set of state variables occurring in the labelled sequent S. A *renaming of* $\mathcal{S}$ is a one-to-one function $f_{\mathcal{S}} : \mathrm{Var}(\mathcal{S}) \mapsto \mathbb{SV}$ (by one-to-one we mean that if $f_{\mathcal{S}}(x) = f_{\mathcal{S}}(y)$ then $x = y$). We write $Dom(f_{\mathcal{S}})$ and $Im(f_{\mathcal{S}})$ to denote the domain and image of $f_{\mathcal{S}}$ respectively.

For any labelled sequent $\mathcal{S}'$ and renaming $f_{\mathcal{S}}$ of the labelled sequent $\mathcal{S}$, let $\mathcal{S}'_{f_{\mathcal{S}}}$ be the labelled sequent obtained from $\mathcal{S}'$ by the simultaneous and uniform substitution $x \mapsto f_{\mathcal{S}}(x)$ for all $x \in Dom(f_{\mathcal{S}}) \cap \mathrm{Var}(\mathcal{S}')$. Notice that $\mathcal{S}'_{f_{\mathcal{S}}}$ need not be an LTS even if $\mathcal{S}'$ is a LTS.

Example 3.4 Consider the following LTS $\mathcal{S}$ (below left) and $\mathcal{S}'$ (below right):

$$x : A \Rightarrow x : A \qquad\qquad Rxy, x : A \Rightarrow y : B$$

Let the renaming $f_{\mathcal{S}}$ of $\mathcal{S}$ be the function mapping $x \mapsto y$. Then $\mathcal{S}'_{f_{\mathcal{S}}}$ is the labelled sequent $Ryy, y : A \Rightarrow y : B$. Clearly this sequent is not a LTS.

However, if $\mathcal{S}$ is a LTS, then for any renaming f_S of $\mathcal{S}$, it is easy to see that $\mathcal{S}_{f_S}$ must be a LTS.

Lemma 3.5 *Let G denote a THS and let $\mathcal{S}$ denote a LTS. Then*

(i) $\mathbb{TL}G$ *is a labelled tree-sequent.*

(ii) $\mathbb{LT}\mathcal{S}$ *is a THS.*

(iii) $\mathbb{LT}(\mathbb{TL}G)$ *is G, and* $\mathbb{TL}(\mathbb{LT}\mathcal{S})$ *is* $\mathcal{S}_{f_{\mathcal{S}}}$ *for some renaming* $f_{\mathcal{S}}$ *of* $\mathcal{S}$.

Proof. The proofs of (i) and (ii) are straightforward, following from an inspection of the functions $\mathbb{TL}$ and $\mathbb{LT}$. In the case of (iii), observe that Definition 2.6(iii) ensures that no labelled formulae are 'lost' when passing from $\mathcal{S}$ to $\mathbb{LT}\mathcal{S}$. However, since the $\mathbb{TL}$ function *assigns* state variables, it may be necessary to 'swap' label names in order to obtain equality of the LTS $\mathbb{TL}(\mathbb{LT}\mathcal{S})$ and $\mathcal{S}$. That is, there is some renaming $f_{\mathcal{S}}$ of $\mathcal{S}$ such that $\mathbb{TL}(\mathbb{LT}\mathcal{S})$ is $\mathcal{S}_{f_{\mathcal{S}}}$. $\dashv$

Lemma 3.6 (substitution lemma) *Suppose that $\mathcal{C}$ is a LTS calculus and $\mathcal{S}$ is a LTS. Let $f_{\mathcal{S}}$ be an arbitrary renaming of $\mathcal{S}$. If $\vdash^{\delta}_{\mathcal{C}} \mathcal{S}$ then there is an effective transformation to a derivation δ' such that $\vdash^{\delta'}_{\mathcal{C}} \mathcal{S}_{f_{\mathcal{S}}}$.*

Proof. Induction on the height of δ. If the height is one, then δ must be an initial sequent. It is easy to see that $\mathcal{S}_{f_{\mathcal{S}}}$ is also an initial sequent.

Now suppose that the last rule in δ is the LTS inference rule ρ, with premises $\mathcal{S}_1, \ldots, \mathcal{S}_n$. The proof is not completely trivial since f $\cup_i \mathrm{Var}(\mathcal{S}_i)$ contains state variables not in $\mathcal{S}$ (for example, due to a standard variable restriction on ρ), then it is possible that $(\mathcal{S}_i)_{f_{\mathcal{S}}}$ is not a LTS even though $\mathcal{S}_i$ is a LTS. For example, suppose that f is the renaming $x \mapsto y$ of the $LTS \Rightarrow x : \Box A$, and consider the following rule instance of ρ:

$$\frac{\mathcal{S}_1 = Rxy, y : \Box A \Rightarrow y : A}{\mathcal{S} = \qquad\qquad \Rightarrow x : \Box A}$$

Then $(\mathcal{S}_1)_f$ is the labelled sequent $Ryy, y : \Box A \Rightarrow y : A$ which is not a LTS (because the relation set contains the cycle Ryy) although $\mathcal{S}_1$ is a LTS.

