Chu's construction: A proof-theoretic approach revisited

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Abstract

The paper Chu's construction: A proof-theoretic approach (Bellin 2003) relates the proof-theoretic question about the meaning of Girard's *long trips* condition for proof nets to research (Hyland and Schalk 2003) on abstract structures with self-duality related to game-semantics, namely, *dialectica categories* (De Paiva 1991) and Chu's construction (Barr 1979). We consider an informal interpretation inspired by htis result, assuming that the role of player and opponent in a dialogue may involve different illocutionary forces (question / answer, assertion/ doubt). We ask how this relates to other game theoretic or dialogical interpretations.

Simple self-duality and the long trip condition.

The abstract of (Bellin 2003) says:
"The essential interaction between classical and intuitionistic features in the system of linear logic is best described in the language of category theory. Given a symmetric monoidal closed category $\mathcal{C}$ with finite products, the category $\mathcal{C} \times \mathcal{C}^{op}$ can be given the structure of a $*$-autonomous category by a special case of the Chu construction. *Girard's trips* induce translations of classical $\mathbf{MLL}^{-}$ proof net into intuitionistic proof in $\mathbf{IMLL}^{-}$ and these translations determine the functor from the free $*$-autonomous category $\mathcal{A}$ on a set of atoms $\{P_1, P_2, \ldots\}$ to $\mathcal{C} \times \mathcal{C}^{op}$, where $\mathcal{C}$ is the free monoidal closed category with products and coproducts on the set of atoms $\{P_1, P_1', P_2, P_2', \ldots\}$ (a pair P, P' in $\mathcal{C}$ for each atom P of $\mathcal{A}$)."

1. CMLL: Classical Multiplicative Linear Logic

Language of CMLL: $A, B \ := \ P \mid P^{\perp} \mid \mathbf{1} \mid \perp \mid A \otimes B \mid A \parr B$
in negation normal form: $\mathbf{1}^{\perp} = \perp, (A \otimes B)^{\perp} = (A^{\perp} \parr B^{\perp})$, etc.

identity rules

logical axiom:
$$\vdash A^{\perp}, A$$

cut:
$$\frac{\vdash \Gamma, A^{\perp} \qquad \vdash \Delta, A}{\vdash \Gamma, \Delta}$$

logical rules

one:
$$\vdash \mathbf{1}$$

nil:
$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp}$$

times:
$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

par:
$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}$$

Table: The sequent calculus **CMLL**.

$\mathbf{CMLL}^{-}$, Classical Multiplicative Linear Logic without units.

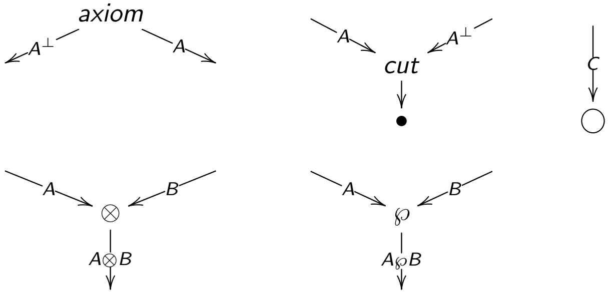


Table: Links of $\mathbf{MLL}^{-}$ proof structures

Alternative notation: $\dfrac{\mathit{axiom}}{A \quad A^\perp}$ $\dfrac{A \quad A^\perp}{\mathit{cut}}$ $\dfrac{A \quad A^\perp}{A \otimes A^\perp}$ $\dfrac{A \quad A^\perp}{A \wp A^\perp}$

Definition

proof-nets A *proof structure* is a directed graph with at least one external point where each edge is typed and every node *(link)* has one of the forms in Table above.

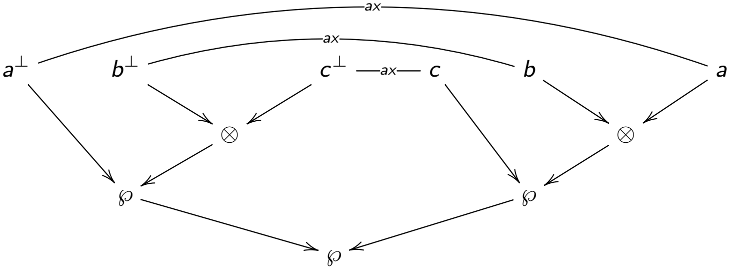
A *Danos-Regnier switching s* is a choice of one of the incoming edges in each *par* node. The (undirected) *D-R graph* $s\mathcal{R}$ results from $\mathcal{R}$ by disconnecting the edge not chosen by s in each *par* node.

