An introduction to the display calculus

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A (very) brief history of proof theory

- The so-called *foundational crisis of mathematics* (early 1900s) arose from various challenges to the implicit assumption that consistency of the 'foundations of mathematics' could be shown within mathematics
- For example, the discovery of Russell's paradox (1902) showed that naive set theory was an inadequate foundation.
- The need for a precise development of the underlying logical systems became apparent.
- The intention is to present a narrative placing the display calculus in a broader context. Introduction not intended to be at all comprehensive!

Hilbert's Program (around 1922)

- Proofs are the essence of mathematics — to establish a theorem.. present a proof!
- Historically, proofs were not the objects of mathematical investigations (unlike numbers, triangles...)
- In Hilbert's *Proof theory*: proofs are mathematical objects.

Hilbert's Program:

1. Formalise the whole of mathematical reasoning in a formal theory T
2. Prove the consistency of T by 'finitistic' means

T is *consistent* if there is no formula A such that $A \wedge \neg A$ is derivable in T.

Hilbert calculus

- Mathematical investigation of proofs $\rightsquigarrow$ formally definition of proof
- Hilbert calculus fulfils this role.

A Hilbert calculus for propositional classical logic. Axiom schemata:

$$\begin{array}{ll} \text{Ax 1:} & A \to (B \to A) \\ \text{Ax 2:} & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\ \text{Ax 3:} & (\neg A \to \neg B) \to ((\neg A \to B) \to A) \end{array}$$

and the rule of *modus ponens*:

$$\frac{A \qquad A \to B}{B}$$

Read $A \leftrightarrow B$ as $(A \to B) \wedge (B \to A)$. More axioms:

$$\text{Ax 4:} \quad A \vee B \leftrightarrow (\neg A \to B) \qquad\qquad \text{Ax 5:} \quad A \wedge B \leftrightarrow \neg(A \to \neg B)$$

Derivation of $A \to A$

Definition

A *formal proof* (derivation) of B is the finite sequence $C_1, C_2, \ldots, C_n \equiv B$ of formulae where each element C_j is an axiom instance or follows from two earlier elements by *modus ponens*.

1	$((A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A)))$	Ax 2
2	$(A \to ((A \to A) \to A))$	Ax 1
3	$((A \to (A \to A)) \to (A \to A))$	MP: 1 and 2
4	$(A \to (A \to A))$	Ax 1
5	$A \to A$	MP: 3 and 4

Not easy to find! Proof has no clear structure (wrt $A \to A$)

Gödel's second incompleteness theorem (1931)

Theorem

Let T be a consistent theory containing arithmetic. Then there is no proof of consistency of T in T (ie. $T \nvdash Con(T)$).

Destroys Hilbert's program (assuming finitistic reasoning can be formalised in arithmetic, as was believed)

If it is the case that

(i) consistency of mathematics can be shown by finitistic reasoning, and

(ii) arithmetic can formalise finitistic reasoning

Then arithmetic can show the consistency of mathematics (and hence arithmetic). Contradiction. So if we believe that (ii) holds, then (i) cannot hold.

Natural deduction and the sequent calculus

- Gentzen: proving consistency of arithmetic in weak extensions of finitistic reasoning.
- Hilbert calculus not convenient for studying the proofs (lack of structure). Gentzen introduces *Natural deduction* which formalises the way mathematicians reason.
- Gentzen introduced a proof-formalism with even more structure: the sequent calculus.
- Sequent calculus built from sequents $X \vdash Y$ where X, Y are lists/sets/multisets of formulae

Sequent calculus

sequent: $$\overbrace{A_1, A_2, \ldots, A_m}^{\text{antecedent}} \underbrace{\;\vdash\;}_{\text{turnstile}} \overbrace{B_1, B_2, \ldots, B_n}^{\text{succedent}}$$

sequent calculus rule:
($\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_k$
are sequents)

$$\frac{\overbrace{\mathcal{S}_1 \qquad \ldots \qquad \mathcal{S}_k}^{k \geq 0 \text{ premises}}}{\underbrace{\mathcal{S}}_{\text{conclusion}}}$$

- Typically a rule for introducing each connective in the antecedent and succedent.
- A 0-premise rule is called an *initial sequent*

Definition (derivation)

A *derivation* in the sequent calculus is an initial sequent or a rule applied to derivations of the premise(s).

The sequent calculus $\mathcal{S}Cp$ for classical logic Cp

$$\frac{}{p, X \vdash Y, p}\ \text{init} \qquad \frac{}{\bot, X \vdash Y}\ \bot\text{l}$$

$$\frac{X \vdash Y, A}{\neg A, X \vdash Y}\ \neg l \qquad \frac{A, X \vdash Y}{X \vdash Y, \neg A}\ \neg r$$

$$\frac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}\ \wedge l \qquad \frac{X \vdash Y, A \qquad X \vdash Y, B}{X \vdash Y, A \wedge B}\ \wedge r$$

$$\frac{A, X \vdash Y \qquad B, X \vdash Y}{A \vee B, X \vdash Y}\ \vee l \qquad \frac{X \vdash Y, A, B}{X \vdash Y, A \vee B}\ \vee r$$

$$\frac{X \vdash Y, A \qquad B, X \vdash Y}{A \to B, X \vdash Y}\ \to l \qquad \frac{A, X \vdash Y, B}{X \vdash Y, A \to B}\ \to\text{r}$$

- Here X, Y are sets of formulae (possibly empty)
- Aside: differs from the calculus Gentzen used (not important)
- Gödel's incompleteness theorem does not apply since this logic does not contain arithmetic

Soundness and completeness of $\mathcal{S}Cp$ for Cp

Need to prove that $\mathcal{S}Cp$ is actually a sequent calculus for Cp.

Theorem

For every formula A we have: $\vdash A$ is derivable in $\mathcal{S}Cp \Leftrightarrow A \in Cp$.

$(\Rightarrow)$ direction is soundness.

$(\Leftarrow)$ direction is completeness.

Proof of completeness

Need to show: $A \in Cp \Rightarrow \vdash A$ derivable in $\mathcal{S}Cp$.

First show that $A, X \vdash Y, A$ is derivable (exercise).

Show that every axiom of Cp is derivable (easy, below) and *modus ponens* can be simulated in $\mathcal{S}Cp$ (not easy)

$$
\dfrac{\dfrac{\dfrac{\dfrac{A, A \to (B \to C) \vdash C, A \qquad \dfrac{B, A \vdash C, A \qquad \dfrac{B, A \vdash C, B \qquad r, B, A \vdash C}{B \to C, B, A \vdash C}}{B, A, A \to (B \to C) \vdash C}}{A, A \to B, (A \to (B \to C)) \vdash C}}{A \to B, (A \to (B \to C)) \vdash (A \to C)}}{(A \to (B \to C)) \vdash (A \to B) \to (A \to C)}}{\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))}
$$

How to simulate *modus ponens*

Gentzen's solution: to simulate *modus ponens* (below left) first add a new rule (below right) to $\mathcal{S}Cp$:

$$\frac{A \qquad A \to B}{B} \qquad\qquad \frac{X \vdash Y, A \qquad A, X \vdash Y}{X \vdash Y}\,cut$$

The following instance of the cut-rule illustrates the simulation of *modus ponens*.

$$\dfrac{\vdash A \qquad \dfrac{\vdash A \to B \qquad \dfrac{A \vdash A \qquad B \vdash B}{A \to B, A \vdash B}}{A \vdash B}}{\vdash B}\,cut$$

So: $A \in Cp \Rightarrow \vdash A$ derivable in $\mathcal{S}Cp + cut$!