Returning to the proof, the solution is to define first a one-to-one function g from $\cup_i \mathrm{Var}(\mathcal{S}_i) \setminus Dom(f_\mathcal{S})$ to fresh state variables (in particular, to variables outside $Im(f_\mathcal{S})$). Then $((\mathcal{S}_i)_g)_{f_\mathcal{S}}$ is a LTS. Then $((\cdot)_g)_{f_\mathcal{S}}$ implicitly defines a renaming $\mathrm{Var}(\mathcal{S}_i) \mapsto \mathbb{SV}$ for $\mathcal{S}_i$ for each i. So $((\mathcal{S}_i)_g)_{f_\mathcal{S}}$ is a LTS.

Continuing the example above, set g as the map $y \mapsto z$, so

$$\frac{((\mathcal{S}_1)_g)_f = Ryz, z : \Box A \Rightarrow z : A}{(\mathcal{S}_g)_f = \qquad\qquad \Rightarrow y : \Box A}$$

and this is a legal rule instance of ρ.

Once again, returning to the proof, by the induction hypothesis we can obtain derivations of $((\mathcal{S}_i)_g)_{f_\mathcal{S}}$ in $\mathcal{C}$. Moreover, observe that

$$\frac{((\mathcal{S}_1)_g)_{f_\mathcal{S}} \ldots ((\mathcal{S}_n)_g)_{f_\mathcal{S}}}{(\mathcal{S}_g)_{f_\mathcal{S}}}$$

is a rule instance of ρ. Hence we have a derivation of $(\mathcal{S}_g)_f$. Since $\mathrm{Var}(\mathcal{S}) \cap Dom(g) = \emptyset$, it follows that $(\mathcal{S}_g)_{f_\mathcal{S}} = \mathcal{S}_{f_\mathcal{S}}$. ⊣

We remind the reader that this substitution lemma pertains to LTS calculi as given in Definition 2.7. In particular, this lemma may not apply to calculi containing pathological rules that are not invariant under renaming, such as the following rule:

$$\frac{x \neq a}{x : A \Rightarrow x : B} \ (a \text{ is some fixed state variable})$$

3.1 Inference rules induced by $\mathbb{TL}$ and $\mathbb{LT}$

It is straightforward to construct an inference rule for THS from an inference rule for LTS and *vice versa* under the maps $\mathbb{LT}$ and $\mathbb{TL}$. We illustrate with some detailed examples. Following standard practice, the active formulae in the conclusion (resp. premise) of an inference rule are called *principal* (*auxiliary*) formulae.

Example 3.7 Consider the following inference rule $R\Box$:

$$\frac{\mathcal{R}, \overbrace{Rxy, y : \Box A}^{\text{principal}}, \Gamma \Rightarrow \Delta, \overbrace{y : A}^{\text{principal}}}{\mathcal{R}, \Gamma \Rightarrow \Delta, \underbrace{x : \Box A}_{\text{principal}}} \ R\Box$$

where y does not appear in the conclusion of the rule. We can write the premise and conclusion, respectively, as

$$(\mathcal{R}, \Gamma \Rightarrow \Delta) \otimes (Rxy, y : \Box A \Rightarrow y : A)$$
$$(\mathcal{R}, \Gamma \Rightarrow \Delta) \otimes (\Rightarrow x : \Box A)$$

The sequent $\mathcal{R}, \Gamma \Rightarrow \Delta$ is an arbitrary LTS except it does not contain y, hence it follows that

$$\mathbb{LT}(\mathcal{R}, \Gamma \Rightarrow \Delta) = G\{\overbrace{X \Rightarrow Y}^{x} / (\underline{X}; \overbrace{\ \emptyset\ }^{y})\}$$

for arbitrary $G, X, Y, \underline{X}$ — the braces indicate the locations corresponding to the state variables, and we use $\emptyset$ as a metalevel symbol to explicate that there is no position in the THS corresponding to y. Meanwhile we have

$$\mathbb{LT}(Rxy, y : \Box A \Rightarrow y : A) = \overbrace{\quad \Rightarrow \quad}^{x} / \overbrace{\Box A \Rightarrow A}^{y}$$
$$\mathbb{LT}(\Rightarrow x : \Box A) = \underbrace{\Rightarrow \Box A}_{x}$$

