Helly Property and Satisfiability of Boolean Formulas Defined on Set Systems

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March 7, 2008

Abstract

We study the problem of satisfiability of Boolean formulas φ in conjunctive normal form whose literals have the form $v \in S$ and express the membership of values to sets S of a given set system $\mathcal{S}$. We establish the following dichotomy result. We show that checking the satisfiability of such formulas (called *$\mathcal{S}$-formulas*) with three or more literals per clause is NP-complete except the trivial case when the intersection of all sets in $\mathcal{S}$ is nonempty. On the other hand, the satisfiability of $\mathcal{S}$-formulas φ containing at most two literals per clause is decidable in polynomial time if $\mathcal{S}$ satisfies the Helly property, and is NP-complete otherwise (in the first case, we present an $O(|\varphi| \cdot |\mathcal{S}| \cdot |D|)$-time algorithm for deciding if φ is satisfiable). Deciding whether a given set family $\mathcal{S}$ satisfies the Helly property can be done in polynomial time. We also overview several well-known examples of Helly families and discuss the consequences of our result to such set systems and its relationship with the previous work on the satisfiability of signed formulas in multiple-valued logic.

1 Introduction

The satisfiability of Boolean formulas in conjunctive normal form (SAT problem) is a fundamental problem in theoretical computer science and discrete mathematics. SAT is not only important because it is the first NP-complete problem but also because many problems in combinatorics and graph theory, planning and scheduling, games, databases, circuit design, and artificial intelligence can be reduced to it. Therefore, SAT is widely investigated both from theoretical and practical point of view. A rich body of important results, algorithms, methods, generalizations, and relationships with other research areas is currently available. For example, SAT can be viewed as the important Boolean case of constraint satisfaction problems (CSP). The research in the complexity of CSP (which gained considerable interest in recent years) is highly guided by the results obtained for SAT. For instance, Schaefer [30] established a dichotomy theorem for SAT by characterizing polynomially tractable Boolean CSP and proving that the rest are NP-complete (see [16] for this and other results on Boolean CSP). Feder and Vardi [22] developed a general and uniform complexity theory of CSPs and conjectured that a dichotomy theorem holds for CSP problems over every finite domain. So far, this important conjecture has been confirmed only for the three-element domain by Bulatov [14].The problem 2-Sat (consisting of Boolean CNF formulas with at most two literals per clause) is an important tractable class in the Schaefer's classification having numerous applications. Aspvall, Plass, and Tarjan [3] presented a linear-time algorithm for deciding if a 2-CNF formula is satisfiable (see the lecture notes of Welzl [33] for other algorithmic and combinatorial aspects of SAT).

We investigate a CSP problem which extends the classical SAT problem and has a strong combinatorial flavor. In this problem the structure is expressed by membership constraints. More precisely, we consider clauses which are disjunctions of atomic propositions of the form $x \in S$, where x is a variable and S is a finite set from a family of sets $\mathcal{S}$ defined on a domain D. Such constraints are quite natural. Our approach is motivated by a variant of finitely-valued logics called *signed logic*; in this context, the sets of $\mathcal{S}$ are called *signs*. A class of membership constraints is characterized by the family of sets that are allowed to occur as signs in the formulas. The study of satisfiability of such formulas was initiated by Manyà [25] and further continued in [1, 9–11, 15] and some other papers. In all these cited papers, the signs have the form $\uparrow a = \{d \in D \mid d \geq a\}$ and $\downarrow a = \{d \in D \mid d \leq a\}$, where D is a totally ordered set or a lattice (formulas defined on such signs are called *regular signed CNF formulas*). In our paper, we aim at a complete classification of membership formulas in conjunctive normal form covering arbitrary families of sets. This general viewpoint allows us to show that the criterion whether a class can be decided in polynomial time or is NP-complete is not related to a specific lattice structure but to a combinatorial property called *Helly property.* For the polynomial cases we show that satisfiability in fact can be checked in linear time. We also show that the Helly property itself can be decided in polynomial time.

In Section 2 we give a precise definition of the problem. Section 3 states the main results of our paper, in particular, a dichotomy theorem for membership constraints. Section 4 presents an overview of known related results on total orders and lattices. The proof of the dichotomy theorem is split into several parts that cover the Sections 5-8. Finally, Section 9 presents several interesting cases of Helly families and 2-Sat membership constraints, well-known in combinatorics and graph theory, for which we obtain a polynomial-time algorithm.

2 Membership Constraints

Let D be a finite set of cardinality at least 2, called a *domain*, and let V be a set of *variables.* For a family $\mathcal{S}$ of subsets of D, the sets of $\mathcal{S}$ are called *signs.* For a variable $x \in V$ and a set $S \in \mathcal{S}$, the expression $x \in S$ is called a *membership literal.* A *membership clause* (or a *signed clause*) is a disjunction of membership literals and of the constant symbols $\top$ and $\bot$. A *membership formula* or a *signed formula* (*formula* for short) φ is a conjunction of membership clauses. A membership formula φ that uses only sets from $\mathcal{S}$ is referred to as an $\mathcal{S}$*-formula.* A clause is called bijunctive if it contains at most two literals; a formula is called *bijunctive* if all its clauses have this property.

An *interpretation* is a mapping $I\colon V \to D$ assigning a domain element $I(x)$ to each variable $x \in V$. An interpretation satisfies a literal $x \in S$, if $I(x) \in S$. It satisfies a clause if it satisfies at least one of its literals, and it satisfies a formula φ if it satisfies every clause of φ. A formula φ is *D-satisfiable* if it is satisfied by some interpretation over the domain D. In the following, $|\varphi|$ stands for the number of occurrences of literals in φ and $|\mathcal{S}|$ for the number of sets in the family of sets $\mathcal{S}$.

We study the complexity of deciding the satisfiability of $\mathcal{S}$-formulas according to the structure of the family $\mathcal{S}$. More precisely, we are interested in the following decision problem.

Problem: MEM-k-SAT($\mathcal{S}$)
Input: An $\mathcal{S}$-formula φ with at most k literals per clause.
Question: Is φ satisfiable?

If we also allow negative literals $x \notin S$, then we denote the resulting decision problem by COMEM-k-SAT($\mathcal{S}$). Since negative literals $x \notin S$ can be replaced by $x \in D \setminus S$, the problem COMEM-k-SAT($\mathcal{S}$) is a particular case of MEM-k-SAT($\mathcal{S}$). Given $\mathcal{S}$, if we set $\bar{\mathcal{S}} = \{D \setminus S \mid S \in \mathcal{S}\}$, then COMEM-$k$-SAT($\mathcal{S}$) = MEM-$k$-SAT($\mathcal{S} \cup \bar{\mathcal{S}}$). Notice that COMEM-$k$-SAT($\mathcal{S}$) captures the k-SAT problem. Indeed, the identity k-SAT = COMEM-k-SAT($\{\{0\}\}$) holds over the Boolean domain $D = \{0, 1\}$.

3 Overview of Main Results

In this section we establish a complexity classification for MEM-k-SAT($\mathcal{S}$). As to be expected, the problem behaves differently for $k = 2$ and $k \geq 3$. For $k \geq 3$ the problem is either trivial or NP-complete, whereas the classification for the bijunctive membership problem is related to a well-studied property in combinatorics and discrete mathematics, namely the Helly property. A family $\mathcal{S}$ of subsets of D is called a *Helly family*, or $\mathcal{S}$ has the *Helly property* [12], if every subfamily $\mathcal{T} \subseteq \mathcal{S}$ satisfying $\bigcap \mathcal{T} = \emptyset$ contains two sets $S, S' \in \mathcal{T}$ such that $S \cap S' = \emptyset$. In the sequel, $\bigcap \mathcal{T}$ stands for $\bigcap_{S \in \mathcal{T}} S$. Now we can formulate the main result of this paper.

Theorem 1 *For $k \geq 3$,* MEM-k-SAT*($\mathcal{S}$) is polynomial if $\bigcap \mathcal{S} \neq \emptyset$, and* NP*-complete otherwise.* MEM-2-SAT*($\mathcal{S}$) is polynomial if $\mathcal{S}$ is a Helly family, and* NP*-complete otherwise. Checking whether $\mathcal{S}$ is a Helly family can be done in polynomial time.*

As an immediate consequence of Theorem 1 we obtain the following observation.

Corollary 2 COMEM-k-SAT*($\mathcal{S}$) is* NP*-complete for $k \geq 3$.* COMEM-2-SAT*($\mathcal{S}$) is polynomial if $\mathcal{S} \cup \bar{\mathcal{S}}$ is a Helly family, and* NP*-complete otherwise.*

According to this result, the problem 2-SAT = COMEM-2-SAT($\{\{0\}\}$) is in P because the set family $\{\{0\}, \{1\}\}$ has the Helly property.