Proof of soundness

Need to show: $\vdash A$ derivable in $\mathcal{S}Cp + cut \Rightarrow A \in \mathcal{S}Cp$.

We need to interpret $\mathcal{S}Cp + cut$ derivations in Cp.

For sequent S	$A_1, A_2, \ldots, A_m \vdash B_1, B_2, \ldots, B_n$
define translation $\tau(S)$	$A_1 \wedge A_2 \wedge \ldots \wedge A_m \rightarrow B_1 \vee B_2 \vee \ldots \vee B_n$

Comma on the left is conjunction, comma on the right is disjunction.

Translations of the intial sequents are theorems of Cp

$$p \wedge X \rightarrow Y \vee p \qquad\qquad \bot \wedge X \rightarrow Y$$

Show for each remaining rule ρ: if the translation of every premise is a theorem of Cp then so is the translation of the conclusion.

$$\text{For} \quad \frac{A, X \vdash B}{X \vdash A \rightarrow B} \qquad \text{need to show:} \quad \frac{A \wedge X \rightarrow B}{X \rightarrow (A \rightarrow B)}$$

The cut-rule is undesirable in $\mathcal{S}Cp + cut$

We have shown

Theorem

For every formula A we have: $\vdash A$ *is derivable in* $\mathcal{S}Cp + cut \Leftrightarrow A \in Cp$.

- The *subformula property* states that every formua in a premise appears as a subformula of the conclusion.
- If all the rules of the calculus satisfy this property, the calculus is *analytic*
- Analyticity is crucial to using the calculus (for consistency, decidability. . .)
- $\mathcal{S}Cp + cut$ is *not* analytic because:

$$\frac{X \vdash Y, A \qquad A, X \vdash Y}{X \vdash Y}\; cut$$

- We want to show: $\vdash A$ is derivable in $\mathcal{S}Cp \Leftrightarrow A \in Cp$

Gentzen's *Hauptsatz* (main theorem): cut-elimination

Theorem

Suppose that δ is a derivation of $X \vdash Y$ in $\mathcal{S}Cp + cut$. Then there is a transformation to eliminate instances of the cut-rule from δ to obtain a derivation δ' of $X \vdash Y$ in $\mathcal{S}Cp$.

Since $\vdash A$ is derivable in $\mathcal{S}Cp + cut \Leftrightarrow A \in Cp$:

Theorem

For every formula A we have: $\vdash A$ is derivable in $\mathcal{S}Cp$ if and only if $A \in Cp$.

Applications: Consistency of classical logic

Consistency of classical logic is the statement that $A \land \neg A \notin Cp$.

Theorem

Classical logic is consistent.

Proof by contradiction. Suppose that $A \land \neg A \in Cp$. Then $A \land \neg A$ is derivable in $\mathcal{S}Cp$ (completeness). Let us try to derive it (read upwards from $\vdash A \land \neg A$):

$$\dfrac{\vdash A \qquad \dfrac{A \vdash}{\vdash \neg A}}{\vdash A \land \neg A}$$

So $\vdash A$ and $A \vdash$ are derivable. Thus $\vdash$ must be derivable in $\mathcal{S}Cp + \mathit{cut}$ (use cut) and hence in $\mathcal{S}Cp$ (by cut-elimination). This is impossible (why?) QED.

Theorem

Decidability of Cp.

Given a formula A, do backward proof search in $\mathcal{S}Cp$ on $\vdash A$. Since termination is guaranteed, we can decide if A is a theorem or not. QED.

Looking beyond the sequent calculus

- Aside from proofs of consistency, proof-theoretic methods enable us to extract information from the proofs and about the logic (a fact already recognised by Gentzen).
- Many more logics of interest than just first-order classical and intuitionistic logic
- How to give a proof-theory to these logics? Want analytic calculi with *modularity*
- In a modular calculus we can add rules corresponding to (suitable) axiomatic extensions and preserve analyticity.

Some nonclassical logics

- *Modal logics* extend classical language with modalities $\Box$ and $\Diamond$. The modalities were traditionally used to qualify statements like "it is *possible* that it will rain today". *Tense logics* include the temporal modalities $\blacklozenge$ and $\blacksquare$. Closed under *modus ponens* and *necessitation* rule ($A/\Box A$).
- An *intermediate logic* L is a set of formulae closed under *modus ponens* such that intuitionistic logic $Ip \subseteq L \subseteq Cp$.
- Starting with the sequent calculus $\mathcal{S}Cp$, if we consider a sequent $X \vdash Y$ to be built from lists X, Y (rather than sets or multisets) then $A, A, X \vdash Y$ and $A, X \vdash Y$ are no longer identical (no contraction). Also $A, B, X \vdash Y$ and $B, A, X \vdash Y$ are not identical (no exchange). The logics obtained by removing these properties are called *substructural logics*.

Sequent calculus inadequate for treating these logics (eg. no analytic sequent calculus for S5 despite analytic sequent calculus for S4)

Display Calculus

- Introduced as *Display Logic* (Belnap, 1982).
- Extends sequent calculus by introducing new structural connectives that interpret the logical connectives (enrich language)
- A *structure* is built from structural connectives and formulae.
- A display sequent: $X \vdash Y$ for structures X and Y
- Display property. A substructure in $X[U] \vdash Y$ *equi-derivable* (displayable) as $U \vdash W$ or $W \vdash U$ for some W.
- Key result. Belnap's general cut-elimination theorem applies when the rules of the calculus satisfy C1–C8 (*display conditions*)
- Display calculi have been presented for substructural logics, modal and poly-modal logics, tense logic, bunched logics, bi-intuitionistic logic...