Thus, the image $\mathbb{LT}(R\Box)$ of $R\Box$ under $\mathbb{LT}$ is the THS inference rule:

$$\frac{G\{X \Rightarrow Y / \Box A \Rightarrow A\}}{G\{X \Rightarrow Y, \Box A\}} \, \mathbb{LT}(R\Box)$$

Example 3.8 For the other direction, consider the THS rule $\Box K_{gl}$.

$$\frac{G\{\overbrace{X \Rightarrow Y}^{x} / \overbrace{\Box A \Rightarrow A}^{y}\}}{G\{\underbrace{X \Rightarrow Y, \Box A}_{x} / \underbrace{\ \emptyset\ }_{y}\}} \, \Box K_{gl}$$

As before, we have used braces to identify location in the THS with state variables, and $\emptyset$ as a metalevel symbol to explicate that there is no position in the conclusion THS corresponding to y. Notice that the sequent $\Box A \Rightarrow A$ in the premise disappears in the conclusion. Equivalently, the location correponding to the variable y is not populated. Applying $\mathbb{TL}_x$ to the premise of $\Box K_{gl}$ we get the LTS

$$\mathcal{R}, Rxy, y : \Box A, \Gamma \Rightarrow \Delta, y : A$$

where $\mathcal{R}, \Gamma$ and Δ are arbitrary. Applying $\mathbb{TL}_x$ to the conclusion of $\Box K_{gl}$ we get the LTS

$$\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A$$

where y does not appear in the conclusion. We thus obtain the LTS rule $\mathbb{TL}\Box K_{gl}$.

$$\frac{\mathcal{R}, Rxy, y : \Box A, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A} \, \mathbb{TL}\Box K_{gl}$$

with the standard variable restriction "y does not occur in the the conclusion of $\mathbb{TL}\Box K_{gl}$".

3.2 Calculi induced by $\mathbb{TL}$ and $\mathbb{LT}$

We can now construct a THS calculus from a LTS calculus and *vice versa.* If $\mathcal{C}$ is a THS calculus, then let $\mathbb{TL}\mathcal{C}$ denote the calculus consisting of the image of every initial sequent and inference rule in $\mathcal{C}$ under $\mathbb{TL}$. Similarly, if $\mathcal{C}$ is a LTS calculus, then let $\mathbb{LT}\mathcal{C}$ denote the calculus consisting of the image of every initial sequent and inference rule in $\mathcal{C}$ under $\mathbb{LT}$.

Lemma 3.9 *Let $\mathcal{C}$ be a THS calculus. Then,* *(i) for any THS G, we have* $\vdash_{\mathcal{C}} G$ *iff* $\vdash_{\mathbb{TLC}} \mathbb{TL}G$

(ii) for any LTS $\mathcal{S}$*, we have* $\vdash_{\mathbb{TLC}} \mathcal{S}$ *iff* $\vdash_{\mathcal{C}} \mathbb{LT}\mathcal{S}$.

In each case, the respective derivations in $\mathcal{C}$ *and* $\mathbb{TL}\mathcal{C}$ *have identical height.*

Proof. Proof of (i). Suppose that $\vdash^{\delta}_{\mathcal{C}} G$. We need to show that $\mathbb{TL}G$ is derivable in $\mathbb{TL}\mathcal{C}$. We can obtain a derivation δ' of $\mathbb{TL}G$ from δ by replacing every *THS* G' appearing in δ with $\mathbb{TL}G'$, and every rule ρ with $\mathbb{TL}\rho$ — by the definition of $\mathbb{TL}\mathcal{C}$, the resulting object is a derivation in the calculus $\mathbb{TL}\mathcal{C}$ with endsequent $\mathbb{TL}G$. In particular, notice that if ρ is a legal rule instance in $\mathcal{C}$, then $\mathbb{TL}\rho$ will obey any relevant standard variable restrictions in $\mathbb{TL}\mathcal{C}$. Moreover, by construction, δ and δ' have identical height.

Proof of (ii) is analogous to the above. ⊣

Corollary 3.10 *For any THS calculus* $\mathcal{C}$ *and formula A we have* $\vdash_{\mathcal{C}} \Rightarrow A$ *iff* $\vdash_{\mathbb{TL}\mathcal{C}} \Rightarrow x : A$.