The proof of Theorem 1 is split into several parts. First, Section 5 presents the intractable cases. Notice that $\bigcap \mathcal{S} \neq \emptyset$ implies that MEM-k-SAT($\mathcal{S}$) is trivially in P, because every $\mathcal{S}$-formula is satisfiable by an interpretation assigning to all variables a value from the intersection $\bigcap \mathcal{S}$. Section 6 establishes an interesting link between the completeness of binary resolution for membership formulas and the Helly property. Section 7 describes polynomial time algorithms for MEM-2-SAT($\mathcal{S}$) when $\mathcal{S}$ is a Helly family. In particular, it presents a linear-time algorithm for evaluating MEM-2-SAT($\mathcal{S}$)-formulas defined on Helly families of constant size, which is a generalization of the Aspvall-Plass-Tarjan algorithm [3] for 2-SAT-formulas. Section 8 shows that the distinction between tractability and intractability, i.e. the Helly property, is polynomially decidable. In the final Section 9 we present several examples of Helly families. One of our basic examples is that of convex sets of a median space. Using various features of median spaces, we establish a link between tractable cases of COMEM-k-SAT($\mathcal{S}$) provided by Corollary 2 and the classical 2-SAT.

4 Known Results

As we noticed already, the study of MEM-k-SAT($\mathcal{S}$) was started in [25] and further continued in [1,9–11,15]. Manyà [25] established that MEM-2-SAT is NP-complete using a reduction from the 3-coloring problem. He also established that MEM-2-SAT($\mathcal{S}$) is polynomially solvable if $\mathcal{S}$ consists of regular signs of the form $\uparrow a$ and $\downarrow a$ of a totally ordered domain D. Béjar, Hähnle, and Manyà [9] reduced the problem of satisfiability of regular signed formulas on totally ordered domains to satisfiability of classical formulas. In particular, a regular MEM-2-SAT($\mathcal{S}$) formula φ is reduced to a 2-CNF formula of size $O(|\varphi| \log |\varphi|)$ [9], which leads to an algorithm of complexity $O(|\varphi| \log |\varphi|)$ to test the satisfiability of a regular MEM-2-SAT($\mathcal{S}$) formula φ in using the linear time algorithm of Aspvall *et al.* for 2-SAT [3]. Baaz and Fermüller [5] established that the MEM-2-SAT($\mathcal{S}$) problem for monosigned CNF formulas φ ($\mathcal{S}$ consisting of signs of the form $\{d\}, d \in D$) is polynomially solvable and Manyà [25] presented a $O(|\varphi| \cdot |D|)$ time algorithm for this problem. Using the binary resolution method, Beckert, Hähnle and Manyà [11] showed that the problem MEM-2-SAT($\mathcal{S}$) is polynomially solvable if D is a lattice and $\mathcal{S}$ consists of regular signs $\uparrow a$ and $\downarrow a$ of D. More recently, Charatonik and Wrona [15] showed that this problem can be solved in quadratic time and in linear time in the size of the formula, if the lattice is fixed. For this, they used a reduction of a many-valued satisfiability problem on a lattice to a classical one. Extending the intractability result of [25], Beckert *et al.* [11] showed that MEM-2-SAT($\mathcal{S}$) is NP-complete (1) if the domain D is a modular lattice and $\mathcal{S}$ consists of complements of regular signs $\uparrow a$ and $\downarrow a$ of D or (2) if the domain D is a distributive lattice and $\mathcal{S}$ consists of regular signs of D and their complements. Notice that neither of cited papers mention Helly property explicitly.

The authors of [11] asked if in (1) modularity of D can be replaced by distributivity. Our Theorem 1 easily shows that this is indeed the case: consider the distributive lattice D_3 of all subsets of the set $\{1,2,3\}$ and let $\mathcal{S}'$ be the family of sets consisting of the complements in D_3 of the sets $\downarrow\{\{a\}\}$ ($a \in \{1,2,3\}$) and the complements of the sets $\uparrow\{\{a,b\}\}$ ($a,b \in \{1,2,3\}, a \neq b$). Then it can be easily seen that $\mathcal{S}'$ consists of pairwise intersecting sets which have an empty intersection (geometrically, each sign of $\mathcal{S}'$ can be viewed as the union of two 2-faces of the 3-cube sharing a common edge). Hence $\mathcal{S}'$ is not Helly and therefore the set system consisting of the complements of upper and down sets of D_3 is not Helly either. In Proposition 20 we present other results in the same vain.

5 Intractable Cases

The case $k \geq 3$. We show that MEM-k-SAT($\mathcal{S}$) is NP-complete if $\bigcap \mathcal{S} = \emptyset$. We encode 3-SAT as an instance of MEM-3-SAT($\mathcal{S}$). Let $\varphi = C_1 \wedge \cdots \wedge C_k$ be a conjunction of clauses, where each clause is of the form $l_1 \vee l_2 \vee l_3$ and the literals l_i are Boolean variables or their negations. Let $\mathcal{T}$ be a minimal subfamily of $\mathcal{S}$ satisfying $\bigcap \mathcal{T} = \emptyset$, and let $\mathcal{T}_0$ and $\mathcal{T}_1$ be disjoint non-empty subfamilies of $\mathcal{T}$ such that $\mathcal{T}_0 \cup \mathcal{T}_1 = \mathcal{T}$. Let f be a function mapping literals to membership constraints as follows:

$$f(l) \quad = \quad \begin{cases} \bigwedge_{T \in \mathcal{T}_0} x \in T & \text{for } l = \neg x \\ \bigwedge_{T \in \mathcal{T}_1} x \in T & \text{for } l = x \ . \end{cases}$$

For a clause C, let $f(C)$ be the conjunctive normal form of $f(l_1) \vee f(l_2) \vee f(l_3)$. For a formula φ, let $f(\varphi)$ be the conjunction $f(C_1) \wedge \cdots \wedge f(C_k)$. Since $f(l)$ consists of at most $|\mathcal{S}|$ conjuncts, $f(\varphi)$

is an $\mathcal{S}$-formula whose length is $O(|\varphi| \cdot |\mathcal{S}|^3)$, where $|\varphi|$ is the number of literals in φ. It remains to show that φ is $\{0,1\}$-satisfiable if and only if $f(\varphi)$ is D-satisfiable. Let I be an interpretation satisfying φ, and for $\varepsilon = 0, 1$ let J be defined as $J(x) = t_\varepsilon$ if $I(x) = \varepsilon$, where t_ε is some fixed element of $\bigcap \mathcal{T}_\varepsilon$. Obviously I satisfies the literal l if and only if J satisfies the formula $f(l)$. Therefore I satisfies φ if and only if J satisfies $f(\varphi)$. Conversely, let J be an interpretation satisfying $f(\varphi)$, i.e., J satisfies the formula $f(l)$ for at least one literal l in every clause of φ. By definition of f, we have $J(x) \in \bigcap \mathcal{T}_0$ if $l = \neg x$ and $J(x) \in \bigcap \mathcal{T}_1$ if $l = x$, for a variable x. Note that the intersections $\bigcap \mathcal{T}_0$ and $\bigcap \mathcal{T}_1$ are disjoint. Let I be defined as $I(x) = 0$ in the first case and $I(x) = 1$ in the second. Then I obviously satisfies φ.

The case $k = 2$. We show that MEM-2-SAT($\mathcal{S}$) is NP-complete if $\mathcal{S}$ is not a Helly family. Notice that $\mathcal{S}$ is not a Helly family if and only if there exists a subfamily $\mathcal{T} \subseteq \mathcal{S}$ of cardinality at least 3 such that $\bigcap \mathcal{T} = \emptyset$ and

$$\gamma(T) = \bigcap (\mathcal{T} \smallsetminus \{T\}) \quad \neq \quad \emptyset$$

for all sets $T \in \mathcal{T}$. Indeed, if $\mathcal{S}$ is not Helly, then it contains a subfamily $\mathcal{T}$ minimal with respect to set inclusion satisfying $\bigcap \mathcal{T} = \emptyset$ and $T \cap T' \neq \emptyset$ for all $T, T' \in \mathcal{T}$. Then removing any subset from $\mathcal{T}$ yields a family whose intersection is no longer empty, i.e., we have $\bigcap(\mathcal{T} \smallsetminus \{T\}) \neq \emptyset$ for all sets $T \in \mathcal{T}$. Obviously $\mathcal{T}$ must contain at least three elements. Notice that such a family $\mathcal{T}$ can be constructed in polynomial time, as we will show in Section 8.