A *proof-net* is a proof-structure such that for any switching s the D-R graph $s\mathcal{R}$ is acyclic and connected.

Girard's theorem: (Girard 1987) *There exists a "context forgetful map" $(\ )^-$ from sequent derivations in* $\mathbf{MLL}^-$ *to proof-nets for with the following properties:*

(i) If d is a sequent derivation of $\vdash \Gamma$, then $(d)^-$ is a proof-net with conclusions Γ;

(ii) *(sequentialization)* If $\mathcal{R}$ is a proof-net with conclusions Γ, then there is a sequent calculus derivation d of $\vdash \Gamma$ such that $\mathcal{R} = (d)^-$.



$$(a^\perp \wp (b^\perp \otimes c^\perp)) \quad \wp \quad (c \wp (b \otimes a))$$

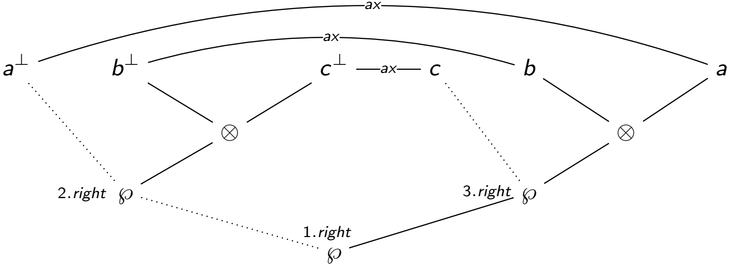


Table: **Example 1:** A proof structure and one of its DR-graph

In **CMLL** without units

- Correctness of proof-nets can be checked in linear time (Guerrini 1999, Murawski and Ong 2006)
- Proof nets solve the problem of identity of proofs in $\mathbf{CMLL}^-$: *two sequent derivations represent the same proof if they are mapped to the same proof net.*
- M.Hyland and L.Ong 1993 gave a full completeness result for $\mathbf{MLL}^-$ with respect to a game semantics with finite, fair games as formulas and uniform, history free strategies as proofs.

In **CMLL** with units, we have an axiom for $\mathbf{1}$ but the unary link for $\perp$ needs to be *attached* somewhere in the proof structure. Then attachments can be moved stepwise (*rewiring*) preserving correctness of the proof net. In 2014 W.Heijltjes and R.Houston showed that the equivalence of proof nets for **MLL** modulo rewiring is PSPACE-complete. Hence a satisfactory representation of proofs in **MLL** with units as proof-nets seems impossible in principle.

2. IMLL$^{\&}$: Intuitionistic MLL with products.

Language: $A, B \ := \ P \mid \mathbf{1} \mid \top \mid A \otimes B \mid A \multimap B \mid A \& B$

identity rules

$$\textit{axiom} \quad A \vdash A \qquad\qquad \dfrac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ \textit{cut}$$

logical rules

$$\textit{axiom}\ \mathbf{1} \quad \vdash \mathbf{1} \qquad\qquad \dfrac{\Gamma \vdash B}{\Gamma, \mathbf{1} \vdash B}\ \mathbf{1}\ \mathrm{L}$$

$$\dfrac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes\ \mathrm{R} \qquad\qquad \dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes\ \mathrm{L}$$

$$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap\ \mathrm{R} \qquad\qquad \dfrac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}\ \multimap\ \mathrm{L}$$

Table: The sequent calculus **IMLL**$^{\&}$.

Sequent calculus $\mathbf{IMLL}^{\&}$ (cont.) additive rules

$\dfrac{\mathit{axiom}\ \top}{\Gamma \vdash \top}$	$\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}\ \&\ \mathrm{R}$	$\dfrac{\Gamma, A_i \vdash B}{\Gamma, A_0 \& A_1 \vdash B}\ \&_i\ \mathrm{L}$ for $i = 0, 1$.

Table: Rules for products ($\&$) and terminal object ($\top$) in $\mathbf{IMLL}^{\&}$.