Display calculi generalise the sequent calculus

Here is the sequent calculus $\mathcal{S}Cp$ once more:

$$\frac{}{p, X \vdash Y, p}\ \text{init} \qquad \frac{}{\bot, X \vdash Y}\ \bot\text{l}$$

$$\frac{X \vdash Y, A}{\neg A, X \vdash Y}\ \neg l \qquad \frac{A, X \vdash Y}{X \vdash Y, \neg A}\ \neg r$$

$$\frac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}\ \wedge l \qquad \frac{X \vdash Y, A \qquad X \vdash Y, B}{X \vdash Y, A \wedge B}\ \wedge r$$

$$\frac{A, X \vdash Y \qquad B, X \vdash Y}{A \vee B, X \vdash Y}\ \vee l \qquad \frac{X \vdash Y, A, B}{X \vdash Y, A \vee B}\ \vee r$$

$$\frac{X \vdash Y, A \qquad B, X \vdash Y}{A \rightarrow B, X \vdash Y}\ \rightarrow l \qquad \frac{A, X \vdash Y, B}{X \vdash Y, A \rightarrow B}\ \rightarrow\text{r}$$

Let's add a new structural connective $*$ for negation.

$$\frac{}{p, X \vdash Y, p}\ \text{init} \qquad \frac{}{\bot, X \vdash Y}\ \bot l$$

$$\frac{*A, X \vdash Y}{\neg A, X \vdash Y}\ \neg l \qquad \frac{X \vdash Y, *A}{X \vdash Y, \neg A}\ \neg r$$

$$\frac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}\ \wedge l \qquad \frac{X \vdash Y, A \qquad X \vdash Y, B}{X \vdash Y, A \wedge B}\ \wedge r$$

$$\frac{A, X \vdash Y \qquad B, X \vdash Y}{A \vee B, X \vdash Y}\ \vee l \qquad \frac{X \vdash Y, A, B}{X \vdash Y, A \vee B}\ \vee r$$

$$\frac{X \vdash Y, A \qquad B, X \vdash Y}{A \to B, X \vdash Y}\ \to l \qquad \frac{A, X \vdash Y, B}{X \vdash Y, A \to B}\ \to \text{r}$$

Add the display rules

The addition of the following rules permit the display property:

Definition (display property)

The calculus has the display property if for any sequent $X \vdash Y$ containing a substructure U, there is a sequent $U \vdash W$ or $W \vdash U$ for some W such that

$$\frac{X \vdash Y}{U \vdash W} \quad \text{or} \quad \frac{X \vdash Y}{W \vdash U}$$

We say that U is *displayed* in the lower sequent.

$$\frac{X, Y \vdash Z}{X \vdash Z, *Y} \qquad \frac{X, Y \vdash Z}{Y \vdash *X, Z} \qquad \frac{X \vdash Y, Z}{X, *Z \vdash Y}$$

$$\frac{X \vdash Y, Z}{*Y, X \vdash Z} \qquad \frac{*X \vdash Y}{*Y \vdash X} \qquad \frac{X \vdash *Y}{Y \vdash *X}$$

$$\frac{**X \vdash Y}{X \vdash Y} \qquad \frac{X \vdash **Y}{X \vdash Y} \qquad \frac{X \vdash \bullet Y}{\bullet X \vdash Y}$$

Using the display rules

Examples:

$$\dfrac{\dfrac{\dfrac{*(A, *B) \vdash *(C, D)}{* * (C, D) \vdash A, *B}}{*A, * * (C, D) \vdash *B}}{B \vdash *(*A, * * (C, D))}$$

B is displayed

$$\dfrac{\dfrac{*(A, *B) \vdash *(C, D)}{C, D \vdash * * (A, *B)}}{D \vdash *C, * * (A, *B)}$$

D is displayed

Exercise. Prove that the display property holds for this calculus.
Also see Kracht, 1996.

Specify the properties of the structural connectives

We want weakening, contraction, exchange, associativity.

Here **I** is a structural constant for the empty list.

$$\begin{array}{c} X \vdash Z \\ \hline\hline \mathbf{I}, X \vdash Z \end{array} \qquad \begin{array}{c} X \vdash Z \\ \hline\hline X \vdash \mathbf{I}, Z \end{array} \qquad \begin{array}{c} \mathbf{I} \vdash Y \\ \hline\hline *\mathbf{I} \vdash Y \end{array}$$

$$\begin{array}{c} X \vdash \mathbf{I} \\ \hline\hline X \vdash *\mathbf{I} \end{array} \qquad \begin{array}{c} X \vdash Z \\ \hline Y, X \vdash Z \end{array} \qquad \begin{array}{c} X \vdash Z \\ \hline X, Y \vdash Z \end{array}$$

$$\begin{array}{c} X, Y \vdash Z \\ \hline Y, X \vdash Z \end{array} \qquad \begin{array}{c} Z \vdash X, Y \\ \hline Z \vdash Y, X \end{array} \qquad \begin{array}{c} X, X \vdash Z \\ \hline X \vdash Z \end{array}$$

$$\begin{array}{c} Z \vdash X, X \\ \hline Z \vdash X \end{array} \qquad \begin{array}{c} X_1, (X_2, X_3) \vdash Z \\ \hline\hline (X_1, X_2), X_3 \vdash Z \end{array} \qquad \begin{array}{c} Z \vdash X_1, (X_2, X_3) \\ \hline\hline Z \vdash (X_1, X_2), X_3 \end{array}$$

Display calculi generalise the sequent calculus

The presence of the display rules permit the following rewriting of the rules:

$$\frac{}{p \vdash p}\ \text{init} \qquad \frac{}{\bot \vdash \mathbf{I}}\ \bot l$$

$$\frac{*A \vdash Y}{\neg A \vdash Y}\ \neg l \qquad \frac{X \vdash *A}{X \vdash \neg A}\ \neg r$$

$$\frac{A, B \vdash Y}{A \wedge B \vdash Y}\ \wedge l \qquad \frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}\ \wedge r$$

$$\frac{A \vdash Y \qquad B \vdash Y}{A \vee B \vdash Y}\ \vee l \qquad \frac{X \vdash A, B}{X \vdash A \vee B}\ \vee r$$

$$\frac{X \vdash A \qquad B \vdash Y}{A \rightarrow B \vdash *X, Y}\ \rightarrow l \qquad \frac{A, X \vdash B}{X \vdash A \rightarrow B}\ \rightarrow r$$

The formulae are called principal formulae. The X, Y are *context* variables.

Sequent calculus to display calculus

From a *procedural* point of view, we obtained the display calculus δCp for Cp from the sequent calculus by

1. Addition of a structural connective $*$ for negation
2. Addition of the display rules to yield the display property
3. Additional structural rules for exchange, weakening, contraction etc.
4. Rewriting the logical rules so the principal formulae in the conclusion are all of the antecedent or succedent

We will consider a *theoretical* viewpoint shortly.

The display calculus δKt for tense logic Kt

- Tense logics extend the classical language with the modal operators $\Diamond, \Box$ and the tense operators $\blacklozenge, \blacksquare$.
- $\Box$ and $\Diamond$ are duals. Similarly $\blacksquare$ and $\blacklozenge$. Ie. axioms:

$$\Box A \leftrightarrow \neg\Diamond\neg A \qquad\qquad \blacksquare A \leftrightarrow \neg\blacklozenge\neg A$$

- The following 'residuation' property holds in the basic tense logic Kt.

$$\blacklozenge A \to B \text{ if and only if } A \to \Box B$$

- Identifying residuation is crucial for constructing a display calculus (more later).