An r-*coloring* of a graph $G = (V, E)$ is a mapping $c\colon V \to C$ such that $|C| = r$ and $c(v) \neq c(w)$ whenever v and w are adjacent in G. The elements of the set C are called the available *colors*. The r-coloring problem r-COL asks whether a graph G admits an r-coloring. It is known to be NP-complete for any $r \geq 3$. We present a reduction from r-COL (for a well-chosen r) to MEM-2-SAT($\mathcal{S}$) with a non-Helly family $\mathcal{S}$. Let $\mathcal{T} \subseteq \mathcal{S}$ be a family of $r \geq 3$ sets such that $\bigcap \mathcal{T} = \emptyset$ and $\gamma(T) \neq \emptyset$ for all $T \in \mathcal{T}$. We use $\mathcal{T}$ as the set of colors for coloring the graph G. Consider the following bijunctive $\mathcal{S}$-formula

$$\varphi_{G,\mathcal{T}} = \bigwedge_{(x,y) \in E} \bigwedge_{T \in \mathcal{T}} (x \in T \vee y \in T).$$

over the variables V and the sets of $\mathcal{T}$. Then the following result holds.

Proposition 3 *A graph $G = (V, E)$ admits an r-coloring if and only if the bijunctive $\mathcal{S}$-formula $\varphi_{G,\mathcal{T}}$ is satisfiable.*

Proof. The function $c\colon V \to \mathcal{T}$ is an r-coloring of G if and only if the formula $\bigwedge_{(x,y)\in E}(c(x) \neq c(y))$ holds, which can be equivalently written as

$$\psi_{G,\mathcal{T}} = \bigwedge_{(x,y) \in E} \bigwedge_{T \in \mathcal{T}} (c(x) \neq T \vee c(y) \neq T) \ .$$

There is a one-to-one correspondence between the inequalities of the formula $\psi_{G,\mathcal{T}}$ and the literals of $\varphi_{G,\mathcal{T}}$. We show that $c(x) \neq T$ holds for a coloring c of G if and only if $x \in T$ evaluates to true in an appropriate interpretation I of $\varphi_{G,\mathcal{T}}$. This equivalence carries over to the whole formula.

Let c be an r-coloring of the graph G, i.e., c satisfies the formula $\psi_{G,\mathcal{T}}$. Define an interpretation I of $\varphi_{G,\mathcal{T}}$ such that $I(x) \in \gamma(c(x))$ holds for every $x \in V$. Such an interpretation exists because the set $\gamma(c(x))$ is nonempty. Let $c(x) \neq T$ be a satisfied literal in $\psi_{G,\mathcal{T}}$, i.e., $c(x) = T'$ for some $T' \neq T$. By definition of I, we have $I(x) \in \gamma(T')$. But since the inclusion $\gamma(T') \subseteq T$ holds, we have $I(x) \in T$, which means that the literal $x \in T$ in $\varphi_{G,\mathcal{T}}$ that corresponds to $c(x) \neq T$ in ψ evaluates to true in I.

Conversely, suppose that $\varphi_{G,\mathcal{T}}$ is satisfied by an interpretation I. For a variable x, consider the literals $x \in T$ in $\varphi_{G,\mathcal{T}}$ that are satisfied by I. Denote by $\mathcal{T}_x$ the family of all sets T participating in these literals. Each $\mathcal{T}_x$ is a proper subfamily of $\mathcal{T}$ because $\bigcap \mathcal{T} = \emptyset$ but $I(x) \in \bigcap \mathcal{T}_x$. Define $c(x) = T$ for any set $T \in \mathcal{T} \setminus \mathcal{T}_x$. Then clearly $c(x) \neq T'$ for any $T' \in \mathcal{T}_x$, i.e., the map c satisfies all literals $c(x) \neq T$ of $\psi_{G,\mathcal{T}}$ that correspond to literals $x \in T$ of $\varphi_{G,\mathcal{T}}$ satisfied by the interpretation I. This shows that the formula $\psi_{G,\mathcal{T}}$ is satisfied by the interpretation c, thus c is an r-coloring of G. □

6 Binary Resolution and the Helly Property

Let $C_1 = (x \in S_1) \vee D_1$ and $C_2 = (x \in S_2) \vee D_2$ be clauses such that $S_1 \cap S_2 = \emptyset$. Then $C = D_1 \vee D_2$ is called the *binary resolvent* of the parent clauses C_1 and C_2. If the binary resolvent contains two redundant literals, i.e., is of the form $(x \in S_1) \vee (x \in S_2)$ with $S_1 \subseteq S_2$, then it is simplified to $x \in S_2$. If C_1 or C_2 contains just one literal, we assume $D_1 = \bot$ or $D_2 = \bot$, respectively. A *proof* of a clause C from a formula φ is a sequence of clauses $C_1, \ldots, C_n$ such that $C_n = C$ and for each k, either C_k is a clause of φ, or C_k is a binary resolvent of C_i and C_j for $i, j < k$. A *refutation* of φ is a proof of $\bot$ from φ.

The proof of the following result is standard and follows the same line as other soundness and completeness proofs for resolution in literature. We adapt the proof from [11] to our situation.

Proposition 4 *Let $\mathcal{S}$ be a Helly family. Then binary resolution is sound and refutationally complete for $\mathcal{S}$-formulas, i.e., an $\mathcal{S}$-formula φ is unsatisfiable if and only if it has a refutation.*

Proof.

Soundness. We prove that a formula φ is unsatisfiable if it has a refutation. Let $C_1, \ldots, C_{n-1}$, $C_n = \bot$ be a refutation of φ. It suffices to show that binary resolution is a sound inference rule, i.e., that every interpretation satisfying two clauses C_1 and C_2 also satisfies their resolvent, or contrapositively, that the unsatisfiability of the resolvent implies the unsatisfiability of the conjunction of the parent clauses. By induction, the unsatisfiability of $\bot$ implies the unsatisfiability of φ. Let $C_1 = (x \in S_1) \vee D_1$ and $C_2 = (x \in S_2) \vee D_2$ be clauses such that $S_1 \cap S_2 = \emptyset$, and let I be an interpretation satisfying both of them. Note that I satisfies at most one of the literals $x \in S_1$ and $x \in S_2$ since S_1 and S_2 are disjoint. Therefore I satisfies either D_1 or D_2, and therefore also the resolvent $D_1 \vee D_2$.

Completeness. We prove that for a family $\mathcal{S}$ of sets with the Helly property, an $\mathcal{S}$-formula φ has a refutation if it is unsatisfiable. Let $e(\varphi)$ denote the number of excess literals of φ, i.e., the total number of literals in φ minus the number of clauses in φ. We show completeness by induction on $e(\varphi)$.

Base case: $e(\varphi) = 0$. All clauses in φ are unit clauses, since the number of literals equals the number of clauses. For a variable x, we denote by φ_x the unit clauses involving x. The unsatisfiability of φ implies that for some x the intersection of all sets in φ_x is empty. Since $\mathcal{S}$ is a Helly family, there must be two sets S_1 and S_2 in φ such that their intersection is empty. By resolving the corresponding literals we obtain a refutation of φ.

Induction step. Suppose that all unsatisfiable $\mathcal{S}$-formulas with at most n excess literals possess refutations, and let φ be an unsatisfiable formula with $n+1$ excess literals. At least one clause in φ, say C, contains two literals. Let C' be the result of removing one literal, say L, from C, and let φ' be the result of replacing C by C' in φ, and let φ'' be the result of replacing C by L in φ. Both formulas φ' and φ'' are unsatisfiable, since any interpretation satisfying φ' or φ'' would also satisfy φ. Clearly $e(\varphi')$ and $e(\varphi'') \leq n$, therefore, by induction hypothesis, there exist refutations of φ' and φ''. Applying the resolution inferences in the refutation of φ' to φ either produces $\perp$ or a clause containing the single literal L. In the first case we are done, in the second case we append the refutation of φ''. □

7 Tractable Case

In this section we prove that MEM-2-SAT($\mathcal{S}$) is in P when $\mathcal{S}$ is a Helly family (as we will show in Section 8, this is covered also by a tractability result obtained by Feder and Vardi [22]), and we describe two algorithms for this purpose. The first algorithm is based on the resolution procedure described above.