Proof. Immediate from Lemma 3.9. ⊣

4 Poggiolesi's *CSGL* and Negri's *G3GL*

Negri [14] has given a labelled sequent calculus *G3GL* for provability logic *GL* as part of a systematic program to present labelled sequent calculi for modal logics. Subsequently Poggiolesi [17] presented the *THS* calculus *CSGL* for *GL* and proved syntactic cut-admissibility. In that work, Poggiolesi [17] states:

> *"As it has probably already emerged in the previous sections, CSGL is quite similar to Negris calculus G3GL [see [14]]: indeed, except for the rule* 4 *that only characterizes CSGL, the propositional and modal rules of the two calculi seem to be based on a same intuition. Given this situation, a question naturally arises: what is the exact relation between the two calculi? Is it possible to find a translation from the THS calculi to the labeled calculi and vice versa?"*

Here we establish the following.

(i) We answer in full the question raised by Poggiolesi. In particular, we give a translation between *CSGL* and *G3GL*; and

(ii) Show that *CSGL* is sound and complete for provability logic *GL* and prove syntactic cut-admissibility utilising the *existing* proofs of these results for *G3GL*. In contrast, Poggiolesi [17] has to provide a new proof for each result, in particular, dealing with the many cases that arise in the proof of syntactic cut-admissibility. Since many proof-theoretical properties (invertibility of the inference rules, for example) are preserved under the notational variants translation, we get these results directly, once again alleviating the need for independent proofs.

A key aspect of our work is the coercion of results from the labelled sequent calculus *G3GL* into the *LTS* calculus $\mathbb{TL}$*CSGL* (Theorem 4.4).

Initial THS**:** $G\{p, X \Rightarrow Y, p\}$ $\quad G\{\Box A, X \Rightarrow Y, \Box A\}$

Propositional rules:

$$\frac{G\{X \Rightarrow Y, A\}}{G\{\neg A, X \Rightarrow Y\}}\,\neg A \qquad \frac{G\{A, X \Rightarrow Y\}}{G\{X \Rightarrow Y, \neg A\}}\,\neg K$$

$$\frac{G\{A, B, X \Rightarrow Y\}}{G\{A \wedge B, X \Rightarrow Y\}}\,\wedge A \qquad \frac{G\{X \Rightarrow Y, A\} \quad G\{X \Rightarrow Y, B\}}{G\{X \Rightarrow Y, A \wedge B\}}\,\wedge K$$

Modal rules:

$$\frac{G\{\Box A, X \Rightarrow Y/(U \Rightarrow V, \Box A/\underline{X})\} \qquad G\{\Box A, X \Rightarrow Y/(A, U \Rightarrow V/\underline{X})\}}{G\{\Box A, X \Rightarrow Y/(U \Rightarrow V/\underline{X})\}}\,\Box A$$

$$\frac{G\{X \Rightarrow Y/\Box A \Rightarrow A\}}{G\{X \Rightarrow Y, \Box A\}}\,\Box K$$

Special logical rule:

$$\frac{G\{\Box A, X \Rightarrow Y/(\Box A, U \Rightarrow V/\underline{X})\}}{G\{\Box A, X \Rightarrow Y/(U \Rightarrow V/\underline{X})\}}\,4$$

Table 1
$CSGL$: the THS calculus of Poggiolesi [17].

Initial LTS**:** $\mathcal{R}, x : p, \Gamma \Rightarrow \Delta, x : p$ $\quad \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta, x : \Box A$

Propositional rules:

$$\frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A}{\mathcal{R}, x : \neg A, \Gamma \Rightarrow \Delta}\,\neg A \qquad \frac{\mathcal{R}, x : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \neg A}\,\neg K$$

$$\frac{\mathcal{R}, x : A, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \wedge B, \Gamma \Rightarrow \Delta}\,\wedge A \qquad \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \quad \mathcal{R}, \Gamma \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \wedge B}\,\wedge K$$

Modal rules:

$$\frac{\mathcal{R}, Rxy, x : \Box A, \Gamma \Rightarrow \Delta, y : \Box A \qquad \mathcal{R}, Rxy, x : \Box A, y : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, Rxy, x : \Box A, \Gamma \Rightarrow \Delta}\,\mathbb{TL}\Box A$$

$$\frac{\mathcal{R}, Rxy, y : \Box A, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}\,\mathbb{TL}\Box K$$

Special logical rule:

$$\frac{\mathcal{R}, Rxy, x : \Box A, y : \Box A, \Gamma \Rightarrow \Delta}{\mathcal{R}, Rxy, x : \Box A, \Gamma \Rightarrow \Delta}\,\mathbb{TL}4$$

Table 2
$\mathbb{TL}CSGL$: the LTS calculus obtained from $CSGL$ under the mapping $\mathbb{TL}$. Rule $\mathbb{TL}\Box K$ has the standard restriction that y does not appear in the conclusion.