Display rules from residuation

- We saw that: $\blacklozenge A \to B \Leftrightarrow A \to \Box B$
- Introduce a new structural connective $\bullet$ for $(\blacklozenge, \Box)$ (ie. $\blacklozenge$ in the antecedent, $\Box$ in the succedent).
- Add the display rules $\dfrac{\bullet X \vdash Y}{X \vdash \bullet Y}$ (double line)
- and additional structural rules (for necessitation)

$$\frac{\mathbf{I} \vdash Y}{\bullet \mathbf{I} \vdash Y}\ (Ml) \qquad\qquad \frac{X \vdash \mathbf{I}}{X \vdash \bullet \mathbf{I}}\ (Mr)$$

A display calculus δKt for tense logic Kt

$$p \vdash p \qquad \bot \vdash I$$

$$\frac{*A \vdash X}{\neg A \vdash X}\neg l \qquad \frac{X \vdash *A}{X \vdash \neg A}\neg r$$

$$\frac{A \circ B \vdash X}{A \wedge B \vdash X}\wedge l \qquad \frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}\wedge r$$

$$\frac{A \vdash X \qquad B \vdash X}{A \vee B \vdash X}\vee l \qquad \frac{X \vdash A \circ B}{X \vdash A \vee B}\vee r$$

$$\frac{X \vdash A \qquad B \vdash Y}{A \to B \vdash *X, Y}\to l \qquad \frac{A, X \vdash B}{X \vdash A \to B}\to \mathrm{r}$$

$$\frac{A \vdash X}{\Box A \vdash \bullet X}\Box l \qquad \frac{X \vdash \bullet A}{X \vdash \Box A}\Box r$$

$$\frac{* \bullet *A \vdash X}{\Diamond A \vdash X}\Diamond l \qquad \frac{X \vdash A}{* \bullet *X \vdash \Diamond A}\Diamond r$$

$$\frac{\bullet A \vdash X}{\blacklozenge A \vdash X}\blacklozenge l \qquad \frac{X \vdash A}{\bullet X \vdash \blacklozenge A}\blacklozenge r$$

$$\frac{A \vdash X}{\blacksquare A \vdash * \bullet *X}\blacksquare l \qquad \frac{X \vdash * \bullet *A}{X \vdash \blacksquare A}\blacksquare r$$

Display Property

Theorem. Every substructure Z appearing in $X \vdash Y$ can be *displayed* as the whole of the antecedent ($Z \vdash U$) or the whole of the succedent ($U \vdash Z$) for suitable U.

$$\frac{X, Y \vdash Z}{\overline{X \vdash Z, *Y}} \qquad \frac{X, Y \vdash Z}{\overline{Y \vdash *X, Z}} \qquad \frac{X \vdash Y, Z}{\overline{X, *Z \vdash Y}}$$

$$\frac{X \vdash Y, Z}{\overline{*Y, X \vdash Z}} \qquad \frac{*X \vdash Y}{\overline{*Y \vdash X}} \qquad \frac{X \vdash *Y}{\overline{Y \vdash *X}}$$

$$\frac{**X \vdash Y}{\overline{X \vdash Y}} \qquad \frac{X \vdash **Y}{\overline{X \vdash Y}} \qquad \frac{X \vdash \bullet Y}{\overline{\bullet X \vdash Y}}$$

Example.

$$\cfrac{\cfrac{\cfrac{*\bullet *p \vdash q}{\overline{*q \vdash \bullet * p}}}{\overline{\bullet * q \vdash *p}}}{\overline{p \vdash * \bullet *q}}$$

p is displayed

$$\cfrac{\cfrac{\cfrac{p \vdash \bullet(q, *\bullet *r)}{\overline{\bullet p \vdash q, *\bullet *r}}}{\overline{*q, \bullet p \vdash *\bullet *r}}}{\overline{*\bullet *(*q, \bullet p) \vdash r}}$$

r is displayed

Translating sequents into formulae

We have seen that a sequent $X \vdash Y$ in δKt is built from *structures* X, Y:

$$\text{Struc} ::= \text{tense formula} \,|\, \mathbf{I} \,|\, (X, X) \,|\, \bullet X \,|\, *X$$

Define the translation functions l and r from structures into tense formulae:

$$\begin{aligned} l(A) &= A & r(A) &= A \\ l(\mathbf{I}) &= \top & r(\mathbf{I}) &= \bot \\ l(*X) &= \neg r(X) & r(*X) &= \neg l(X) \\ l(X, Y) &= l(X) \wedge l(Y) & r(X, Y) &= r(X) \vee r(Y) \\ l(\bullet X) &= \blacklozenge l(X) & r(\bullet X) &= \Box r(X) \end{aligned}$$

The sequent $X \vdash Y$ is interpreted as $l(X) \rightarrow r(Y)$.

Soundness of δKt for Kt

It suffices to prove that if the translations of the premises are theorems of Kt, then so is the translation of the conclusion (this has to be done for all rules).

Eg. consider the display rule (below left) and its formula translation (below right)

$$\frac{\bullet X \vdash Y}{\overline{X \vdash \bullet Y}} \qquad\qquad \frac{\blacklozenge l(X) \to r(Y)}{\overline{l(X) \to \Box r(Y)}}$$

We already noted that above right holds in Kt.

Aside. Helpful to prove soundness with respect to frame semantics for Kt.

Completeness of δKt for Kt

As in the case of the sequent calculus, we can prove completeness by

- deriving all the axioms, and
- simulating *modus ponens* and *necessitation* $\dfrac{A}{\Box A}$

The following suffices:

$$\frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ \text{cut} \qquad\qquad \frac{\mathbf{I} \vdash Y}{\bullet\mathbf{I} \vdash Y}\ (MI)$$

The cut-rule applies only to a formula (and not a structure!)

To obtain an analytic calculus we need to eliminate cut.

Belnap's general cut-elimination theorem

Belnap showed that any display calculus satisfying the *display conditions* has cut-elimination. The display conditions C1–C8 are syntactic conditions on the rules of the calculus.

Theorem

A display calculus that satisfies the Display Conditions C2–C8 has cut-elimination. If C1 is satisfied, then the calculus has the subformula property.

Proof 'follows' Gentzen's cut-elimination, uses display property.

Only C8 is non-trivial to verify.

Display conditions

$$\frac{*\bullet *A \vdash X}{\Diamond A \vdash X}\ (\Diamond l) \qquad \frac{X, Y \vdash Z}{X \vdash Z, *Y}\ (*r) \qquad \frac{\Box\Box A \vdash X}{\Box A \vdash X}$$

(C1) Each schematic formula variable occurring in a premise of a rule $\rho \neq$ *cut* is a sub-formula of some schematic formula variable in the conclusion of ρ.

(C2) A parameter is an occurrence of a schematic structure variable in the rule schema. Occurrences of the identical structure variable are said to be *congruent* to one another (really a definition)

(C3) Each parameter is congruent to at most one structure variable in the conclusion. Ie. no two structure variables in the conclusion are congruent to each other.

Display conditions

$$\frac{* \bullet * A \vdash X}{\Diamond A \vdash X}\ (\Diamond l) \qquad \frac{X, Y \vdash Z}{X \vdash Z, * Y}\ (* r) \qquad \frac{X \vdash Y}{X, X \vdash Y}$$

(C1) Each schematic formula variable occurring in a premise of a rule $\rho \neq cut$ is a sub-formula of some schematic formula variable in the conclusion of ρ.

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(C3) Each parameter is congruent to at most one structure variable in the conclusion. Ie. no two structure variables in the conclusion are congruent to each other.