Proposition 5 *If the set family $\mathcal{S}$ has the Helly property, then the satisfiability of bijunctive $\mathcal{S}$-formulas can be decided in $O(|\varphi|^3 \cdot |D|)$ time by binary resolution.*

Proof. To check whether an $\mathcal{S}$-formula is satisfiable we compute all binary resolvents. By Proposition 4, the formula is unsatisfiable if and only if the empty clause can be derived. Each resolvent has at most two literals and consists of literals already present in the initial formula. Hence the total number of resolvents is quadratic in the number of literals in φ. In order to decide if two clauses $C_1 = (x \in S_1) \vee D_1$ and $C_2 = (x \in S_2) \vee D_2$ define a binary resolvent $C = D_1 \vee D_2$, we should test if $S_1 \cap S_2 = \emptyset$. This test can be done in $O(|D|)$ time. To derive all binary resolvents, each time when a new resolvent C_1 is detected, we check if C_1 together with an existing binary resolvent C_2 define a new binary resolvent. Since C_2 must contain a literal sharing a common variable with one of the literals of C_1, at each step of the algorithm only $O(|\varphi|)$ existing binary resolvents can occur in the role of C_2. Therefore all binary resolvents derived from C_1 can be computed in $O(|\varphi| \cdot |D|)$ time. For each such resolvent, we can test in constant time if it was already derived. For this, we use a two-dimensional $|\varphi| \times |\varphi|$-array indexed by the literals of φ : an entry of this array corresponding to the literals D_1 and D_2 indicates whether $D_1 \vee D_2$ was already detected as a binary resolvent. Since φ contains $O(|\varphi|^2)$ binary resolvents, the total complexity of this algorithm is $O(|\varphi|^3 \cdot |D|)$. □

The second algorithm is a modification of the linear algorithm of Aspvall *et al.* [3] for 2-SAT. Given a 2-CNF-formula φ over the variables V and the clauses C, this algorithm constructs a directed graph $G(\varphi)$ with $2\,|V|$ vertices $v, \neg v$ and $2\,|C|$ arcs $\neg u \rightarrow v$ and $\neg v \rightarrow u$ for each clause $u \vee v$. The formula φ is satisfiable if and only if each pair of vertices $u, \neg u$ belong to different strongly connected components of the graph $G(\varphi)$. The satisfying assignment for φ can be computed by traversing the strongly connected components of $G(\varphi)$ in reverse topological order.

Now, let $\mathcal{S}$ be a Helly family defined over a finite domain D and let φ be a bijunctive $\mathcal{S}$-formula over the variables V. Let $\mathcal{S}(\varphi)$ denote the family of all sets of $\mathcal{S}$ occurring in the literals of φ. In order to capture the Helly property of $\mathcal{S}$ and the satisfiability of φ, we define the following directed graph $G(\varphi)$.

(1) For each literal $x \in S$, we add two vertices xSt and xSf to $G(\varphi)$, interpreted respectively as "$x \in S$ is true" and "$x \in S$ is false".

(2) For each clause $(x \in S) \vee (y \in S')$ of φ, add the arcs $xSf \to yS't$ and $yS'f \to xSt$ to $G(\varphi)$.

(3) For each pair of literals of φ of the form $x \in S$ and $x \in S'$, such that $S \cap S' = \emptyset$, we add the arcs $xSt \to xS'f$ and $xS't \to xSf$ to $G(\varphi)$.

As in the case of the 2-Sat problem, the graph $G(\varphi)$ has the following *duality property:* $G(\varphi)$ is isomorphic to the graph obtained by reversing all arcs and all nodes of $G(\varphi)$. By this property, every strongly connected component H of $G(\varphi)$ has a dual component $\bar{H}$ induced by the complements of the vertices in H (two vertices u, v belongs to the same strongly connected component if there exist directed paths from u to v and from v to u).

Suppose that φ is satisfied by an interpretation I. We say that the vertex xSt of $G(\varphi)$ is *satisfied* by I if $I(x) \in S$; then xSf is said to be *unsatisfied.* Otherwise, if $I(x) \notin S$, then we say that xSf is *satisfied* and xSt is *unsatisfied.* Notice that

(a) exactly one of the vertices xSt and xSf is satisfied by I,

(b) no arc $u \to v$ of $G(\varphi)$ has u satisfied and v unsatisfied, or equivalently, no directed path leads from a satisfied vertex to an unsatisfied vertex.

Vice versa, if we partition all vertices of $G(\varphi)$ into *satisfied* and *unsatisfied* vertices and this assignment obeys the conditions (a) and (b), then we can define an interpretation I of φ compatible with this assignment. Indeed, for each variable x, let $\mathcal{S}_x$ denote the subfamily of $\mathcal{S}(\varphi)$ consisting of all S such that the vertex xSt is satisfied. We assert that a non-empty $\mathcal{S}_x$ implies $\bigcap \mathcal{S}_x \neq \emptyset$. In view of the Helly property, it suffices to show that the sets of $\mathcal{S}_x$ pairwise intersect. Indeed, if $\mathcal{S}_x$ contains two disjoint sets S, S', since $xSt \to xS'f$ is an arc of $G(\varphi)$ and the vertex xSt is satisfied, condition (a) implies that $xS'f$ must be satisfied as well, yielding that $xS't$ is not satisfied. This contradicts the choice of S'. Thus $\bigcap \mathcal{S}_x$ is indeed non-empty. Now define an interpretation I of φ by letting $I(x) \in \bigcap \mathcal{S}_x$ for all variables x with nonempty $\mathcal{S}_x$. We assert that the $\mathcal{S}$-formula is satisfied by I. Pick an arbitrary clause $(x \in S) \vee (y \in S')$ of φ. If $S \in \mathcal{S}_x$, then the first literal of this clause is satisfied, and we are done. Otherwise, if $S \notin \mathcal{S}_x$ then the vertex xSf is satisfied. Since $xSf \to yS't$ is an arc of $G(\varphi)$, condition (b) yields that the vertex $yS't$ must be satisfied, thus $S' \in \mathcal{S}_y$ establishing our assertion.

Proposition 6 *Given a Helly family $\mathcal{S}$ on D, the bijunctive $\mathcal{S}$-formula φ is satisfiable if and only if no vertex xSt is in the same strong component as its complement xSf. Deciding whether a bijunctive $\mathcal{S}$-formula is satisfiable can be done in time $O(|\varphi| \cdot |\mathcal{S}(\varphi)| \cdot |D|)$. Computing a satisfying interpretation requires $O(|\varphi| \cdot |D|)$ extra time.*

Proof. First, let φ have a satisfying interpretation I. By condition (a), exactly one of the vertices xSt and xSf is satisfied by I. Now if xSt and xSf belonged to the same strong component then we would obtain a directed path running from a satisfied vertex to an unsatisfied one, contrary to condition (b).

Conversely, suppose that all xSt and xSf belong to different strong components. We provide an algorithm "à la" Aspvall-Plass-Tarjan for finding a satisfying interpretation for φ. We traverse the strong components H of $G(\varphi)$ in reverse topological order and perform the following operation: If H is already marked, do nothing. Else if H coincides with its dual component $\bar{H}$ then stop and return "φ is unsatisfiable". Otherwise mark H as *satisfied* and $\bar{H}$ as *unsatisfied*. We can easily see that every component marked *satisfied* has only *satisfied* components as successors and every component marked *unsatisfied* has only *unsatisfied* components as predecessors. Thus the algorithm marks complementary components with complementary values and no directed path leads from a *satisfied* component to an *unsatisfied* one. Hence by assigning to each vertex of $G(\varphi)$ the mark of its component, we obtain an assignment satisfying the conditions (a) and (b). From our discussion preceding this proposition we conclude that this assignment can be turned into a satisfying interpretation I for φ.

Notice that the graph $G(\varphi)$ contains $2\,|\varphi|$ vertices and at most $2\,|\varphi| \cdot (1 + |\mathcal{S}(\varphi)|)$ edges because any vertex of $G(\varphi)$ is incident to at most $|\mathcal{S}(\varphi)|$ arcs of type (3). The arcs of type (2) can be easily defined in $O(|\varphi|)$ time. To construct the arcs of type (3) of $G(\varphi)$, first, for each variable x of φ, we define the family $\mathcal{S}_x^+$ of all sets $S \in \mathcal{S}$ such that $x \in S$ is a literal of φ. This can be done in total $O(|\varphi|)$ time. Given a pair of literals $x \in S$ and $x \in S'$, we can decide in $O(|D|)$ time if $S \cap S' = \emptyset$. Therefore, the arcs of $G(\varphi)$ running between two vertices involving the variable x can be identified in $O(|\mathcal{S}_x^+|^2 \cdot |D|)$ time. Since $\sum_{x \in V} |\mathcal{S}_x^+| = |\varphi|$ and $|\mathcal{S}_x| \le |\mathcal{S}(\varphi)|$, we conclude that $\sum_{x \in V} |\mathcal{S}_x^+|^2 \cdot |D| \le |\mathcal{S}(\varphi)| \cdot |D| \cdot |\varphi|$, thus establishing that $G(\varphi)$ can be constructed in $O(|\varphi| \cdot |\mathcal{S}(\varphi)| \cdot |D|)$ time. Finding strongly connected components requires time linear in the size of the graph $G(\varphi)$. Thus deciding if φ is satisfiable can be done in time linear in the size of $G(\varphi)$. If the algorithm returns an assignment of vertices satisfying the conditions (a) and (b), then, with the help of this assignment, for each $\mathcal{S}_x$ we compute in $O(|\mathcal{S}_x| \cdot |D|)$-time the (nonempty) intersection $\bigcap \mathcal{S}_x$ (in fact this can be done in time linear in the total size of the sets from $\mathcal{S}_x$) and pick any element of $\bigcap \mathcal{S}_x$ as $I(x)$. Since $\sum_{x \in V} |\mathcal{S}_x| \le |\varphi|$, we obtain the claimed time complexity. This concludes the proof and the description of the algorithm. □

When we consider Mem-2-Sat($\mathcal{S}$), the family $\mathcal{S}$ and the domain D are not part of the input, but $\mathcal{S}$ and D parameterize the problem, implying the following complexity result.