The sequent calculi **CMLL** and $\mathbf{IMLL}^{\&}$ satisfy cut-elimination. Since there is no left rule for $\top$, there is no symmetric cut reduction for it But any permutation of a cut with the $\top$ axiom up into a derivation d yields the cancellation of d:

$$\dfrac{\begin{array}{c} d \\ \Gamma \vdash A \end{array} \qquad \begin{array}{c} \mathit{axiom}\ \top \\ A, \Delta \vdash \top \end{array}}{\Gamma, \Delta \vdash \top} \quad \text{reduces to} \quad \begin{array}{c} \mathit{axiom}\ \top \\ \Gamma, \Delta \vdash \top \end{array} \ .$$

Categorical semantics of **IMLL** is given by symmetric monoidal closed categories $\langle \mathcal{C}, \otimes, I \rangle$., where
- $\otimes$ is a bifunctor, associative and symmetric up to natural isomorphsms α and σ,
- I is the identity of $\otimes$ up to natural isomorphisms λ and ρ, and
- α, σ, λ and ρ satisfy coherence axioms.
- $\mathcal{C}$ has a closed structure if for every A, the functor $B \mapsto A \otimes B$ has a right adjoint $C \mapsto A \multimap C$, i.e., if there is a bijection

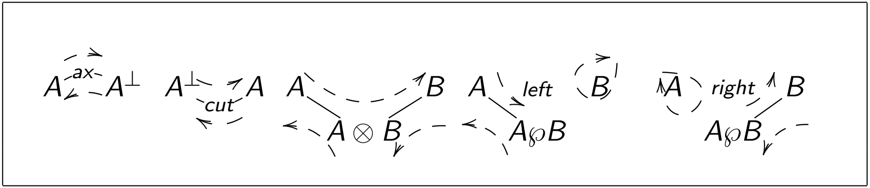
$$\mathcal{C}(A \otimes B, C) \cong C(B, A \multimap C)$$

natural in B and C.
Categorical semantics of **CMLL** is given by $*$-autonomous categories (Seely 1989);
where a symmetric monoidal closed category $\mathcal{C}$ is $*$-autonomous if it has a dualizing object $\perp$ such that writing $A^* = A \multimap \perp$, there is an isomorphism $A^{**} \cong A$ (Barr 1979).

3. Trip Translation of $\mathbf{MLL}^-$ proof-nets into $\mathbf{IMLL}^{\&}$ proofs.

Let the language of $\mathbf{MLL}^-$ be based on atoms $\{P_1, P_2, \ldots\}$ and that of $\mathbf{IMLL}^-$ on $\{P_1, P_1', P_2, P_2', \ldots\}$ (two atoms P_i, P_i' for each P_i in $\mathbf{MLL}^-$).

Trip translation algorithm. (Bellin and Scott 1994) Given a proof structure $\mathcal{R}$ with conclusions Γ, C and a switching s, *traverse* $s\mathcal{R}$ *from* C*, following the right hand rule*:



1. To a directed edge A of $\mathcal{R}$ assign **O** if the direction of the trip coincides with that of the edge *at the second visit to* A, assign **I** to A otherwise.
2. If some edge has not been visited twice, $s\mathcal{R}$ is cyclic or disconnected, return NO.
3. Otherwise, $\mathcal{R}$ is given a polarization $\delta : (\mathcal{R}, \Gamma, C, s) \to \{\mathbf{I}, \mathbf{O}\}$ (see table below) where the polarized conclusions are $\Gamma_{\mathbf{I}}, C_{\mathbf{O}}$
4. Polarization determines a translation $(\;)^{\delta} : \mathbf{MLL}^{-} \to \mathbf{IMLL}^{-}$ of the fomulas in $\mathcal{R}$.
5. Suppose $\mathcal{R}$ is cut-free: Sequentialize $\mathcal{R}^{\delta}$: this yields an $\mathbf{IMALL}^{-}$ derivation of $\Gamma^{\delta} \vdash C^{\delta}$.

Polarization and $\mathbf{IMLL}^{-}$ Translation

Assume that axiom links contain only atomic formulas P_i.

