Combining fuzziness and context sensitivity in game based models of vague quantification

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Motivation

- How to model reasoning with quantifiers like many, few, almost all, or even with classical quantifiers applied to vague arguments?
- fuzzy logic alone is linguistically inadequate, in general
- Thesis: adequate models need to incorporate contexts
  game semantics provides an appropriate framework

Overview

- a useful classification of (vague) quantification: types I, II, III, IV
- four desiderata on FL-based quantifiers models
- problems with fuzzy models of type II quantification
- incorporating contexts to respect intensionality
- type III quantification: random choices of witness constants
- type IV (fully vague) quantification: combining the previous lessons
- conclusion and further topics

A useful classification of vague quantification

NB: In natural language quantifiers have not only a scope,
but also a range predicate (restriction), called arguments here.

Originally for fuzzy logic, but applicable generally to vague quantification:

- type I: precise quantifier, precise arguments
  'More than 3 doors are locked'
- type II: precise quantifier, vague arguments
  'All children are nice'
- type III: vague quantifier, precise arguments (semi-fuzzy)
  'Many doors are locked'
- type IV: vague quantifier, vague arguments (fully fuzzy)
  'Many children are nice'

Four desiderata for models of vague quantification

- Embeddability in t-norm based fuzzy logics:
  – of particular particularly: Łukasiewizc logic (**Ł**)
  – But: Glöckner's approach is incompatible with **Ł**-implication
- Interpretability of truth degrees:
  – 'degree of truth' is highly ambiguous
  – meaning of values should be derived from models of reasoning
- Guidance for the choice of truth functions:
  – embarrassment of riches: how to justify the choice of particular functions w.r.t. first principles of reasoning?
- Respecting vagueness specific intensionality:
  – range and scope predicates are semantically linked therefore knowing only their (fuzzy) extensions is insufficient
  – evaluation w.r.t. contexts of precifisifications is needed

Problems with type II quantification

Consider 'All As are Bs', where A and B are vague (here: fuzzy sets)

Two options for formalization in **Ł** from the literature:

O1: $F_1 = \forall x(\neg A(x) \vee B(x))$ [standard in FQ-Theory, eg. Glöckner]
$\|F_1\|_{\mathcal{J}} = \inf_{c \in D} \max(1 - \mu_A(c), \mu_B(c))$

O2: $F_2 = \forall x(A(x) \rightarrow B(x))$ [in analogy to classical logic]
$\|F_2\|_{\mathcal{J}} = \inf_{c \in D} \max(1, 1 - \mu_A(c) + \mu_B(c))$
equivalent to $\forall x(\neg A(x) \oplus B(x))$, but not to F_1

Example

All $c \in D$ are borderline cases of A (being a child) and of B (being poor):
O1: $\|F_1\|_{\mathcal{J}} = 0.5$
O2: $\|F_2\|_{\mathcal{J}} = 1$
Option O1 looks more reasonable (at least to Glöckner, Zadeh, ...).

But consider 'All children are children' ($A = B =$ being a child):
Now option O2 looks more plausible!

Respecting intensionality

Claim: The above 'puzzle' demonstrates that (truth-functional) fuzzy logic is inadequate as 'logic of vagueness'.

Zadeh's Sogan: '<u>fuzziness</u> is different from <u>vagueness</u>'

'All children are poor' versus 'All children are children'

illustrates <u>intensionality</u>: context specific dependence between A and B is relevant for evaluating sentences like 'All As are Bs'.

Modeling intensionality:

Definition: (context of precisifications)

A context C is a finite set of classical (0/1-valued) interpretations.

Extracting truth degrees from contexts:

$$\|F\|_C(a) = \frac{|\{\mathcal{J} \in C : \|F(a)\|_{\mathcal{J}} = 1\}|}{|C|}$$

Contextual evaluation is not truth-functional

Example (continued)

Let C be a context, where all 4 possible combinations of 'being a child' and of 'being poor' occur equally often:

O1: $\|\forall x(\neg A(x) \vee B(x))\|_C = \|\forall x(\neg A(x) \vee A(x))\|_C = 0.5$

O2: $\|\forall x(A(x) \to B(x))\|_C = \|\forall x(A(x) \to A(x))\|_C = 1$

To make the reference to contexts explicit we introduce connective $\circ$:

$$\|\circ F\|_C = \frac{|\{\mathcal{J} \in C : \|F\|_{\mathcal{J}} = 1\}|}{|C|}$$

Proposition [corr.])

O1 and O2 appear as bounds to contextual evaluation:

$\|\forall x(\neg A(x) \vee B(x))\|_C \ \leq \ \|\forall x \circ (\neg A(x) \vee B(x))\|_C =$

$\|\forall x \circ (A(x) \to B(x))\|_C \ \leq \ \|\forall x(A(x) \to B(x))\|_C$

A model of reasoning: Giles's game for Ł

State: multisets of formulas $[F_1, \ldots, F_m \mid G_1, \ldots, G_n]$ (*Your*|*My* tenet)

Rules of the $\mathcal{G}$-game: (*Myself* and *You* may be in role **P** or **O**]

$F \wedge G$: **O** chooses whether to replace $F \wedge G$ with F or with G in **P**'s tenet

$F \vee G$: **P** chooses whether to replace $F \vee G$ with F or with G in **P**'s tenet