Corollary 7 *For a Helly family $\mathcal{S}$,* Mem-2-Sat*($\mathcal{S}$) can be decided and solved in linear time $O(|\varphi|)$.*

8 Complexity of the Meta-Problem and the 2-Mapping Property

In this section we discuss the complexity of deciding for a given family $\mathcal{S}$, whether the problem Mem-2-Sat($\mathcal{S}$) is in P or is NP-complete. According to Theorem 1 this is equivalent to recognize if the set family $\mathcal{S}$ has the Helly property. We present two polynomial in $|D|$ and $|\mathcal{S}|$ algorithms for this task following from two classical characterizations of Helly families given by Berge and Duchet [12, pp. 22-23] and [13].

Proposition 8 *A family $\mathcal{S}$ of subsets of D has the Helly property if and only if for any three elements $a, b, c \in D$, the subfamily $\mathcal{S}(a, b, c)$ of all sets $S \in \mathcal{S}$ containing at least two of the elements a, b, c has a non-empty intersection.*

According to this characterization, it suffices to generate the family $\mathcal{S}(a, b, c)$ for each triplet $a, b, c \in D$ and to test if $\bigcap \mathcal{S}(a, b, c) \neq \emptyset$. A straightforward way is to construct for all pairs of elements $a, b \in D$ the families $\mathcal{S}(a, b)$ consisting of all $S \in \mathcal{S}$ which contain both a and b. This can be done in time $O(|D|^2 \cdot |\mathcal{S}|)$. For a fixed pair a, b, we find the intersection $\bigcap \mathcal{S}(a, b)$ in time $O(|D| \cdot |\mathcal{S}(a, b)|)$. All such intersections taken over all pairs of D can be computed in time $O(|D|^3 \cdot |\mathcal{S}|)$. Now having the intersections $\bigcap \mathcal{S}(a, b)$, $\bigcap \mathcal{S}(b, c)$, and $\bigcap \mathcal{S}(c, a)$ at hand, it takes $O(|D|)$ time to find $\bigcap \mathcal{S}(a, b, c)$, requiring time $O(|D|^4)$ to compute all such intersections. According to Proposition 8, the algorithm returns "NO" if a family $\mathcal{S}(a, b, c)$ is found where $\bigcap \mathcal{S}(a, b, c) = \emptyset$. We obtain the following observation which concludes the proof of Theorem 1.

Proposition 9 *Given a set family $\mathcal{S}$ over a domain D, we can decide in $O(|D|^4 + |D|^3 |\mathcal{S}|)$ time whether $\mathcal{S}$ is a Helly family.*

Now, we present the second algorithm which which has better complexity than the first one in the case when the size of D is significantly larger than the size of $\mathcal{S}$. For this, we need a few notions from hypergraph theory.

For a set family $\mathcal{S}$, an element $d \in D$ dominates another element $d' \in D$ if for all $S \in \mathcal{S}$, $d' \in S$ implies $d \in S$; in this case, the element d' is called *redundant*. A family of sets $\mathcal{S}$ is called *reduced* if it does not contain redundant elements. According to the following lemma, further we may assume that the family of sets $\mathcal{S}$ is reduced.

Lemma 10 *Let d and d' be two distinct elements of D such that d dominates d'. Let h be a homomorphism defined by setting $h(d') = d$ and $h(x) = x$ for $x \neq d'$. Then an $\mathcal{S}$-formula φ is satisfiable if and only if the corresponding $h(\mathcal{S})$-formula $h(\varphi)$ is satisfiable.*

Proof. Note that $h(S) \subseteq S$ for every set $S \in \mathcal{S}$ because $d' \in S$ implies $d \in S$. It suffices to show that single literals are equivalent with respect to satisfiability. Let I be an interpretation satisfying a literal $x \in S$. Since h is a homomorphism, $I(x) \in S$ implies $(h \circ I)(x) \in h(S)$. Conversely, let I be an interpretation satisfying a literal $x \in h(S)$. Then I also satisfies $x \in S$ because of $h(S) \subseteq S$. □

For a set system $\mathcal{S}$ on D, let $\overline{\mathcal{S}} = \{D \setminus S \mid S \in \mathcal{S}\}$. The *dual* of $\mathcal{S}$ is a set system $\mathcal{S}^*$ defined on the domain whose elements are the sets of $\mathcal{S}$ and, for each element $d \in D$, $\mathcal{S}^*$ contains a set of the form $\{S \in \mathcal{S} \mid d \in S\}$. A set $T \subset D$ is a *transversal* of $\mathcal{S}$ if it intersects all sets of $\mathcal{S}$, i.e., $T \cap S \neq \emptyset$ for all $S \in \mathcal{S}$. The family of all minimal (by inclusion) transversals of $\mathcal{S}$ is denoted by $Tr(\mathcal{S})$. Then the second characterization of Helly families given by Berge and Duchet [12, 13] can be rephrased in the following way:

Proposition 11 *A family $\mathcal{S}$ of non-empty subsets of D has the Helly property if and only if all minimal transversals of the set system $\overline{\mathcal{S}^*} = \{\{S \in \mathcal{S} \mid d \notin S\} : d \in D\}$ have size 2.*

For notational simplicity, further we set $n = |D|$ and $m = |\mathcal{S}|$. Then $\mathcal{S}^*$ and $\overline{\mathcal{S}^*}$ contain n sets each and are defined on the domain $\mathcal{S}$ of size m. The set system $\overline{\mathcal{S}^*}$ can be constructed in $O(n \cdot m)$ time by first transposing the incidence $(0, 1)$-matrix of $\mathcal{S}$ (in this way we define the dual family $\mathcal{S}^*$) and then switching the 0 and the 1 values of the resulting matrix. Then we compute in $O(m^2 \cdot n)$ time the set E of all minimal transversals of size 2 of $\overline{\mathcal{S}^*}$. Let $G = (\mathcal{S}, E)$ be the non-oriented simple graph defined by the set E. According to Proposition 11, $\mathcal{S}$ is a Helly family if and only if $Tr(\overline{\mathcal{S}^*}) = E$ holds. The following result shows that instead of testing if $Tr(\overline{\mathcal{S}^*}) = E$ it suffices to check if $Tr(E) = \overline{\mathcal{S}^*}$.

Lemma 12 *$Tr(\overline{\mathcal{S}^*}) = E$ if and only if $Tr(E) = \overline{\mathcal{S}^*}$.*

Proof. The initial set-family $\mathcal{S}$ is a reduced family, thus the dual family $\mathcal{S}^*$ and its complement $\overline{\mathcal{S}^*}$ are both *clutters*, i.e., they do not contain pairs of sets one included into another. Therefore, by a well-known result of Edmonds and Fulkerson [21], the following equality holds $Tr(Tr(\overline{\mathcal{S}^*})) = \overline{\mathcal{S}^*}$. If $Tr(\overline{\mathcal{S}^*}) = E$, then previous equality implies that $Tr(E) = \overline{\mathcal{S}^*}$. Conversely, let $Tr(E) = \overline{\mathcal{S}^*}$. Since E is a clutter as well, applying the idempotent rule of Edmonds and Fulkerson to E, we conclude that $E = Tr(Tr(E)) = Tr(\overline{\mathcal{S}^*})$, yielding the required property $Tr(\overline{\mathcal{S}^*}) = E$. □

Notice that $Tr(E)$ consists of all minimal by inclusion subsets of vertices of the graph G meeting all edges of E (i.e., all minimal vertex covers of G). The complements of minimal vertex covers are the maximal by inclusion stable sets of the graph G. Johnson, Yannakakis, and Papadimitriou [24] developed an algorithm which enumerates all maximal independent sets of a graph with m vertices with delay $O(m^3)$ between two subsequent maximal independent sets. We run this algorithm on the graph G until it returns the first $n+1 = \left|\overline{\mathcal{S}^*}\right| + 1$ maximal independent sets of G (this can be done in overall $O(m^3 \cdot n)$ time). Let $\mathcal{I}$ be the returned collection of independent sets. If $\overline{\mathcal{I}} = \overline{\mathcal{S}^*}$ (or, more simply, if $\mathcal{I} = \mathcal{S}^*$), then $Tr(\overline{\mathcal{S}^*}) = E$ and $\mathcal{S}$ is a Helly family. Otherwise, by what has been shown above, $\overline{\mathcal{S}^*}$ has a minimal transversal of size at least 3 and therefore $\mathcal{S}$ is not Helly. The last test can be performed in $O(n^2 \cdot m)$ time, while the total complexity of the algorithm is $O(m^3 \cdot n + n^2 \cdot m)$.