4.1 The calculus $CSGL$ and $\mathbb{TL}CSGL$

Poggiolesi's THS calculus $CSGL$ [17] is given in Table 1. From this calculus we construct the LTS calculus $\mathbb{TL}CSGL$ (Table 2) following the procedure given in the previous section.

For a relation term or labelled formula α, define the left and right weakening rules as follows:

$$\frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \alpha, \Gamma \Rightarrow \Delta}\, LW \qquad \frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta, \alpha}\, RW$$

We remind the reader that each of the above rules when viewed as a LTS inference rule (as opposed to a labelled sequent inference rule) has the restriction that the premise and conclusion is a LTS. The left and right contraction rules are defined as follows:

$$\frac{\mathcal{R}, x : A, x : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A, \Gamma \Rightarrow \Delta}\, LC \qquad \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A, x : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A}\, RC$$

Lemma 4.1 *The rules LW and RW for weakening and the rules LC and RC for contraction are height-preserving syntactically admissible in $\mathbb{TL}CSGL$.*

Proof. Poggiolesi [17] shows that the corresponding THS rules (ie. the weakening and contraction rules under $\mathbb{LT}$) are height-preserving syntactically admissible in $CSGL$. By Theorem 3.9, the mapping between derivations in $CSGL$ and $\mathbb{TL}CSGL$ is height-preserving. Hence the analogous results apply to $\mathbb{TL}CSGL$ too, so we are done. ⊣

4.2 Negri's calculus $G3GL$

If we compare the $\mathbb{TL}CSGL$ calculus with Negri's labelled sequent calculus $G3GL$, the only differences are that

(i) in $G3GL$, the treelike condition on the relation set of every labelled sequent is removed, and

(ii) $G3GL$ does not contain the inference rule $\mathbb{TL}4$, and

(iii) $G3GL$ contains the initial sequent (*Irrefl*) and inference rule (*Trans*):

$$\mathcal{R}, Rxx, \Gamma \Rightarrow \Delta \ (\textit{Irref}) \qquad \frac{\mathcal{R}, Rxz, Rxy, Ryz, \Gamma \Rightarrow \Delta}{\mathcal{R}, Rxy, Ryz, \Gamma \Rightarrow \Delta}\, (\textit{Trans})$$

For those rules in $G3GL$ that also occur in $\mathbb{TL}CSGL$, we will use the rule labelling of $\mathbb{TL}CSGL$. For example, we write $\mathbb{TL}\Box K$ instead of the label $R\Box{-}L$ used in [14]. Strictly speaking, the calculus $G3GL$ also contains rules for the disjunction and implication connectives. Since these connectives can be written in terms of negation and conjunction, for our purposes there is no harm in this omission. The rules LW and RW as well as the contraction rules LC and RC are height-preserving admissible in $G3GL$ [14].

Theorem 4.2 (Negri) *The labelled sequent calculus $G3GL$ (i) has syntactic cut-admissibility, and (ii) is sound and complete for the logic GL.*

Proof. See Negri [14]. ⊣

4.3 Results

Let $G3GL + \mathbb{TL}4$ be the calculus obtained by the addition of the rule $\mathbb{TL}4$ to $G3GL$, where the rule $\mathbb{TL}4$ will no longer be subject to the restriction that its premise and conclusion are LTS. Suppose that ρ is the following instance of (*Trans*) in a derivation in $G3GL + \mathbb{TL}4$:

$$\frac{\mathcal{R}, Rxy, Ryz, Rxz, \Gamma \Rightarrow \Delta}{\mathcal{R}, Rxy, Ryz, \Gamma \Rightarrow \Delta}\ (\mathit{Trans})$$

Define the *width* of ρ to be the number of rule occurrences above ρ that make the term Rxz principal. Observe that Rxz can be principal in this way only due to a rule from $\{\mathbb{TL}\Box A, \mathbb{TL}4, (\mathit{Trans}), (\mathit{Irref})\}$.

Lemma 4.3 *Let δ be a derivation in $G3GL{+}\mathbb{TL}4$ not containing (Irref). Then the (Trans) rule is eliminable from δ.*

Proof. The non-trivial case is when there is a positive number $s+1$ of occurrences of (Trans) in δ. Let ρ be an arbitrary topmost occurrence of (Trans) in δ:

$$\begin{array}{c}\text{no } (\mathit{Trans}) \text{ rules}\\ \dfrac{Rxy, Ryz, Rxz, \mathcal{R}, \Gamma \Rightarrow \Delta}{Rxy, Ryz, \mathcal{R}, \Gamma \Rightarrow \Delta}\ \rho \\ \vdots \\ \mathcal{R}_0, X \Rightarrow Y\end{array}$$

Let γ denote the subderivation of δ deriving $Rxy, Ryz, \mathcal{R}, \Gamma \Rightarrow \Delta$. We claim that ρ is eliminable. Proof by induction on the width n of ρ.