Display conditions

$$\frac{* \bullet * A \vdash X}{\Diamond A \vdash X}\ (\Diamond l) \qquad \frac{X \circ Y \vdash Z}{X \vdash Z \circ * Y}\ (*r) \qquad \frac{X \circ Y \vdash Z}{X \vdash Z \circ Y}$$

(C4) Congruent parameters are all either a-part or s-part structures.

(C5) A schematic formula variable in the conclusion of an inference rule ρ is either the entire antecedent or the entire succedent. This formula is called a *principal formula* of ρ.

(C6/7) Each inference rule is closed under simultaneous substitution of arbitrary structures for congruent parameters.

Display conditions

$$\frac{* \bullet * A \vdash X}{\Diamond A \vdash X} (\Diamond l) \qquad \frac{X \circ Y \vdash Z}{X \vdash Z \circ * Y} (*r) \qquad \frac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}$$

(C4) Congruent parameters are all either a-part or s-part structures.

(C5) A schematic formula variable in the conclusion of an inference rule ρ is either the entire antecedent or the entire succedent. This formula is called a *principal formula* of ρ.

(C6/7) Each inference rule is closed under simultaneous substitution of arbitrary structures for congruent parameters.

Display conditions

$$\frac{*\bullet *A \vdash X}{\Diamond A \vdash X}\ (\Diamond\mathrm{l}) \qquad\qquad \frac{X \circ Y \vdash Z}{X \vdash Z \circ *Y}\ (*\mathrm{r})$$

(C8) If there are inference rules ρ and σ with respective conclusions $X \vdash A$ and $A \vdash Y$ with formula A principal in both inferences (in the sense of C5) and if *cut* is applied to yield $X \vdash Y$, then $X \vdash Y$ is identical to either $X \vdash A$ or $A \vdash Y$; or it is possible to pass from the premises of ρ and σ to $X \vdash Y$ by means of inferences falling under *cut* where the cut-formula is always a proper sub-formula of A.

$$\dfrac{\dfrac{X \vdash A}{*\bullet *X \vdash \Diamond A}\ \Diamond\mathrm{r} \qquad \dfrac{*\bullet *A \vdash Y}{\Diamond A \vdash Y}\ \Diamond\mathrm{l}}{*\bullet *X \vdash Y} \qquad\qquad \dfrac{\dfrac{X \vdash A \qquad \dfrac{*\bullet *A \vdash Y}{A \vdash *\bullet *Y}\ \mathrm{drs}}{X \vdash *\bullet *Y}}{*\bullet *X \vdash Y}\ \mathrm{drs}$$

Display calculi for modal logics

- A display calculus δK for the modal logic K can be obtained by deleting the introduction rules for the tense operators $\blacklozenge$ and $\blacksquare$.
- By cut-elimination and the conservativity of tense logic over modal logic, any derivation of a modal formula does not require the use of tense operators.
- However, using the display rules in δK may result in $\bullet$ in the antecedent.

$$\dfrac{\dfrac{p \vdash p}{\Box p \vdash \bullet p}\ \Box\mathrm{I}}{\bullet\Box p \vdash p}\ \mathrm{dr} \qquad\qquad \begin{array}{c} p \to p \\ \Box p \to \Box p \\ \blacklozenge\Box p \vdash p \end{array}$$

Recall that cut-elimination in the display calculus relies on the display property.

Kracht's structural rule extensions of *Kt*

Definition

A primitive tense axiom has the form $A \rightarrow B$ where both A and B are constructed from propositional variables and $\top$ using $\{\wedge, \vee, \Diamond, \blacklozenge\}$ and A contains each propositional variable at most once.

Some examples of primitive tense axioms

$$\Diamond\Diamond A \rightarrow \Diamond A \qquad \blacklozenge\Diamond A \rightarrow \Diamond\blacklozenge A \qquad \Diamond(A \wedge \blacklozenge B) \rightarrow \blacklozenge\Diamond(A \vee \Diamond B)$$

First two axioms are primitive tense equivalents of $\Box A \rightarrow \Box\Box A$ (transitivity) and $\Diamond\Box A \rightarrow \Box\Diamond A$ (connectedness) respectively.

Theorem (Kracht)

Let L be a tense logic. Then L is an axiomatic extension of Kt by primitive tense axioms iff there is a proper structural rule extension of δKt for L.

A *proper structural rule* is a structural rule satisfying C1–C8.

Structural rules from primitive tense axioms

Let $A \to B$ be a primitive tense axiom. Since

$$
\begin{aligned}
&\Diamond(C \vee D) \leftrightarrow \Diamond C \vee \Diamond D \\
&\blacklozenge(C \vee D) \leftrightarrow \blacklozenge C \vee \blacklozenge D \\
&(C \vee D) \wedge E \leftrightarrow (C \wedge E) \vee (D \wedge E)
\end{aligned}
$$

So $A \to B \leftrightarrow (\bigvee_{i \leq m} C_i) \to \left(\bigvee_{j \leq n} D_j\right)$ where C_i, D_j built from $\top$, $\wedge$, $\blacklozenge$ and $\Diamond$.

Then $A \to B$ is equivalent to the following axioms:

$$
C_1 \to \bigvee_{j \leq n} D_j \qquad \ldots \qquad C_m \to \bigvee_{j \leq n} D_j
$$

Structural rules from primitive tense axioms (ctd)

Now $C_j \to \bigvee_{j \leq n} D_j$ is equivalent to the rule

$$\frac{\sigma(D_1) \vdash Y \qquad \ldots \qquad \sigma(D_n) \vdash Y}{\sigma(C_j) \vdash Y} \rho_i$$

where

$$\begin{aligned}
\sigma(\top) &= \mathbf{I} \\
\sigma(p) &= X_p \\
\sigma(A \wedge B) &= \sigma(A), \sigma(B) \\
\sigma(\blacklozenge B) &= \bullet \sigma(B) \\
\sigma(\Diamond B) &= * \bullet * \sigma(B)
\end{aligned}$$

(recall: C_i, D_j built from $\top$, $\wedge$, $\blacklozenge$ and $\Diamond$)

Example

$$\sigma(\top) = \mathbf{I}$$
$$\sigma(p) = X_p$$
$$\sigma(A \wedge B) = \sigma(A), \sigma(B)$$
$$\sigma(\blacklozenge B) = \bullet\sigma(B)$$
$$\sigma(\Diamond B) = * \bullet * \sigma(B)$$

Consider $\blacklozenge\Diamond p \to \Diamond\blacklozenge p$. We compute the rule

$$\frac{(* \bullet *) \bullet X \vdash Y}{\bullet(* \bullet *)X \vdash Y}\ \rho$$

By Kracht's theorem

$\delta Kt + \rho$ is a display calculus for the tense logic $Kt + \blacklozenge\Diamond p \to \Diamond\blacklozenge p$

Since $\blacklozenge\Diamond p \to \Diamond\blacklozenge p \leftrightarrow \Diamond\Box p \to \Box\Diamond p$

$\delta K + \rho$ is a display calculus for the modal logic $K + \Diamond\Box p \to \Box\Diamond p$

Another look at constructing display calculi

We have seen how to construct a display calculus for δKt. This raises several questions.