Summarizing, we obtain the following algorithm of testing for Hellyness of $\mathcal{S}$. First, construct the dual family $\mathcal{S}^*$ and its complement $\overline{\mathcal{S}^*}$, and compute the set E of minimal transversals of size 2 of $\overline{\mathcal{S}^*}$. Then, using the algorithm of Johnson *et al.* [24], compute $\left|\overline{\mathcal{S}^*}\right| + 1$ maximal independent sets of the graph $G = (\mathcal{S}, E)$. If the returned family of independent sets coincides with $\mathcal{S}^*$, then return the answer "$\mathcal{S}$ is Helly", otherwise return the answer "$\mathcal{S}$ is not Helly." We obtain the following result.

Proposition 13 *Given a set family $\mathcal{S}$ over a domain D, we can decide in $O(|\mathcal{S}|^3 \cdot |D| + |D|^2 \cdot |\mathcal{S}|)$ time whether $\mathcal{S}$ is a Helly family.*

The first result of Berge and Duchet can be also used to show that the problem Mem-2-Sat($\mathcal{S}$) for Helly families $\mathcal{S}$ is a subclass of the tractable class of CSPs having the 2-mapping property identified by Feder and Vardi [22, Theorem 25]. According to [22], a CSP has the *2-mapping property* if there exists a function $g\colon D^3 \to D$, satisfying the majority rule $g(u, v, v) = g(v, u, v) = g(v, v, u) = v$ for all elements u and v in D, and such that whenever we are given 3 satisfying assignments of a given constraint then the tuple obtained by applying the function g component-wise also satisfies this constraint. In the case of Mem-2-Sat($\mathcal{S}$) with a Helly family S, the mapping g can be defined in the following way: for a triplet of pairwise distinct elements (a, b, c), let $g(a, b, c)$ be equal to any point from $\bigcap \mathcal{S}(a, b, c)$ (according to Proposition 8 this intersection is non-empty). In addition, set $g(a, a, b) = g(a, b, a) = g(b, a, a) = a$ for all a and b in D. By definition, g is a majority operation. To show that g satisfies the closure property pick any binary clause $C = (x \in S \vee y \in S')$, with $S, S' \in \mathcal{S}$ and let (a_1, b_1), (a_2, b_2), and (a_3, b_3) be three satisfying assignments for C. For each $i = 1, 2, 3$, we have $a_i \in S$ or $b_i \in S'$. Suppose without loss of generality that $a_1 \in S$ and $a_2 \in S$. Then $S \in \mathcal{S}(a_1, a_2, a_3)$, yielding that $a = g(a_1, a_2, a_3) \in \bigcap \mathcal{S}(a_1, a_2, a_3) \subseteq S$. Therefore, independently of the value b of $g(b_1, b_2, b_3)$, the couple (a, b) is a satisfying assignment for C. Thus we obtain the following observation.

Proposition 14 *For Helly families $\mathcal{S}$,* Mem-2-Sat*($\mathcal{S}$) satisfies the 2-mapping property.*

9 Particular Cases

Many instances of Helly families have been explored in the literature. An interested reader can find several examples in the book by Berge [12] on hypergraphs. We review here the consequences of our results on most important examples of Helly families. While our classification was stated for finite domains, the polynomial time algorithms carry over to set systems over infinite domains provided that the set family is finitely presented and any finite intersection can be computed in polynomial time, like is the case for *concepts* in PAC-learning or for *ranges* in computational geometry. This allows us to present a unified treatment of several results obtained for particular domains.

Interval Structures. A mapping $J\colon D \times D \to 2^D$ is an *interval structure* on D in the sense of [26, 27] if $J(x, y) \subseteq J(u, v)$ holds whenever $x, y \in J(u, v)$ and $J(u, v) \cap J(v, w) \cap J(w, u) \neq \emptyset$ for all $u, v, w \in D$. Each set $J(u, v)$ is called an *interval.* A subset S of D is called J*-convex* if $J(u, v) \subseteq S$ holds whenever $u, v \in S$. Obviously each interval in D is J-convex. Denote by $\mathcal{C}_J$ the family of all J-convex sets of an interval structure. Given a set family $\mathcal{S}$ on D, for two elements $u, v \in D$, we write $J_{\mathcal{S}}(u, v) = \bigcap\{S \in \mathcal{S} \mid u, v \in S\}$. Then Gilmore's characterization of duals of Helly families (see [12, p. 31]) can be rephrased in the following way.

Proposition 15 ([26, 27]) *$\mathcal{S}$ is a Helly family if and only if $J_{\mathcal{S}}$ is an interval structure on D. If J is an interval structure on D then the collection $\mathcal{C}_{J_{\mathcal{S}}}$ of all J-convex sets is a Helly family.*

Let $L = \langle D; \wedge, \vee\rangle$ be a finite lattice with the induced partial order $\leq$ defined by $a \leq b$ if $a \vee b = b$. As noticed in [32], we obtain an interval structure on D by taking $J(u, v) = \{w \in D \mid u \wedge v \leq w \leq u \vee w\}$. From Proposition 15 we infer that the convex sets of L define a Helly family (see also [12, 32]). This shows that Mem-2-Sat($\mathcal{S}$) is in P if $\mathcal{S}$ is a family of intervals or a family of convex sets of a lattice. This observation extends an analogous result of Ansótegui and Manyà [1] about intervals in a totally ordered domain. Since for each element $a \in D$, the *upper-set* $\uparrow a = \{d \in D \mid d \geq a\}$ and the *lower-set* $\downarrow a = \{d \in D \mid d \leq a\}$ of a are convex (see for example [32, p. 6]), from previous remark we also derive the result of Beckert *et al.* [11] showing that $\mathcal{S}$ consisting of upper- and lower-sets of a lattice implies that the satisfiability of bijunctive clause sets is polynomial-time decidable.

Another example of an interval structure is obtained by taking a tree $T = (D, E)$ and defining $J(u, v)$ to be the set of all vertices on the unique path in T connecting u and v. The J-convex sets of T are exactly the sets $\mathcal{T}$ of all sub-trees of T. By Proposition 15, $\mathcal{T}$ is a Helly family [12]. Let $\mathcal{T}^0$ be the subfamily of $\mathcal{T}$ consisting of all pairs of complementary subtrees T_u, T_v obtained by removing an edge uv of T : T_u is the subtree of T induced by all vertices x such that the unique path connecting x with v passes via u (T_v is defined in a similar way). Clearly $\mathcal{T}^0$ is closed by taking complements, thus from Corollary 2 and our discussion we conclude that CoMem-k-Sat($\mathcal{S}$) is polynomial if $\mathcal{S} = \mathcal{T}^0$.

Copair Helly Families, Median Structures, and CoMem-2-Sat($\mathcal{S}$). The last observation about the family $\mathcal{T}^0$ can be generalized in an interesting and non-trivial way. A set family $\mathcal{H}$ on D is called a *copair family* if $D \setminus H \in \mathcal{H}$ holds for any set H of $\mathcal{H}$. A *Helly copair family* is a copair family with the Helly property [27]. Then Corollary 2 can be rephrased in the following way: CoMem-k-Sat*($\mathcal{S}$) is polynomial if and only if $k = 2$ and $\mathcal{S} \cup \bar{\mathcal{S}}$ is a Helly copair family, otherwise* CoMem-k-Sat*($\mathcal{S}$) is* NP-*complete.*

Notice that all sets $H, D \setminus H$ of a Helly copair family $\mathcal{H}$ are convex sets of the interval structure $J_{\mathcal{H}}$. A Helly copair family $\mathcal{H}$ is *maximal* if adding any couple $(Q, D \setminus Q)$ to $\mathcal{H}$ leads to a non-Helly family. It is shown in [27] that maximal Helly copair families are exactly those families which separate any pair of elements u, v of D, i.e., $\mathcal{H}$ contains complementary sets $(H, D \setminus H)$ such that $u \in H$ and $v \in D \setminus H$. In view of this result, every Helly copair family $\mathcal{H}$ on D can be re-defined on the equivalence classes of D so that $\mathcal{H}$ becomes a maximal Helly copair family. For this, set $x \sim y$ if $\mathcal{H}_x = \mathcal{H}_y$, where $\mathcal{H}_x = \{H \in \mathcal{H} \mid x \in H\}$. We call a domain D *reduced* if every equivalence class of $\sim$ is a singleton.