axiom 1:	axiom 2:	cut 1:	cut 2:
$\overline{P_{\mathbf{I}}^{\perp} \quad P_{\mathbf{O}}}$	$\overline{P_{\mathbf{I}} \quad P_{\mathbf{O}}^{\perp}}$	$\underline{A_{\mathbf{O}} \quad A_{\mathbf{I}}^{\perp}}$	$\underline{A_{\mathbf{O}}^{\perp} \quad A_{\mathbf{I}}}$

times 1:	times 2:	times 3:
$\dfrac{A_{\mathbf{O}} \quad B_{\mathbf{O}}}{(A \otimes B)_{\mathbf{O}}}$	$\dfrac{A_{\mathbf{O}} \quad B_{\mathbf{I}}}{(A \otimes B)_{\mathbf{I}}}$	$\dfrac{A_{\mathbf{I}} \quad B_{\mathbf{O}}}{(A \otimes B)_{\mathbf{I}}}$

par 1:	par 2:	par 3:
$\dfrac{A_{\mathbf{I}} \quad B_{\mathbf{I}}}{(A \wp B)_{\mathbf{I}}}$	$\dfrac{A_{\mathbf{I}} \quad B_{\mathbf{O}}}{(A \wp B)_{\mathbf{O}}}$	$\dfrac{A_{\mathbf{O}} \quad B_{\mathbf{I}}}{(A \wp B)_{\mathbf{O}}}$

Table: Polarization.

We have

$$P_{\mathbf{O}}^{\perp} = P_{\mathbf{I}} \quad P_{\mathbf{I}}^{\perp} = P_{\mathbf{O}}$$

In the intuitionistic translation of the atoms we let $P_{\mathbf{O}} = P$ and $P_{\mathbf{I}} = P'$.

Suppose the proof net is cut free. The translation is defined by induction of complexity of formulas in the proof net.

axiom 1: $\dfrac{}{P \quad P}$

times 1: $\dfrac{A^\delta_{\mathbf{O}} \quad B^\delta_{\mathbf{O}}}{A^\delta_{\mathbf{O}} \otimes B^\delta_{\mathbf{O}}} \otimes \mathrm{R}$

times 2: $\dfrac{A^\delta_{\mathbf{O}} \quad B_{\mathbf{I}}}{A^\delta_{\mathbf{O}} \multimap B_{\mathbf{I}}} \multimap \mathrm{L}$

times 3: $\dfrac{A^\delta_{\mathbf{I}} \quad B^\delta_{\mathbf{O}}}{B^\delta_{\mathbf{O}} \multimap A^\delta_{\mathbf{I}}} \multimap \mathrm{L}$

axiom 2: $\dfrac{}{P' \quad P'}$

par 1: $\dfrac{A^\delta_{\mathbf{I}} \quad B^\delta_{\mathbf{I}}}{A^\delta_{\mathbf{I}} \otimes B^\delta_{\mathbf{I}}} \otimes \mathrm{L}$

par 2: $\dfrac{A^\delta_{\mathbf{I}} \quad B^\delta_{\mathbf{O}}}{A^\delta_{\mathbf{I}} \multimap B^\delta_{\mathbf{O}}} \multimap \mathrm{R}$

par 3: $\dfrac{A^\delta_{\mathbf{O}} \quad B^\delta_{\mathbf{I}}}{B^\delta_{\mathbf{I}} \multimap A^\delta_{\mathbf{O}}} \multimap \mathrm{R}$

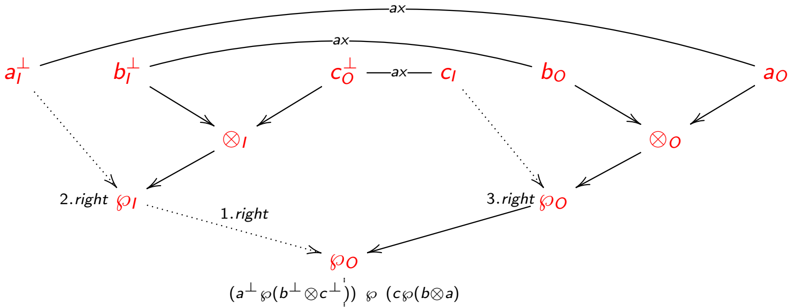
Claim: the relabelled proof net $\mathcal{R}^\delta$ is a representation of a proof in $\mathbf{IMLL}^-$. (cfr. Lamarche's *essential nets* about 1994).
This is seen by sequentializing $\mathcal{R}^\delta$.

In the case of a cut 1 $\underline{A^\delta_{\mathbf{O}} \quad A^{\perp\delta}_{\mathbf{I}}}$ we may have $A^\delta_{\mathbf{O}} \neq A^{\perp\delta}_{\mathbf{I}}$ and similarly for cut 2. So the representation fails. To cover the case of cut, we need the functorial trip translation of **MLL** into **IMLL** with products.