- How did we know which structural connectives to add?
- How to choose the display rules to ensure display property?
- Why did we consider Kt and not K?
- Under what conditions can the program of adding structural connectives, display rules be used to obtain analytic calculi?

Residuation crucial to constructing new calculi

- To add the tense operators we used the observation:

$$\blacklozenge A \to B \in Kt \quad \Leftrightarrow \quad A \to \Box B \in Kt$$

- We then assigned the structural connective $\bullet$ to $(\blacklozenge, \Box)$.
- Residuation is the key to constructing new display calculi.
- Let us illustrate by constructing a calculus for bi-intuitionistic logic...

Bi-intuitionistic logic

- Intuitionistic logic $Ip \subset Cp$ and $Ip + p \vee \neg p = Cp$.
- Aside: Gentzen observed that restricting the succedent of the sequent calculus $\mathcal{S}Cp$ to at most one formula gives a sequent calculus $\mathcal{S}Ip$ for Ip
- Ie. use sequents of the form $X \vdash A$ and $X \vdash$ instead of $X \vdash Y$
- (try to derive $\vdash p \vee \neg p$ in $\mathcal{S}Ip$ and see what happens!)
- The language of bi-intuitionistic logic $Bi{-}Ip$ extends the language of Ip with the connective $\rightarrow_d$ (*dual-implication*).
- Axiomatisation for $Bi{-}Ip$ were given by Rauszer, 1974.

Residuated pairs for bi-intuitionistic logic

The following are theorems of *Bi−Ip*.

$$A \to (B \to C) \quad \Leftrightarrow \quad A \wedge B \to C \quad \Leftrightarrow \quad B \to (A \to C)$$

$$B \to (A \vee C) \quad \Leftrightarrow \quad (A \to_d B) \to C \quad \Leftrightarrow \quad A \to (B \vee C)$$

Assign the structural connective $\circ$ for $(\wedge, \to)$ and $\bullet$ for $(\to_d, \vee)$. This immediately gives us the display rules:

$$\cfrac{\cfrac{X \vdash Y \circ Z}{X \circ Y \vdash Z}}{Y \vdash X \circ Z} \qquad \cfrac{\cfrac{X \vdash Y \bullet Z}{X \bullet Y \vdash Z}}{Y \vdash X \bullet Z}$$

and the following *rewrite* rules:

$$\frac{A \circ B \vdash Y}{A \wedge B \vdash Y}\ \wedge\mathrm{l} \qquad \frac{X \vdash A \circ B}{X \vdash A \to B}\ \to\mathrm{r}$$

$$\frac{A \bullet B \vdash Y}{A \to_d B \vdash Y}\ \to_d\mathrm{l} \qquad \frac{X \vdash A \bullet B}{X \vdash A \vee B}\ \vee\mathrm{r}$$

Computing the *decoding* rules

$$\frac{A \circ B \vdash Y}{A \wedge B \vdash Y}\ \wedge\mathrm{l} \qquad \frac{X \vdash A \circ B}{X \vdash A \rightarrow B}\ \rightarrow\mathrm{r}$$

$$\frac{A \bullet B \vdash Y}{A \rightarrow_d B \vdash Y}\ \rightarrow_d\mathrm{l} \qquad \frac{X \vdash A \bullet B}{X \vdash A \vee B}\ \vee\mathrm{r}$$

Here are the missing introduction rules (*decoding rules* in the terminology of Goré, 1998).

$$\frac{X \vdash A \qquad Y \vdash B}{X \circ Y \vdash A \wedge B} \qquad \frac{X \vdash A \qquad B \vdash Y}{A \rightarrow B \vdash X \circ Y}$$

$$\frac{X \vdash B \qquad A \vdash Y}{X \bullet Y \vdash B \rightarrow_d A} \qquad \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X \bullet Y}$$

Constructing the decoding rules is systematic (but not obvious, reasoning not shown here) and enforces:

Lemma

Every rewrite rule is invertible.

Some technical points

- To be really precise, the semantics we used would actually lead to the non-associative Bi-Lambek (substructural) logic
- Ie. the first residuation property is
  $a \leq (c \leftarrow b) \quad \Leftrightarrow \quad (a \otimes b) \leq c \quad \Leftrightarrow \quad b \rightarrow (a \rightarrow c)$
- Since we were aiming for $Bi{-}Ip$ we have used the properties of exchange, contraction, weakening and associativity…
- …to collapse

$$\leftarrow \text{ and } \rightarrow$$

$$\otimes \text{ and } \wedge$$

$$\oplus \text{ and } \vee$$

- The point is that we need to add structural rules for exchange, contraction, weakening and associativity!

Adding weakening, contraction, exchange, associativity

$$\frac{X \vdash Y}{X \vdash Y \bullet Z} \qquad \frac{X \vdash Y}{X \circ Z \vdash Y} \qquad \frac{X \vdash Y \bullet Z}{X \vdash Z \bullet Y}$$

$$\frac{X \circ Z \vdash Y}{Z \circ X \vdash Y} \qquad \frac{X \vdash Y \bullet Y}{X \vdash Y} \qquad \frac{X \circ X \vdash Y}{X \vdash Y}$$

$$\frac{X \vdash (Y \bullet Z) \bullet U}{X \vdash Y \bullet (Z \bullet U)} \qquad \frac{(X \circ Y) \circ Z \vdash U}{X \circ (Y \circ Z) \vdash U}$$

Finally, the following structural rules are the unit rules for conjunction, disjunction.

$$\begin{array}{c} \mathbf{I} \circ X \vdash Y \\ \hline\hline X \vdash Y \\ \hline\hline X \circ \mathbf{I} \vdash Y \end{array} \qquad \begin{array}{c} X \vdash Y \bullet \mathbf{I} \\ \hline\hline X \vdash Y \\ \hline\hline X \vdash \mathbf{I} \bullet Y \end{array}$$

A display calculus for *Bi–Ip*

A display sequent $X \vdash Y$ is constructed from structures

$$\text{Struc} ::= \text{Bi-Int formula} \mid \mathbf{I} \mid (X \circ X) \mid (X \bullet X)$$

$$\frac{\mathbf{I} \vdash X}{\top \vdash X}\ \top l \qquad \frac{X \vdash \mathbf{I}}{X \vdash \bot}\ \bot r$$

$$\frac{A \circ B \vdash X}{A \wedge B \vdash X}\ \wedge l \qquad \frac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B}\ \wedge r$$

$$\frac{A \vdash X \quad B \vdash X}{A \vee B \vdash X}\ \vee l \qquad \frac{X \vdash A \bullet B}{X \vdash A \vee B}\ \vee r$$

$$\frac{X \vdash A \quad Y \vdash B}{A \to B \vdash X \circ Y}\ \to l \qquad \frac{X \vdash A \circ B}{X \vdash A \to B}\ \to r$$

$$\frac{B \bullet A \vdash X}{B \to_d A \vdash X}\ \to_d l \qquad \frac{X \vdash B \quad Y \vdash A}{X \bullet Y \vdash B \to_d A}\ \to_d r$$