The main result of [27] establishes that $\mathcal{H}$ is a maximal Helly copair family if and only if $J_{\mathcal{H}}$ is a median interval structure. Recall that an interval structure $J\colon D \times D \to 2^D$ is called *median* if $m(u, v, w) = J(u, v) \cap J(v, w) \cap J(w, u)$ is a singleton for all $u, v, w \in D$. The underlying graph $G = (D, E)$ of a median interval structure defined by setting $uv \in E$ whenever $J(u, v) = \{u, v\}$ holds is called a *median graph*.

Median structures have been investigated in several contexts ranging from universal algebra and geometry of spaces of non-positive curvature to discrete mathematics and satisfiability problems. We present here a brief account of the properties of median structures related to the subject of our paper. For more detailed information, the interested reader can consult the book [32] and the paper [6]. The survey [6] also presents several examples of median structures. An *abstract median operator* on a (not necessarily finite) set D is a function $m\colon D^3 \to D$ satisfying the following conditions:

(1) $$m(u, v, v) = v \qquad \text{(majority)}$$

(2) $$m(u, v, w) = m(u, w, v) \qquad \text{(right symmetry)}$$

(3) $$m(u, v, w) = m(v, u, w) \qquad \text{(left symmetry)}$$

(4) $$m(u, v, m(u, w, x)) = m(u, m(u, v, w), x) \qquad \text{(transitivity)}$$

The resulting pair (D, m) is called a *median algebra*. All median algebras are subdirect products of the two-element algebra $\{0, 1\}$. The median operator of a median interval structure satisfies the axioms (1-4). These axioms are also satisfied by semilattices $(D, \wedge)$ characterized by the property that all lower sets $\downarrow a = \{x \in D \mid x \leq a\}$ are distributive lattices and three elements have an upper bound whenever each pair of them does. The median operator can be retrieved from such semilattice by the identity $m(u, v, w) = (u \wedge v) \vee (v \wedge w) \vee (w \wedge u)$, just as in the classical case of distributive lattices. There is a bijection between finite (and more generally, discrete) median algebras, median graphs, and median interval structures. With any discrete median algebra (D, m) one can associate a connected graph by taking D as the vertex set and the pairs xy, such that $m(x, y, z) \in \{x, y\}$ holds for all $z \in D$, as edges. Avann [4] proved that median graphs and finite median algebras constitute the same objects.

A subset Y of a median algebra is called *median-stable*, or a subalgebra, if the membership $m(x, y, z) \in Y$ for all $x, y, z \in Y$. For any subset M there exists the smallest median-stable set containing M (the *median closure* of M). Every median subalgebra of a hypercube generated by a subset D is determined by the bipartitions of D into complementary subsets induced by each coordinate of the hypercube. These bipartitions define a maximal Helly copair family $\mathcal{H}$ on D. Vice versa, the median interval structure on a finite domain D induced by a maximal Helly copair family $\mathcal{H}$ can be encoded as a median subalgebra of a hypercube of dimension $\frac{1}{2}|\mathcal{H}|$. For this, we enumerate all pairs of complementary sets $(H, D \setminus H)$ of $\mathcal{H}$. Now, if $(H, D \setminus H)$ is the i-th such pair then we simply set the i-th coordinate of all elements belonging to H to 1 and the i-th coordinate of all elements belonging to $D \setminus H$ to 0. Median-stable subsets of Boolean algebras naturally arise as solution sets of 2-SAT instances:

Proposition 16 ([30]) *Median-stable subsets of Boolean algebras are exactly the solution sets of instances of the* 2-SAT *problem.*

The aforementioned results establish an interesting link between the classical 2-SAT and our CoMem-2-SAT problem.

Proposition 17 CoMem-2-SAT *can be solved in polynomial time only for Helly copair families* $\mathcal{H}$ *defined on reduced domains* D*, so that* D *can be encoded as the set of solutions of some* 2-SAT-*formula with* $\frac{1}{2}|\mathcal{H}|$ *variables and* $\mathcal{H}$ *consists of the subsets of* D *which can be obtained as canonical bipartitions of* D *on each coordinate.*

To give an illustrative example of this one-to-one correspondence, Let $D = \{0, 1, \ldots, 10\}$ and let the Helly copair family $\mathcal{H}$ consist of the following 5 pairs of complementary halfspaces

$$\begin{array}{ll} (\{1,2,3\},\ \{0,4,5,6,7,8\}), & (\{3,4,5\},\ \{0,1,2,6,7,8\}), \\ (\{5,6,7\},\ \{0,1,2,3,4,8\}), & (\{7,8,9\},\ \{0,1,2,3,4,5,6,10\}), \\ (\{1,9,10\},\ \{0,2,3,4,5,6,7,8\})\ . & \end{array}$$

Then D can be encoded as the median-stable subset of the 5-dimensional hypercube using the map ρ defined in the following way.

$$\begin{array}{lll} \rho(0) = (0,0,0,0,0), & \rho(1) = (1,0,0,0,1), & \rho(2) = (1,0,0,0,0), \\ \rho(3) = (1,1,0,0,0), & \rho(4) = (0,1,0,0,0), & \rho(5) = (0,1,1,0,0), \\ \rho(6) = (0,0,1,0,0), & \rho(7) = (0,0,1,1,0), & \rho(8) = (0,0,0,1,0), \\ \rho(9) = (0,0,0,1,1), & \rho(10) = (0,0,0,0,1). & \end{array}$$

The 2-SAT-formula φ whose set of solutions is $\rho(D)$ has the form $\varphi = (\neg x_1 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_5) \wedge (\neg x_3 \vee \neg x_5)$. Geometrically speaking, the median graph of D can be viewed as the fragment of the 5-dimensional cube consisting of 5 squares glued together to form a bipartite 5-wheel. All squares share the vertex $\rho(0)$ and two consecutive squares share an edge between $\rho(0)$ and $\rho(2i)$ for $i = 1, \ldots, 5$.

Gated Sets. A subset W of a metric space (D, d) is called *gated* if for every point $x \notin W$ there exists a point x' (the *gate* of x) in W such that $d(x, y) = d(x, x') + d(x', y)$ for each point y of W. Examples of gated sets are subtrees of a tree and the subsets of the l_1-space $(\mathbb{R}^m, d_1)$ which can be represented as intersections of halfspaces defined by axis parallel hyperplanes, in particular, axis-parallel boxes. More generally, convex sets in median metric spaces are gated as well. For other examples of gated sets see [6, 32]. It is shown in [20] that gated sets of a metric space enjoy the *finite* Helly property, that is, every finite family of gated sets that pairwise intersect has a nonempty intersection.

Balls. A metric space (D, d) is called *hyperconvex* (or *injective*) [2, 23] if any family of closed balls $B_{r_i}(x_i) = \{x \in D \mid d(x_i, x) \le r_i\}$ with centers x_i and radii r_i for $i \in I$, satisfying $d(x_i, x_j) \le r_i + r_j$ for all $i, j \in I$, has a nonempty intersection, that is, (X, d) is a Menger-convex space such that the family $\mathcal{B}$ of balls has the (*infinite*) Helly property. It is well known that (X, d) is hyperconvex if and only if it is an absolute retract, that is, (X, d) is a retract of every metric space into which it embeds isometrically. For every metric space (D', d') there exists the smallest injective space (X, d) extending (X', d'), referred to as the *injective hull* [23] or *tight span* [19] of (X', d').

Helly graphs are the discrete analogues of hyperconvex spaces, where the requirement that radii of balls are nonnegative reals is modified by replacing the reals by the integers. A graph $G = (D, E)$ is called a *Helly graph* if the family $\mathcal{B}$ of balls of G has the Helly property. Helly graphs have been characterized in [7, 8, 28], leading to an $O(|D|^2 \cdot |E|)$-time algorithm for recognizing if a graph $G = (D, E)$ is Helly. There exists a close relationship between median graphs and Helly graphs. Let H^{Δ} be the graph having the same vertex set as H, where two vertices are adjacent if and only if they belong to a common cube of H. If H is a median graph then H^{Δ} is a Helly graph. For a detailed account on Helly graphs and for other constructions leading to such graphs, see [6].