If $n = 0$ then there is no rule above ρ that makes Rxz principal. It is clear that we can transform γ by deleting the Rxz term from every sequent above ρ to obtain directly a derivation γ' of $Rxy, Ryz, \mathcal{R}, \Gamma \Rightarrow \Delta$. Replacing the subderivation γ in δ with γ' we have eliminated ρ and thus the new derivation contains only s occurrences of (Trans).

Now suppose that $n = k+1$. Since δ does not contain (Irref) and because ρ is a topmost occurrence of (Trans), the Rxz term must be principal due to either a $\mathbb{TL}\Box A$ rule or a $\mathbb{TL}4$ rule.

Case I (Rxz principal by $\mathbb{TL}\Box A$). Then δ has the following form, where we have written the two premises of the $\mathbb{TL}\Box A$ rule one above the other:

$$\begin{array}{c}\dfrac{\begin{array}{c}Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, \Gamma' \Rightarrow \Delta', z : \Box B\\ Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, z : B, \Gamma' \Rightarrow \Delta'\end{array}}{Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, \Gamma' \Rightarrow \Delta'}\ \mathbb{TL}\Box A\\ \vdots \\ \dfrac{Rxy, Ryz, Rxz, \mathcal{R}, \Gamma \Rightarrow \Delta}{Rxy, Ryz, \mathcal{R}, \Gamma \Rightarrow \Delta}\ \rho \\ \vdots \\ \mathcal{R}_0, X \Rightarrow Y\end{array}$$

Apply the admissible rule LW with $y : \Box B$ to each of the premises of $\mathbb{TL}\Box A$. Then apply $\mathbb{TL}\Box A$ to these sequents and proceed as follows (once again we write the two premises of the $\mathbb{TL}\Box A$ rule one above the other):

$$\dfrac{\dfrac{\dfrac{Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, y : \Box B, \Gamma' \Rightarrow \Delta', z : \Box B \qquad Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, z : B, y : \Box B, \Gamma' \Rightarrow \Delta'}{Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, y : \Box B, \Gamma' \Rightarrow \Delta'}\ \mathbb{TL}\Box A}{Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, \Gamma' \Rightarrow \Delta'}\ \mathbb{TL}4}{}$$

$$\vdots$$

$$\dfrac{Rxy, Ryz, Rxz, \mathcal{R}, \Gamma \Rightarrow \Delta}{Rxy, Ryz, \mathcal{R}, \Gamma \Rightarrow \Delta}\ \rho$$

$$\vdots$$

$$\mathcal{R}_0, X \Rightarrow Y$$

Notice that in the $\mathbb{TL}\Box A$ and $\mathbb{TL}4$ rules in the above proof diagram, it is the Ryz and Rxy term respectively that is principal (and not the Rxz term). As a result the width of ρ is reduced to k. Eliminate ρ using the induction hypothesis.

Case II (Rxz principal by $\mathbb{TL}4$). Then δ has the following form:

$$\dfrac{Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, z : \Box B, \Gamma' \Rightarrow \Delta'}{Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, \Gamma' \Rightarrow \Delta'}\ \mathbb{TL}4$$

$$\vdots$$

$$\dfrac{Rxy, Ryz, Rxz, \mathcal{R}, \Gamma \Rightarrow \Delta}{Rxy, Ryz, \mathcal{R}, \Gamma \Rightarrow \Delta}\ \rho$$

$$\vdots$$

$$\mathcal{R}_0, X \Rightarrow Y$$

Apply the admissible rule LW with $y : \Box B$ to the premise of $\mathbb{TL}4$. Then apply $\mathbb{TL}4$ to this sequent and proceed as follows:

$$\dfrac{\dfrac{\ldots}{Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, y : \Box B, \Gamma' \Rightarrow \Delta'}\ \mathbb{TL}4}{Rxy, Ryz, Rxz, \mathcal{R}', x : \Box B, \Gamma' \Rightarrow \Delta'}\ \mathbb{TL}4$$

$$\vdots$$

$$\dfrac{Rxy, Ryz, Rxz, \mathcal{R}, \Gamma \Rightarrow \Delta}{Rxy, Ryz, \mathcal{R}, \Gamma \Rightarrow \Delta}\ \rho$$

$$\vdots$$

$$\mathcal{R}_0, X \Rightarrow Y$$

Notice that in the two $\mathbb{TL}4$ rules in the above proof diagram, it is the Ryz and Rxy term respectively that is principal (and not the Rxz term). As a result the width of ρ is reduced to k. Eliminate ρ using the induction hypothesis.