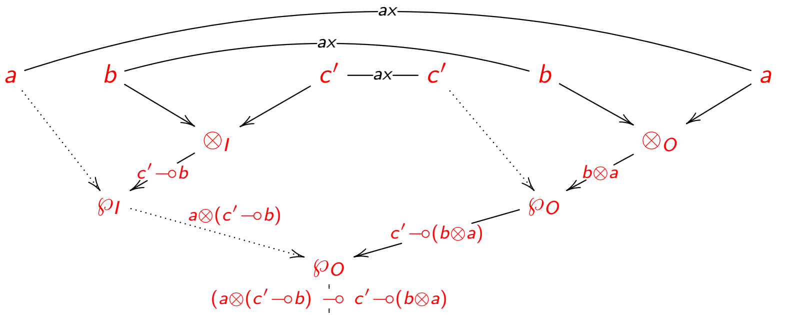
Example 1 (cont.) Consider the proof net $\mathcal{R}$ with switching s. $\mathcal{R}$ results by the *context forgetful map* of Girard's theorem from the following derivation:

$$
\dfrac{\vdash c^{\perp}, c' \qquad \dfrac{\vdash b^{\perp}, b \qquad \vdash a^{\perp}, a}{\vdash b^{\perp}, , a^{\perp}, b \otimes a}}{\dfrac{\vdash c^{\perp}, c \otimes b^{\perp}, a^{\perp}, b \otimes a}{\dfrac{\vdash a^{\perp}, c \otimes b^{\perp}, c^{\perp} \wp (b \otimes a)}{\dfrac{\vdash a^{\perp} \wp (c \otimes b^{\perp}), c^{\perp} \wp (b \otimes a)}{\vdash (a^{\perp} \wp (c \otimes b^{\perp})) \wp (c^{\perp} \wp (b \otimes a))}}}}
$$

The trip on $\mathcal{R}$ induced by the switching s yieds the following polarization:



Applying the trip translation we have:



After sequentialization, we obtain the following **IMLL**$^-$ derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{c' \vdash c' \qquad \dfrac{b \vdash b \qquad a \vdash a}{b, a \vdash b \otimes a}}{c', c' \multimap b, a \vdash b \otimes a}}{a, c' \multimap b \vdash c' \multimap (b \otimes a)}}{a \otimes (c' \multimap b) \vdash c' \multimap (b \otimes a)}}{\vdash (a \otimes (c' \multimap b)) \multimap (c' \multimap (b \otimes a))}$$

4. Functorial trip translation and Chu's construction.

$(P^\perp)_O = P_I$ (P atomic)	$(P^\perp)_I = P_O$;
$\mathbf{1}_O = \mathbf{1}, \quad \mathbf{1}_I = \top$	$\bot_I = \mathbf{1} \quad \bot_O = \top$;
$(A \otimes B)_O = A_O \otimes B_O$	$(A \wp B)_I = A_I \otimes B_I$;
$(A \otimes B)_I = (A_O \multimap B_I) \& (B_O \multimap A_I)$	$(A \wp B)_O = (A_I \multimap B_O) \& (B_I \multimap A_O)$

Table: Functorial trip translation, the propositions.

$\vdash \mathbf{1} \quad \Rightarrow \quad \vdash \mathbf{1}$	$\bot_O \dfrac{\vdash \Gamma}{\vdash \Gamma, \bot} \quad \Rightarrow \quad \Gamma_I \vdash \top$
$\bot_I \dfrac{\vdash \Gamma, \mathbf{A}}{\vdash \bot, \Gamma, \mathbf{A}} \quad \Rightarrow$	$\dfrac{\Gamma_I \vdash A_O}{\mathbf{1}, \Gamma_I \vdash A_O}$ 1-L

Table: Functorial trip translation, the proofs for the units.

$$\vdash P^{\perp}, \mathbf{P} \quad \Rightarrow \quad P_{\mathsf{O}} \vdash P_{\mathsf{O}} \qquad\qquad \vdash \mathbf{P}^{\perp}, P \quad \Rightarrow \quad P_{\mathsf{I}} \vdash P_{\mathsf{I}}$$

$$cut\ \frac{\vdash \Gamma, \mathbf{A} \qquad \vdash A^{\perp}, \Delta, \mathbf{C}}{\vdash \Gamma, \Delta, \mathbf{C}} \quad \Rightarrow \quad \frac{\Gamma_{\mathsf{I}} \vdash A_{\mathsf{O}} \qquad A_{\mathsf{I}}^{\perp}, \Delta_{\mathsf{I}} \vdash C_{\mathsf{O}}}{\Gamma_{\mathsf{I}}, \Delta_{\mathsf{I}} \vdash C_{\mathsf{O}}}\ cut$$