Cliques. As we noticed before, 2-Sat is defined by the Helly family $\{\{0\}, \{1\}\}$ on the binary domain $D = \{0, 1\}$. More generally, [5,25] establish that Mem-2-Sat($\mathcal{S}$) is always in P if $\mathcal{S}$ consists of one-element subsets of a finite (or discrete) domain D, i.e., $\mathcal{S} = \{\{d\} \mid d \in D\}$ (monosigned CNF formulas). Notice that such $\mathcal{S}$ can be viewed as the family of cliques of the empty graph $G = (D, \emptyset)$. Now, if instead of one-element subsets of D we take two-element subsets $\mathcal{E}$, then we can easily see that $\mathcal{E}$ is a Helly family if and only if the graph $G = (D, \mathcal{E})$ is triangle-free. Again in this case, $\mathcal{E}$ is the family of cliques of G. This leads to the following general concept.

A *clique-Helly graph* is a graph $G = (D, E)$ in which the collection $\mathcal{C}$ of cliques has the Helly property [29]. Every Helly family $\mathcal{S}$ on D leads to a clique-Helly graph $G_{\mathcal{S}} = (D \cup \mathcal{S}, E)$: we draw an edge between any $u, v \in D$ and we draw an edge between a set $S \in \mathcal{S}$ and an element $u \in D$ if and only if $u \in S$. Then $G_{\mathcal{S}}$ contains exactly $|\mathcal{S}| + 1$ cliques. These are D and the cliques defined by each $S \in \mathcal{S}$ together with all elements belonging to S. Helly graphs constitute another example of clique-Helly graphs. In fact, Helly graphs are the dismantlable clique-Helly graphs [8]. A vertex x of G is *dominated* by a vertex y if $B_1(y)$ includes $B_1(x)$. Then a graph G is *dismantlable* if its vertices can be linearly ordered, $v_1, v_2, \ldots, v_n$, so that for each v_i, where $i > 1$ there is a neighbor v_j of v_i for a $j < i$, dominating the vertex v_i in the subgraph G_i of G induced by the vertices $v_1, \ldots, v_i$. See [8] for other characterizations and other examples of clique-Helly graphs.

In general, a clique-Helly graph may contain an exponential number of cliques. Take for example the so-called cocktail party graph, which is the complete graph on $2n$ vertices minus a perfect matching. This graph is clique-Helly and contains 2^n cliques. Nevertheless, thanks to the characterization of Helly families given by Berge and Duchet (Proposition 8) reformulated in graph-theoretical terms, recognizing if a graph $G = (D, E)$ is clique-Helly can be done in $O(|D| \cdot |E|^2)$ time [18, 31]. Unfortunately, applying this algorithm to the graph $G_{\mathcal{S}}$ does not improve the complexity of recognizing if $\mathcal{S}$ is Helly, because $G_{\mathcal{S}}$ contains $|D|^2 + \sum_{S \in \mathcal{S}} |S|$ edges.

Summarizing the results of previous subsections, we obtain the following consequences of Theorem 1 and Corollary 2. Notice that every subfamily of a Helly family still has the Helly property. Therefore, the larger the family $\mathcal{S}$ is, the more meaningful is the tractable problem Mem-2-Sat($\mathcal{S}$).

Proposition 18 *The problem* Mem-2-Sat*($\mathcal{S}$) is in* P *if $\mathcal{S}$ is one of the following set-families: (1) convex sets of an interval structure; (2) convex sets and intervals of a finite lattice; (3) convex sets of a median structure; (4) gated sets of a finite metric space; (5) balls of a finite Helly graph; (6) cliques of a finite clique-Helly graph; (7) subtrees of a tree.*

Proposition 19 CoMem-2-Sat*($\mathcal{S}$) is in* P *if* $\mathcal{S}\cup\bar{\mathcal{S}}$ *is one of the following set-families: (1) subtrees of a tree; (2) halfspaces of a median structure.*

Complexity of CoMem-2-Sat($\mathcal{S}$) for Specific Helly Families. In general, given a Helly family $\mathcal{S}$, there is no reason to assume that the complement set $\bar{\mathcal{S}}$ and, a fortiori, $\mathcal{S}\cup\bar{\mathcal{S}}$ remains Helly. In fact, for most families $\mathcal{S}$ with Mem-2-Sat($\mathcal{S}$) in P, the problem CoMem-2-Sat($\mathcal{S}$) is NP-complete. In order to illustrate this observation, we consider two specific examples from the list above.

Let us start with a lattice $L = \langle D; \wedge, \vee\rangle$ containing (at least) two incomparable elements a and b. Let $\overline{\uparrow a}$ and $\overline{\downarrow a}$ denote the complements of the upper- and lower-set with respect to D, respectively. Consider the family $\mathcal{U}$ of all upper-sets in L. The sets $\uparrow a$, $\uparrow b$, and $\overline{\uparrow(a\vee b)}$ form a subfamily of $\mathcal{U}\cup\overline{\mathcal{U}}$ which is not Helly: The pairwise intersections are non-empty, since $a \in {\uparrow a} \cap \overline{\uparrow(a\vee b)}$, $b \in {\uparrow b} \cap \overline{\uparrow(a\vee b)}$, and $a\vee b \in {\uparrow a}\cap{\uparrow b}$, but the intersection of all three sets is empty because ${\uparrow a}\cap{\uparrow b} = {\uparrow(a\vee b)}$. By Corollary 2 we conclude that CoMem-2-Sat($\mathcal{U}$) is NP-complete.

Now let A be an antichain (i.e., a set of pairwise incomparable elements) that is maximal in the following sense: (1) every other element in the domain is comparable to some element in A, and (2) every domain element greater than some element in A is in fact greater than at least two elements in A. Such a maximal set always exists if the lattice contains at least two incomparable elements. Consider the family of sets $\mathcal{F} = \{{\downarrow e} \mid e\in A\}\cup\{{\uparrow(e\vee e')} \mid e,e'\in A, e\neq e'\}$. It contains at least three sets because A has at least two elements. We have $A\smallsetminus\{e\}\subseteq\overline{\downarrow e}$, $A\subseteq\overline{\uparrow(e\vee e')}$, $\top\in\overline{\downarrow e}$, and $\bot\in\overline{\uparrow(e\vee e')}$ for all $e\neq e'$, where$\bot$ and $\top$ denote the bottom and top element, respectively. Hence the pairwise intersection of any two sets in $\bar{\mathcal{F}}$ is non-empty. The intersection of all sets, however, is empty. Every domain element is either inferior or equal to some $e\in A$ and therefore does not occur in $\overline{\downarrow e}$, or it is greater than some $e\in A$ and therefore does not occur in $\overline{\uparrow(e\vee e')}$ for some $e'\in A$. Hence $\bar{\mathcal{F}}$ is not a Helly family. By Theorem 1 we conclude that Mem-2-Sat($\bar{\mathcal{F}}$) and therefore also CoMem-2-Sat($\mathcal{F}$) are NP-complete.

For a tree $T = (D,E)$, consider the set $\mathcal{T}$ of all subtrees of T. We have seen that $\mathcal{T}$ is a Helly family. Suppose that T is not a path. Then it contains a vertex v of degree at least 3, say (v,x), (v,y) and (v,z) are in E. Hence the sets $\{x,y\}$, $\{x,z\}$ and $\{z,y\}$ form a subfamily of $\bar{\mathcal{T}}$ which is not Helly. Therefore $\bar{\mathcal{T}}$ cannot be Helly either. We conclude that Mem-2-Sat($\bar{\mathcal{T}}$) and CoMem-2-Sat($\mathcal{T}$) are NP-complete. These aforementioned observations are summarized in the following proposition.

Proposition 20 CoMem-2-Sat*($\mathcal{S}$) is* NP*-complete if $\mathcal{S}$ consists of the upper-sets of a finite lattice with at least 2 incomparable elements. The problems* Mem-2-Sat($\bar{\mathcal{S}}$) *and* CoMem-2-Sat*($\mathcal{S}$) are* NP*-complete if $\mathcal{S}$ is one of the following Helly families: (1) upper- and lower-sets of a finite lattice with at least 2 incomparable elements; (2) subtrees of a tree with a vertex of degree at least 3.*

Our proposition subsumes results obtained in the context of signed logic. Beckert *et al.* show in [11] the NP-completeness of CoMem-2-Sat($\mathcal{S}$) where $\mathcal{S}$ consists of the upper-sets of the 4-element lattice M_2 (which contains two incomparable elements), and of Mem-2-Sat($\bar{\mathcal{S}}$) where $\mathcal{S}$ consists of the upper- and lower-sets of the 5-element lattice M_3 (which contains three incomparable elements).

Acknowledgment. The second algorithm for testing the Helly property is a fruit of a discussion of the authors with Gérard Cornuejols and Yann Vaxès.

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