We have shown how to reduce the number of occurrences of (*Trans*) from $s + 1$ to s. As ρ was an arbitrary topmost occurrence, the result follows from an induction argument. $\dashv$

The following result connects Negri's labelled sequent calculus $G3GL$ and the LTS calculus $\mathbb{TL}CSGL$. Together with Corollary 3.10, this completely answers the question posed in Poggiolesi [17].

Theorem 4.4 *For any formula A, $\vdash_{\mathbb{TL}CSGL} \Rightarrow x : A$ iff $\vdash_{G3GL} \Rightarrow x : A$. Moreover the translation between the corresponding derivations is effective.*

Proof. For the left-to-right direction it suffices to show that $\mathbb{TL}4$ is syntactically admissible in $G3GL$. First, working in $G3GL$ (because we are working in a labelled sequent calculus, the relation sets that occur in the derivation need not be treelike), observe that:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z : \Box A \Rightarrow z : \Box A}{Rxz, x : \Box A, z : \Box A \Rightarrow z : A, z : \Box A} \qquad \dfrac{z : A \Rightarrow z : A}{Rxz, z : A, x : \Box A, z : \Box A \Rightarrow z : A}}{Rxz, x : \Box A, z : \Box A \Rightarrow z : A}\,\mathbb{TL}\Box A}{Rxy, Ryz, Rxz, x : \Box A, z : \Box A \Rightarrow z : A}\,LW}{Rxy, Ryz, x : \Box A, z : \Box A \Rightarrow z : A}\,(\mathit{Trans})}{Rxy, x : \Box A \Rightarrow y : \Box A}\,\mathbb{TL}\Box K$$

Suppose that we are given a derivation of the premise $\mathcal{R}, Rxy, y : \Box A, x : \Box A, X \Rightarrow Y$ of $\mathbb{TL}4$. From the cut-rule and the above derivation we get a derivation of $\mathcal{R}, Rxy, x : \Box A, x : \Box A, X \Rightarrow Y$. By Theorem 4.2 we can obtain a cut-free derivation of this sequent. Since the left contraction rule LC is admissible in $G3GL$ [14], we get $\mathcal{R}, Rxy, x : \Box A, X \Rightarrow Y$ and thus $\mathbb{TL}4$ is syntactically admissible in $G3GL$.

Now for the right-to-left direction. First observe that the derivation of $\Rightarrow x : A$ does not contain any occurrences of the initial sequent (*Irref*). To see this, observe that in any $G3GL$ derivation, viewed downwards, a state variable occurrence y can disappear from premise sequent to conclusion sequent only via the $\mathbb{TL}\Box K$ rule — all the other rules preserve the set of state variables in the relation set. Moreover, for this to occur, the variable y must occur exactly *once* in the relation set of the premise of $\mathbb{TL}\Box K$ (in a term of the form Rxy for some variable x distinct from y). Now, if the given derivation contains the initial sequent (*Irref*) $\mathcal{R}, Ryy, X \Rightarrow Y$, then the relation set of the initial sequent contains at least two occurrences of y. It follows that the relation set of every sequent below this initial sequent in δ will contain these two occurrences of y, contradicting the fact that the endsequent has the form $\Rightarrow x : A$.

Suppose that we are given a derivation δ of the $LTS \Rightarrow x : A$ in $G3GL$. We need to obtain a derivation of $\Rightarrow x : A$ in $\mathbb{TL}CSGL$. By Lemma 4.3, there is a derivation δ' of $\Rightarrow x : A$ in $G3GL + \mathbb{TL}4$ containing no occurrences of (*Trans*). To complete the proof, we will show that δ' is a derivation in $\mathbb{TL}CSGL$. It suffices to show that every labelled sequent in δ' is a LTS. By inspection, every rule in $G3GL + \mathbb{TL}4$ with the exception of the (*Trans*) rule has the property that if the conclusion is a LTS, then so are the premise(s). Since $\Rightarrow x : A$ is a LTS by assumption, every sequent in δ' must be a LTS so we are done. ⊣

A comment regarding the initial sequent $\mathcal{R}, Rxx, \Gamma \Rightarrow \Delta$ (*Irref*). Negri uses this initial sequent in the proof of cut-admissibility for $G3GL$ to argue that there cannot be a labelled sequent with a relation set (in our terminology) containing $\{Rxx_1, Rx_1x_2, \ldots Rx_nx\}$ (a 'loop'). We saw above that (*Irref*) cannot occur in any $G3GL$ derivation of a sequent of the form $\Rightarrow x : A$. By definition, the relation set of a LTS can never contain such a loop so there is no initial LTS in $\mathbb{TL}CSGL$ corresponding to (*Irref*) in $G3GL$.