The display rules:

$$\dfrac{\dfrac{Y \vdash X \circ Z}{X \circ Y \vdash Z}}{X \vdash Y \circ Z} \quad \dfrac{\dfrac{X \vdash Y \circ Z}{X \circ Y \vdash Z}}{Y \vdash X \circ Z} \quad \dfrac{\dfrac{X \bullet Z \vdash Y}{X \vdash Y \bullet Z}}{X \bullet Y \vdash Z} \quad \dfrac{\dfrac{X \bullet Y \vdash Z}{X \vdash Y \bullet Z}}{X \bullet Z \vdash Y}$$

Define the interpretation functions l and r from structures into Bi-lp formulae:

$$\begin{aligned} l(A) &= A & r(A) &= A \\ l(\mathbf{I}) &= \top & r(\mathbf{I}) &= \bot \\ l(X \circ Y) &= l(X) \wedge l(Y) & r(X \circ Y) &= l(X) \rightarrow r(Y) \\ l(X \bullet Y) &= l(X) \rightarrow_d l(Y) & r(X \bullet Y) &= r(X) \vee r(Y) \end{aligned}$$

A sequent $X \vdash Y$ is interpreted as $l(X) \rightarrow r(Y)$.

Another look at constructing display calculi

- Suitable *gaggle semantics* for a logic can be used to construct display calculi via the residuation property (Goré, 1998). Think non-associative Bi-Lambek calculus.
- The residuation property gives the display rules.
- Add new structural connectives and interpret as logical connectives (rewriting rules).
- Add remaining introduction rules (decoding rules).
- axioms for weakening, contraction etc. are converted to structural rules.

This approach provides an answer to: which structural connectives to add? how to choose display rules?

Why did we consider Kt and not K?

Consider the residuation property once more in $Bi\!-\!Ip$:

$$B \to (A \vee C) \quad \Leftrightarrow \quad (A \to_d B) \to C \quad \Leftrightarrow \quad A \to (B \vee C)$$

- Addition of the corresponding display rules leads to the addition of the dual-connective $\to_d$.
- As noted before: we can delete the introduction rules in $\delta Bi\!-\!Ip$ to get the display calculus δIp for Ip.
- Result follows by cut-elimination and conservativity of $Bi\!-\!Ip$ over Ip.
- However, the translation of sequents in δIp is into $Bi\!-\!Ip$.

In the same way, the property $\blacklozenge A \to B \Leftrightarrow A \to \Box B$ (and the ensuing display rules) necessitate a detour into Kt.

Structural rule extensions of display calculi: a general recipe

$$\begin{tikzcd}
\text{Logic } L \arrow[rr, leftrightarrow] \arrow[d, "\text{suitable axiomatic extension}"] & & \text{base display calculus } \mathcal{C} \arrow[d, dashed, "\textit{structural} \text{ rule extension}"] \\
L + \{A_1, \ldots, A_n\} \arrow[rr, leftrightarrow, dashed] & & \mathcal{C} + \rho_1 + \ldots + \rho_m
\end{tikzcd}$$

- Generalises method for obtaining hypersequent structural rules from axioms (Ciabattoni *et al.*, 2008)
- The approach is language and logic independent; purely syntactic conditions on the base calculus
- Extends Kracht's theorem on primitive tense formulae.

Obtaining a structural rule from a Hilbert axiom

δKt is a display calculus for the tense logic Kt satisfying C1–C8. Let us obtain the structural rule extension of δKt for the logic $Kt \oplus \Diamond\Box p \rightarrow \Box\Diamond p$.

STEP 1. Start with the axiom (below left) and apply all possible *invertible* rules backwards (below right).

$$\dfrac{}{\mathbf{I} \vdash \Diamond\Box p \rightarrow \Box\Diamond p}$$

stop here: $\Box l, \Diamond r$ *not invertible*

$$\dfrac{\dfrac{\dfrac{* \bullet *\Box p \vdash \bullet\Diamond p}{\Diamond\Box p \vdash \bullet\Diamond p}\ \Diamond l}{\Diamond\Box p \vdash \Box\Diamond p}\ \Box r}{\mathbf{I} \vdash \Diamond\Box p \rightarrow \Box\Diamond p}\ \rightarrow r$$

So it suffices to introduce a structural rule equivalent to $* \bullet *\Box p \vdash \bullet\Diamond p$.

STEP 2. Apply Ackermann's Lemma.

Lemma

The following rules are pairwise equivalent

$$\boxed{\frac{\mathcal{S}}{X \vdash A}\rho_1 \quad \frac{\mathcal{S} \quad A \vdash \mathcal{L}}{X \vdash \mathcal{L}}\rho_2} \qquad \boxed{\frac{\mathcal{S}}{A \vdash X}\delta_1 \quad \frac{\mathcal{S} \quad \mathcal{L} \vdash A}{\mathcal{L} \vdash X}\delta_2}$$

where $\mathcal{S}$ is a set of sequents, $\mathcal{L}$ is a fresh schematic structure variable, and A is a tense formula.

$$\frac{}{* \bullet *\Box p \vdash \bullet\Diamond p} \overset{\text{d.p.}}{\Leftrightarrow} \frac{}{\Box p \vdash (* \bullet *) \bullet \Diamond p} \overset{\text{lem}}{\Leftrightarrow} \frac{\mathcal{L} \vdash \Box p}{\mathcal{L} \vdash (* \bullet *) \bullet \Diamond p}$$

$$\overset{\text{d.p.}}{\Leftrightarrow} \frac{\mathcal{L} \vdash \Box p}{\bullet(* \bullet *)\mathcal{L} \vdash \Diamond p} \overset{\text{lem}}{\Leftrightarrow} \frac{\mathcal{L} \vdash \Box p \qquad \Diamond p \vdash \mathcal{M}}{\bullet(* \bullet *)\mathcal{L} \vdash \mathcal{M}}$$

Stop when there are no more formulae in the conclusion

STEP 3. Apply all possible invertible rules backwards.

$$\frac{\mathcal{L} \vdash \Box p \qquad \Diamond p \vdash \mathcal{M}}{\bullet(* \bullet *)\mathcal{L} \vdash \mathcal{M}} \quad \Leftrightarrow \quad \dfrac{\dfrac{\mathcal{L} \vdash \bullet p}{\mathcal{L} \vdash \Box p} \qquad \dfrac{* \bullet * p \vdash \mathcal{M}}{\Diamond p \vdash \mathcal{M}}}{\bullet(* \bullet *)\mathcal{L} \vdash \mathcal{M}}$$

The following rule is not a structural rule.

$$\frac{\mathcal{L} \vdash \bullet p \qquad * \bullet * p \vdash \mathcal{M}}{\bullet(* \bullet *)\mathcal{L} \vdash \mathcal{M}}\rho$$

By Belnap's general cut-elimination theorem, $\delta Kt + \rho$ has cut-elimination but not subformula property.