$F \to G$: **O** chooses whether to dismiss $F \to G$ in **P**'s tenet or whether to to replace it by G there and add F to **O**'s tenet

$F \& G$: **P** chooses whether to replace $F \& G$ with F, G or with $\perp$ in **P**'s tenet

$\forall x F(x)$: **O** chooses a $c \in D$ and $F(c)$ replaces $\forall x F(x)$ in **P**'s tenet

$\exists x F(x)$: **P** chooses a $c \in D$ and $F(c)$ replaces $\forall x F(x)$ in **P**'s tenet

Pay-off for is defined as inverted <u>risk</u> of loosing money for a false claim:
Pay-off for *Myself* in atomic state $[A_1, \ldots, A_m \mid B_1, \ldots, B_n]$:
$m - n + 1 + \sum_{1 \le i \le n} \|B_i\|_{\mathcal{J}} - \sum_{1 \le i \le m} \|A_i\|_{\mathcal{J}}$.

Adequateness Theorem of the $\mathcal{G}$-game for Ł:

The $\mathcal{G}$-game for F under $\mathcal{J}$ has value w for *Myself* iff $\|F\|_{\mathcal{J}} = w$.

Game semantics for context based reasoning

Giles's model of evaluating $[A_1, \ldots, A_m \mid B_1, \ldots, B_n]$:
A dispersive experiment E_A with failure probability $\langle A \rangle$ is associated with each atomic formula A. For each atomic claim in our tenets the associated experiment is performed and a unit of money is to be paid to the other player for each failed experiment.

Our interpretation of 'dispersive experiment' E_A:
random choice of a precisification $\mathcal{J} \in C$, experiment fails if $\|A\|_{\mathcal{J}} = 0$

A $\mathcal{G}$-game-rule for $\circ$:

$(R_\circ)$ If **P** asserts $\circ F$ then, in reply to **O**'s attack, some precisification $\mathcal{J} \in C$ is chosen randomly and $\circ F$ is replaced with $F{\uparrow}\mathcal{J}$ in **P**'s tenet.

Final evaluation in $[A_1\xi_1, \ldots, A_m\xi_m \mid B_1\xi'_1, \ldots, B_n\xi'_n]$, where either
– ξ_i, ξ'_j is empty indicating final random choice of precisification, or
– ξ_i, ξ'_j is ${\uparrow}\mathcal{J}$, for some previously chosen precisification $\mathcal{J}$

Type III (semi-fuzzy) quantification

vague quantifiers like ‘many’, ‘about half’ applied to precise arguments call for an additional extension of game semantics:

(D) defender chooses the witness ($\Rightarrow\ \exists x A(x)$)

(A) attacker chooses the witness ($\Rightarrow\ \forall x A(x)$)

(R) the witness is chosen randomly (finite domain, uniform distribution!)

The simplest quantifier rule with type-R challenge:

- If **P** asserts $\Pi x A(x)$ then, upon **O**’s attack, a $c \in D$ is chosen randomly and $A(c)$ replaces the attacked formula in **P**’s tenet

Truth function for Π for finite domains and precise (=classical) scope:

$$\Rightarrow\ \|\Pi x A(x))\|_{\mathcal{J}} = Prop_x A(x) = \frac{\sum_{c \in D} \|A(c)\|_{\mathcal{J}}}{|D|}$$

Betting for and against random experiments

Bet for A: assert A, risking $1 - \|A\|_{\mathcal{J}}$ units of money

Bet against A: assert $\bot$ in exchange for the opponent's assertion of A, risking $\|A\|_{\mathcal{J}}$ units of money

Example: proportionality quantifiers Π^k_m:

- If I assert $\Pi^k_m x A(x)$ then $k + m$ constants are chosen randomly. I have partition those constants into $\{c_1, \ldots, c_k\} \,\dot{\cup}\, \{d_1, \ldots, d_m\}$ and to bet for $A(c_1), \ldots, A(c_k)$ and against $A(d_1), \ldots, A(d_m)$.

$$\|\Pi^k_m x A(x)\|_{\mathcal{J}} = \binom{k+m}{k} (Prop_x A(x))^k (1 - Prop_x A(x))^m$$

This leads to models of 'about half', 'about a quarter', 'many', 'few', and other vague quantifiers in combination with threshold selections modeled by promises to pay certain amounts in particular situations.

Type IV (fully vague) quantification

(vague quantifiers like ‘many’, ‘about half’ applied to vague arguments)

Two sources of vagueness that should be kept separate:

(1) vagueness due to imprecise boundaries of range and scope predicates

(2) vagueness due to imprecise meaning of quantifier expressions

Claim:

The combination of random choice of precisifications and random choice of witnessing constants (together with negotiable thresholds) adequately reflects the two-faced nature of type IV quantification.

Conclusion

- truth functional fuzzy logic is inadequate for vague quantification
- game semantics comes to the rescue by modeling evaluation in contexts of precisifications and random sampling of finite domains
- the specific combination of 'fuzziness' (truth-functional degrees) and 'contextuality' (intensional) can be analyzed using connective $\circ$

Topics for further investigation

- fuzzy logics as limit cases, providing appropriate bounds
- incomplete information: **P** and **O** might target different precisifications, without communicating their choices
- compatibility with other fuzzy logics
- relating random choices and incomplete information: equilibrium semantics for imperfect information games