Theorem 4.5 *The calculus $CSGL$ (i) is sound and complete for the logic GL, and (ii) has syntactic cut-admissibility.*

Proof. Follows from Theorem 4.2 using Corollary 3.10 and Theorem 4.4. $\dashv$

Note that although the above proofs make use of the results for $G3GL$ [14], these results are syntactic because the proofs for $G3GL$ are syntactic.

5 Conclusion

We have shown that THS and LTS are notational variants, allowing us to transfer proof-theoretic results including syntactic cut-admissibility between these formalisms, thus alleviating the need for independent proofs in each system. We have answered in full Poggiolesi's question regarding the relationship between the THS calculus $CSGL$ and the labelled sequent calculus $G3GL$.

It is straightforward to construct mappings between THS and the nested (deep) sequents that Fitting [5] refers to in his work (adapting to Brünnler's [2] formulation of nested sequents is analogous). Define a nested sequent to be of the form $\Gamma, [\mathcal{N}_1], \ldots, [\mathcal{N}_k]$ where Γ is a formula multiset and $\mathcal{N}_1, \ldots, \mathcal{N}_k$ $(k \geq 0)$ are nested sequents. Nested sequent calculi consist of initial sequents and inference rules built from nested sequents. Following the formulations used by Brünnler and Fitting, nested sequent inference rules are permitted to operate at any level of nesting. That is, the auxiliary formulae — the 'active' formulae in the premise — of a rule instance are permitted to occur inside the scope of [ ]. For example, here is a rule instance of Fitting's $\wedge$-introduction rule:

$$\frac{p, [q, [r]] \qquad p, [q, [s]]}{p, [q, [r \wedge s]]}$$

However, a nested sequent inference rule is not permitted to operate *inside* a formula — ie. a proper subformula cannot be the auxiliary formula of a rule instance. For example, the following is *forbidden* as the auxiliary formulae are the proper subformulae r (of $q \vee r$) and s (of $q \vee s$):

$$\frac{p, [q \vee r] \qquad p, [q \vee s]}{p, [q \vee (r \wedge s)]}$$

Informally speaking, this means that the deep inference applies to the nesting but not to subformulae. This restriction leads to straightforward maps between THS and nested sequents as shown below.

Given a traditional sequent $X \Rightarrow Y$, we will assume that we have at hand a suitable concrete representation $X \Rightarrow_{1S} Y$ of $X \Rightarrow Y$ as a formula multiset — think of $X \Rightarrow_{1S} Y$ as a one-sided sequent. Also asssume that given a formula multiset Γ, we have at hand a suitable concrete representation Γ^s of Γ as a traditional sequent. Then the maps $\mathbb{TN}$ and $\mathbb{NT}$ respectively map THS to nested sequents and *vice versa*. In the following: $X \Rightarrow Y$ is a traditional sequent; $G_1, \ldots, G_k$ are THS; Γ is a formula multiset; and $\mathcal{N}_1, \ldots, \mathcal{N}_k$ are nested sequents.

$$\mathbb{TN}(X \Rightarrow Y/(G_1; \ldots; G_k)) = (X \Rightarrow_{1S} Y, [\mathbb{TN}G_1], \ldots, [\mathbb{TN}G_k])$$
$$\mathbb{NT}(\Gamma, [\mathcal{N}_1], \ldots, [\mathcal{N}_k]) = (\Gamma^s/(\mathbb{NT}\mathcal{N}_1; \ldots; \mathbb{NT}\mathcal{N}_k))$$

In this way we can show that THS and nested sequents are notational variants. By computing the calculi induced by these mappings our results can be extended to these systems. Since THS and LTS are notational variants, this provides an answer to the question posed by Fitting concerning how labelled systems and nested (deep) sequent systems relate.

Negri [14,15] has identified a large class of modal and intermediate logics that can be presented using cut-free labelled sequent calculi. We would like to identify the subclass of such labelled sequent calculi that can be coerced into the LTS framework. In this way we could directly obtain proofs of syntactic cut-admissibility for LTS and THS calculi for suitable logics from the existing proofs for labelled sequent calculi. Theorem 4.4 in this paper is an example of such a result. Hein [8] has conjectured that such a coercion is possible for modal logics axiomatised by 3/4 Lemmon-Scott formulae $\{\Diamond^h \Box^i p \supset \Box^j p | h, i, j \geq 0\}$. This investigation is the subject of future work.

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