$$\otimes_{\mathsf{O}}\ \frac{\vdash \Gamma, \mathbf{A} \qquad \vdash \Delta, \mathbf{B}}{\vdash \Gamma, \Delta, \mathbf{A} \otimes \mathbf{B}} \quad \Rightarrow \quad \frac{\Gamma_{\mathsf{I}} \vdash A_{\mathsf{O}} \qquad \Delta_{\mathsf{I}} \vdash B_{\mathsf{O}}}{\Gamma_{\mathsf{I}}, \Delta_{\mathsf{I}} \vdash A_{\mathsf{O}} \otimes B_{\mathsf{O}}}\ \otimes\text{-R}$$

$$\otimes_{\mathsf{I}}\ \frac{\vdash \Gamma, \mathbf{A} \qquad \vdash B, \Delta, \mathbf{C}}{\vdash \Gamma, \Delta, A \otimes B, \mathbf{C}} \quad \Rightarrow \quad \frac{\dfrac{\Gamma_{\mathsf{I}} \vdash A_{\mathsf{O}} \qquad B_{\mathcal{I}}, \Delta_{\mathsf{I}} \vdash C_{\mathsf{O}}}{\Gamma_{\mathsf{I}}, A_{\mathsf{O}} \multimap B_{\mathsf{I}}, \Delta_{\mathsf{I}} \vdash C_{\mathsf{O}}}\ \multimap\text{-L}}{\Gamma_{\mathsf{I}}, (A_{\mathsf{O}} \multimap B_{\mathsf{I}}) \& (B_{\mathsf{O}} \multimap A_{\mathsf{I}}), \Delta_{\mathsf{I}} \vdash C_{\mathsf{O}}}$$

$$\wp_{\mathsf{O}}\ \frac{\vdash \Gamma, A, \mathbf{B}}{\vdash \Gamma, \mathbf{A} \wp \mathbf{B}} \ \text{ and } \ \frac{\vdash \Gamma, \mathbf{A}, B}{\vdash \Gamma, \mathbf{A} \wp \mathbf{B}} \quad \Rightarrow \quad \frac{\dfrac{\Gamma_{\mathsf{I}}, A_{\mathsf{I}} \vdash B_{\mathsf{O}}}{\Gamma_{\mathsf{I}} \vdash A_{\mathsf{I}} \multimap B_{\mathsf{O}}} \qquad \dfrac{\Gamma_{\mathsf{I}}, A_{\mathsf{O}} \vdash B_{\mathsf{I}}}{\Gamma_{\mathsf{I}} \vdash B_{\mathsf{I}} \multimap A_{\mathsf{O}}}}{\Gamma_{\mathsf{I}} \vdash (A_{\mathsf{I}} \multimap B_{\mathsf{O}}) \& (A_{\mathsf{I}} \multimap B_{\mathsf{O}})}\ \multimap\text{-R}$$

$$\wp_{\mathsf{I}}\ \frac{\vdash A, B, \Gamma, \mathbf{C}}{\vdash A \wp B, \Gamma, \mathbf{C}} \quad \Rightarrow \quad \frac{A_{\mathsf{I}}, B_{\mathsf{I}}, \Gamma_{\mathsf{I}} \vdash C_{\mathsf{O}}}{A_{\mathsf{I}} \otimes B_{\mathsf{I}}, \Gamma_{\mathsf{I}} \vdash C_{\mathsf{O}}}\ \otimes\text{-L}$$

Table: Functorial trip translation, the proofs.

Theorem

(Bellin 2003) Let $\mathcal{A}$ *be the free* $*$*-autonomous category on a set of objects* $\{P, P', \ldots\}$ *and let* $\mathcal{C}$ *be the symmetric monoidal closed category with products, free on the set of objects* $\{P_{\mathbf{O}}, P_{\mathbf{I}}, P'_{\mathbf{O}}, P'_{\mathbf{I}}, \ldots\}$ *(a pair* $P_{\mathbf{O}}, P_{\mathbf{I}}$ *in* $\mathcal{C}$ *for each* P *in* $\mathcal{A}$*).*
We can give $\mathcal{C} \times \mathcal{C}^{op}$ *the structure of a* $*$*-autonomous category thus:*