STEP 4. Apply all possible cuts (and verify termination)

$$\frac{\mathcal{L} \vdash \bullet p \qquad * \bullet * p \vdash \mathcal{M}}{\bullet(* \bullet *)\mathcal{L} \vdash \mathcal{M}}\rho \quad \overset{\text{d.p.}}{\Leftrightarrow} \quad \frac{\bullet\mathcal{L} \vdash p \qquad p \vdash * \bullet * \mathcal{M}}{\bullet(* \bullet *)\mathcal{L} \vdash \mathcal{M}}$$

$$\Leftrightarrow \quad \frac{\bullet\mathcal{L} \vdash * \bullet * \mathcal{M}}{\bullet(* \bullet *)\mathcal{L} \vdash \mathcal{M}}\rho'$$

One direction is cut, the other direction is non-trivial.

We conclude:

$\delta Kt + \rho'$ is a calculus for $Kt + \Diamond\Box p \rightarrow \Box\Diamond p$ with cut-elimination and subformula property.

Summary of the recipe

(1) Invertible rules (2) Ackermann's lemma (3) invertible rules (4) all possible cuts

Only certain axioms are suitably decomposable

Suppose we start with the axiom $\Diamond\Box p \to \Box\Diamond\Box p$.

$$\frac{}{\mathbf{I} \vdash \Diamond\Box p \to \Box\Diamond\Box p} \quad \overset{(1)}{\Leftrightarrow} \quad \frac{}{* \bullet * \Box p \vdash \bullet\Diamond\Box p}$$

$$\overset{(2)}{\Leftrightarrow} \frac{\mathcal{L} \vdash \Box p \qquad \Diamond\Box p \vdash \mathcal{M}}{\bullet(* \bullet *)\mathcal{L} \vdash \mathcal{M}} \quad \overset{(3)}{\Leftrightarrow} \quad \frac{\mathcal{L} \vdash \bullet p \qquad * \bullet * \Box p \vdash \mathcal{M}}{\bullet(* \bullet *)\mathcal{L} \vdash \mathcal{M}} \Leftrightarrow ??$$

From the display property, the last rule is equivalent to the following. Observe that we can no longer apply cut.

$$\frac{\bullet\mathcal{L} \vdash p \qquad \Box p \vdash (* \bullet *)\mathcal{M}}{\bullet(* \bullet *)\mathcal{L} \vdash \mathcal{M}}$$

Definition. Amenable calculus

Let $\mathcal{C}$ be a display calculus satisfying C1–C8. l and r are functions from structures into formulae s.t. $\mathsf{l}(A) = \mathsf{r}(A) = A$. Also:

(i) $X \vdash \mathsf{l}(X)$ and $\mathsf{r}(X) \vdash X$ are derivable.

(ii) $X \vdash Y$ derivable implies $\mathsf{l}(X) \vdash \mathsf{r}(Y)$ is derivable.

There is a structure constant **I** such that the following are admissible:

$$\frac{\mathbf{I} \vdash X}{Y \vdash X}\ \mathbf{I}l \qquad\qquad \frac{X \vdash \mathbf{I}}{X \vdash Y}\ \mathbf{I}r$$

There are associative and commutative binary logical connectives $\vee, \wedge$ in $\mathcal{C}$ such that

(a)$_\vee$ $A \vdash X$ and $B \vdash X$ implies $\vee(A, B) \vdash X$

(b)$_\vee$ $X \vdash A$ implies $X \vdash \vee(A, B)$ for any formula B.

(a)$_\wedge$ $X \vdash A$ and $X \vdash B$ implies $X \vdash \wedge(A, B)$

(b)$_\wedge$ $A \vdash X$ implies $\wedge(A, B) \vdash X$ for any formula B.

The intermediate logic $Ip + Bd_2$

Consider the intuitionistic axiom Bd_2: $p_2 \vee (p_2 \to (p_1 \vee \neg p_1))$.

Here is the corresponding structural rule

$$\frac{\mathcal{M} \vdash \mathcal{L} \qquad \mathcal{K} \vdash \mathcal{N}}{\vdash \mathcal{L} \bullet (\mathcal{M} \circ (\mathcal{N} \bullet (\mathcal{K} \circ \mathbf{I})))}\ \rho$$

such that $\delta Ip + \rho$ is a cut-free calculus for $Ip + Bd_2$.

Recovering the display calculus δCp

- Work out the structural rule ρ corresponding to $p \vee \neg p$ (ie. $p \vee p \to \bot$).
- Observe that this is not exactly the calculus δCp we presented before.
- Fun exercise. Work out how to obtain δCp from $\delta Ip + \rho$.

Summary I

- The display calculus generalises the sequent calculus by the addition of new structural connectives.
- Display rules yield the display property.
- The display property is used to prove Belnap's general cut-elimination theorem.
- Residuation property central to choosing structural connectives, display rules.
- Relationship between cut-elimination and algebraic completions (recall Terui's remark)
- Remember: the display calculus is one of several proof-frameworks proposed to address the (lack of) analytic sequent calculi for logics of interests.
- Some other frameworks include hypersequents, nested sequents, labelled sequents.

Summary II

- In some frameworks such as the calculus of structures, we can operate 'inside' formulae (deep inference). The display calculus (below right) seems to mimic some notion of deep inference.

$$\frac{\vdash \Box B}{\vdash \Box(B \vee B')} \qquad \frac{\dfrac{\dfrac{\mathbf{I} \vdash \bullet B}{\bullet\mathbf{I} \vdash B}}{\bullet\mathbf{I} \vdash B, B'}}{\mathbf{I} \vdash \bullet(B, B')}$$

- Structural rules in the display calculus means that it is difficult to control proof-search (for example)
- However it is a good starting point for constructing an analytic calculus.
- Recent work used a display calculus as the startting point for an analytic calculus for Full intuitionistic linear logic (MILL extended with par). A (deep inference) nested sequent calculus is then constructed to obtain complexity, conservativity results (Clouston *et al.*, 2013).

- N.D. Belnap. Display Logic. *Journal of Philosophical Logic*, 11(4), 375–417, 1982.
- A. Ciabattoni, N. Galatos and K. Terui. From axioms to analytic rules in nonclassical logics. Proceedings of LICS 2008, pp. 229–240, 2008.
- A. Ciabattoni and R. Ramanayake. Structural rule extensions of display calculi: a general recipe. Proceedings of WOLLIC 2013.
- R. Clouston, R. Goré, and A. Tiu Annotation-Free Sequent Calculi for Full Intuitionistic Linear Logic. Proceedings of CSL 2013.
- G. Gentzen. The collected papers of Gerhard Gentzen. Edited by M. E. Szabo. Studies in Logic and the Foundations of Mathematics. Amsterdam, 1969.
- R. Goré. Substructural Logics On Display. Logic Journal of the IGPL, 6(3):451-504, 1998.
- R. Goré. Gaggles, Gentzen and Galois: how to display your favourite substructural logic. Logic Journal of the IGPL, 6(5):669-694, 1998.
- M. Kracht. Power and weakness of the modal display calculus. In: *Proof theory of modal logic*, 93–121. Kluwer. 1996.
- C. Rauszer. A formalization of propositional calculus of H-B logic. Studia Logica, 33. 1974.

The slides can be found at <www.logic.at/revantha>