$$(X_{\mathbf{O}}, X_{\mathbf{I}}) \otimes (Y_{\mathbf{O}}, Y_{\mathbf{I}}) =_{df} (X_{\mathbf{O}} \otimes Y_{\mathbf{O}}, (X_{\mathbf{O}} \multimap Y_{\mathbf{I}}) \times (Y_{\mathbf{O}} \multimap X_{\mathbf{I}})$$

with unit $(\mathbf{1}, \top)$ *and involution* $(X_{\mathbf{O}}, X_{\mathbf{I}})^{\perp} = (X_{\mathbf{I}}, X_{\mathbf{O}})$

where $\mathbf{1}$ *is the unit of* $\otimes$ *and* $\top$ *the terminal object of* $\mathcal{C}$*.*

Therefore there is a functor F *from* $\mathcal{A}$ *to* $\mathcal{C} \times \mathcal{C}^{op}$ *sending an object* P *to* $(P_{\mathbf{O}}, P_{\mathbf{I}})$*.*
If $\pi : I \to \wp(\Gamma)$ *is a morphism of* $\mathcal{A}$ *represented as a proof-net* $\mathcal{R}$ *with conclusions* Γ*, then the morphism* $(\mathbf{1}, \top) \to (\wp(\Gamma)_{\mathbf{O}}, \wp(\Gamma)_{\mathbf{I}})$ *encodes all* Girard's trips (in a sense specified in Bellin 2003).

Example (i) Writing the $\mathbf{IMLL}^{\&}$ derivation D given by the *functorial trip translation* applied to $\mathcal{R}$ is impractical. However the derivation D has a multiplicative skeleton, which may be represented by the collection of all $\mathbf{IMLL}^{-}$ derivations given by the trip translations of $(\mathcal{R}, s)$, for all switching s. (As $\mathcal{R}$ is cut free, there is no danger in doing this.) Such skeleton is given by derivation of the following sequents.

$$
\begin{array}{lll}
\vdash (a \otimes (c' \multimap b)) \ \multimap \ (c' \otimes (b \multimap a)) & & \mathrm{s} = (\mathrm{r\ r\ r}) \text{ or } (\mathrm{r\ l\ r}) \\
\vdash (a \otimes (b' \multimap c)) \ \multimap \ ((a \multimap b') \multimap c) & & \mathrm{s} = (\mathrm{r\ r\ l}) \text{ or } (\mathrm{r\ l\ l}) \\
\vdash (c' \otimes (a \multimap b')) \ \multimap \ (a \multimap (b \otimes c)) & & \mathrm{s} = (\mathrm{l\ r\ r}) \text{ or } (\mathrm{l\ r\ l}) \\
\vdash (c' \otimes (b \multimap a')) \ \multimap \ ((c' \multimap b) \multimap a') & & \mathrm{s} = (\mathrm{l\ l\ r}) \text{ or } (\mathrm{l\ l\ l}).
\end{array}
$$

Example (*ii*) In intuitionistic **IMLL** we have the proofs

$$\frac{\dfrac{\dfrac{A \vdash A}{A, \mathbf{1} \vdash A}}{A \otimes \mathbf{1} \vdash A}}{} \qquad \frac{A \vdash A \qquad \vdash \mathbf{1}}{A \vdash A \otimes \mathbf{1}} \tag{1}$$

representing the fact that in the Symmetric Monoidal Category $\mathcal{C}$ the map $\rho : A \otimes \mathbf{1}$ is an isomorphaism In the $*$-autonomous category $(\mathcal{C}, \mathcal{C}^{op})$ the unit of the tensor $(A_{\mathbf{O}}, A_{\mathbf{I}}) \otimes (B_{\mathbf{O}}, B_{\mathbf{I}})$. is $(\mathbf{1}, \top)$ where $\top$ is the unit of product & and the terminal object of the category. Thus in $(\mathcal{C}, \mathcal{C}^{op})$ we have isomorphisms

$$a : A_{\mathbf{O}} \otimes \mathbf{1} \to A_{\mathbf{O}} \qquad \langle b, c \rangle : A_{\mathbf{I}} \to [A_{\mathbf{O}} \multimap \top]\&[\mathbf{1} \multimap A_{\mathbf{I}}] \tag{2}$$

The following is a derivation corresponding to $\langle b, c \rangle$:

$$\frac{\dfrac{\dfrac{\top \ \mathit{axiom}}{A_{\mathbf{I}}, A_{\mathbf{O}} \vdash \top}}{A_{\mathbf{I}} \vdash A_{\mathbf{O}} \multimap \top} \qquad \dfrac{A_{\mathbf{I}} \vdash A_{\mathbf{I}}}{A_{\mathbf{I}} \vdash \mathbf{1} \multimap A_{\mathbf{I}}}}{A_{\mathbf{I}} \vdash [A_{\mathbf{O}} \multimap \top]\&[\mathbf{1} \multimap A_{\mathbf{I}}]}$$

Clearly, for no other object X different from $\top$ the sequent $A_{\mathbf{I}}, A_{\mathbf{O}} \vdash X$ is a valid axiom in linear logic.

5. **"Dialectic" interpretations: making illocutionary forces explicit.** If Chu's construction is an abstract form of game semantics, does it have natural language interpretations? If yes, how does such an informal interpretation relate to those of other approaches e.g., dialogical logic and ludics?

- A.Blass (Blass 1995) speaks of *questions* and *answers*, evoking different illocutionary forces for the operations of the opponent and of the player in game semantics.
- J-Y.Girard (Girard 2007, Tome 2, pp.293-4) reinterprets the functional interpretation by distinguishing between *proofs* and *tests*: a proof of A is a function θ that passes certain tests τ for A. E.g., a test τ for $A \multimap B$ is a pair (θ', τ') where θ' is a proof of A and τ' is a test for B, etc.
- In *ludics*, consideration of atomic propositions is abandoned: one can develop a theory of dialogical exchange including acts which do not require consideration of the propositional content.

The following is an informal natural language interpretation in terms of *assertions* and *doubts*.

- Let P_i express the *assertion* $\vdash p_i$ of the proposition p_i. $\vdash p_i$ is justified if *conclusive evidence* is available that proposition p_i is true.
- Let P'_i express a doubt that $\vdash p_i$ is justified, i.e., the hypothesis $\mathcal{H}\neg p_i$ that p_i may be false. *Some evidence* that p_i may be false suffices to justify $\mathcal{H}\neg p_i$ and is *evidence against* $\vdash p_i$.

Here we use the *illocutionary force operators* $\vdash$ and $\mathcal{H}$ of a logic for pragmatics (Bellin 2015). The dual of $\vdash p$ is $\mathcal{H}\neg p$. The negation sign may be safely assumed to be classical negation. Now we can define **"dialectic" semantics** for $\mathbf{MLL}^-$ as a particular interpretation of Chu's construction.

1. Let the elementary formulas of **IMALL** be $E = \{P_1, P'_1, P_2, P'_2, \ldots\}$.
2. $\mathcal{C}$ is the free symmetric monoidal closed cateogory generated by E.
3. Evidence *for* $A \wp B$ is given by a pair of morphism $\langle f, g\rangle$, where f transforms evidence *against* A into evidence *for* B and g

evidence *against* B into evidence *for* A.

4. Evidence *against* $A \wp B$ is given by evidence *against* A together with evidence *against* B.
5. Evidence *for* $A \otimes B$ is evidence *for* A together with evidence *for* B.
6. Evidence *against* $A \otimes B$ is given by a pair of morphisms $\langle f, g \rangle$, where f transforms evidence *for* A into evidence *against* B and g evidence *for* B into evidence *against* A.
7. Evidence for [against] $A^{\perp}$ is evidence against [for] A;

Then we have an interpretation of $\mathbf{MLL}^{-}$ without units. The interpretation of the units is more problematic.

The Chu functor acts as follows:

$$\mathbf{1} \quad \mapsto \quad (\mathbf{1}, \top) \qquad\qquad \perp \quad \mapsto \quad (\top, \mathbf{1})$$

We may regard $\mathbf{1}$ as an assertion which is always justified and such that there can be evidence against it; dually, for $\perp$. This means that if $\top$ is regarded as an assertion, there is no evidence for it. How do we interpret this?

Since $\top$ is the terminal object on $\mathcal{C}$, for any $f : A \to B$, $t_B \circ f = t_A : A \to \top$. This basic fact corresponds to the cancellation property of the $\top$ axiom in cut elimination mantioned above. Similarly in proof search when $\top$ occurs in the succedent, one can simply give up the search for a proof on that branch and apply the $\top$ axiom (the *daimon* in ludics).

It seems therefore that our *assertions / doubts interpretation* does not apply to the $\top$ operator and that it should be part of a larger theory where pragmatic and dialogic operators may not depend on the propositional content, as in ludics.

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