Dialogue Games for Fuzzy Logic

2. Diplomarbeitsvortrag

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Outline

Giles Style Dialogue Games

Overview of Giles's Game I

Motivation

- introduced by Robin Giles in the 1970s
- aim: model reasoning in physical theories
- provide a *tangible meaning* to (compound) propositions
- corresponds to Łukasiewicz Logic

Overview

- atomic propositions are identified with binary experiments
- experiments may show *dispersion*
- at any point in the game each player asserts a (multi)set of propositions
- game is divided into two seperate parts:
  - deconstruction of complex propositions
  - evaluation of atomic game states

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Giles Style Dialogue Games

Overview of Giles's Game II

Risk Values

- after playing the game both players have to pay a certain amount of money to each other
- the expected amount a player has to pay is called his *risk value*
- both players aim to minimize their risk

Game Interpretation

- primarily an evaluation game
- fixed assignment of probability values to experiments
- finite two-player zero-sum game with perfect information
- truth of a proposition F is identified with the existence of a winning strategy for a player asserting F

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Assume that both players assert only atomic propositions.

Betting for Positive Results

Let a be an atomic proposition. He who asserts a agrees to pay his opponent 1 € if a trial of E_a yields the outcome "no".

- for each assertion of an atomic proposition a trial of the associated experiment is done
- for an atomic proposition *a* the corresponding experiment is denoted E_a
- the risk value for one player is the expected amount of money he has to pay in this game state

In the following let the players be called *you* and *me*.

Example

Let a and b be atomic propositions associated with the experiments E_a and E_b and $\pi(E_a) = 0.3$ and $\pi(E_b) = 0.9$. Assume that you assert a and I assert both a and b.

When evaluating this final game state, the experiment E_a is conducted twice and E_b once. In the expected case you have to pay me 0.7 € and I have to pay you 0.8 €. Thus, my risk value for this game state is 0.1 €.

Giles Style Dialogue Games

Decomposing Complex Propositions

Assume that both players assert a (multi)set of arbitrary propositions.

General Game Rule

One player chooses a compound proposition asserted by the other one. Either

- he attacks it according to the corresponding dialogue rule. Then the other player has to defend his claim as indicated by the rule.
- or he grants the proposition to his opponent.

Afterwards the proposition is deleted from the game.

- The order in which the players attack each others' assertions is not specified.

Implication

He who asserts $A \to B$ agrees to assert B if his opponent will assert A

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He who asserts $A \rightarrow B$ agrees to assert B if his opponent will assert A

Giles Style Dialogue Games

Other Rules

Disjunction

He who asserts $A \vee B$ undertakes to assert either A or B at his own choice if challenged

Conjunction

He who asserts $A \wedge B$ undertakes to assert either A or B at his opponent's choice

- Negation can be expressed using $\neg A \equiv A \to \bot$.
- Other rules suitable for conjunction and disjunction as well.
- Dialogue rules refer to Lorenzen (1960s).

Giles Style Dialogue Games

Łukasiewicz Logic Ł

- many-valued, truth functional fuzzy logic
- domain of truth values: unit interval $[0, 1]$

Connectives of Łukasiewicz Logic

Connectives: $\rightarrow$, $\&$, $\wedge$, $\vee$, $\neg$ with truth functions:

- $f_{\rightarrow}(x, y) = \min(1, 1 - x + y)$,
- $f_{\&}(x, y) = \max(0, x + y - 1)$,
- $f_{\wedge}(x, y) = \min(x, y)$,
- $f_{\vee}(x, y) = \max(x, y)$,
- $f_{\neg}(x) = 1 - x$.

A formula is called *true* in Ł under given interpretation iff it evaluates to 1.

Adequateness of Giles's Game for Ł

For a fixed assignment of probability values to atomic propositions and a corresponding interpretation, I have a strategy to ensure that my risk is 0 when asserting a formula A, if and only if A is true in Łukasiewicz Logic.

Correspondence Between Risk Values and Valuations

Let v be an interpretation corresponding to the assignment of probability values to atomic propositions, A be an arbitrary formula, and $\langle A \rangle$ be the risk value (for me) for the game starting with me asserting A.
Then the valuation of A under v in Ł and the inverted risk value $1 - \langle A \rangle$ coincide.

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t-Norm Based Fuzzy Logics

Definition: t-Norm

Continuous t-norm

A continuous t-norm is a continuous, associative, monotonically increasing function $* : [0,1]^2 \longrightarrow [0,1]$ where $1 * x = x \quad \forall x \in [0,1]$.

Residuum of a continuous t-norm $*$

The residuum of $*$ is a function $\Rightarrow_*: [0,1]^2 \longrightarrow [0,1]$ where $x \Rightarrow_* y := \max\{z | x * z \leq y\}$.

- $*$ is used as truth function for (strong) conjunction.
- $\Rightarrow_*$ is used for as truth function implication.

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t-Norm Based Fuzzy Logics

Popular t-Norms

The three most important t-norms are:

	t-Norm	Residuum
Łukasiewicz	$x *_Ł y = \max(0, x + y - 1)$	$x \Rightarrow_Ł y = \min(1, 1 - x + y)$
Gödel	$x *_G y = \min(x, y)$	$x \Rightarrow_G y = \begin{cases} 1 \text{ if } x \leq y \\ y \text{ otherwise} \end{cases}$
Product	$x *_\Pi y = x \cdot y$	$x \Rightarrow_\Pi y = \begin{cases} 1 \text{ if } x \leq y \\ y/x \text{ otherwise} \end{cases}$

- Any continuous t-norm can be constructed from these three ones.

t-Norm Based Fuzzy Logics

Defining Connectives

Using $*$ and its residuum $\Rightarrow_*$ a logic L_* can be defined containing of

- the binary connective & (strong conjunction),
- the binary connective $\rightarrow$,
- the constant $\bot$.

We can, furthermore, define the following derived connectives:

- $\neg A := A \rightarrow \bot$
- $A \wedge B := A \& (A \rightarrow B)$
- $A \vee B := ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A)$

Variants for Other Logics

Changing Evaluation Strategy

Joint Bets

A player has to pay 1 € to his opponent, unless all experiments associated with his assertions test positively.

$\longrightarrow$ Product Logic

Selecting Representatives

Each player picks one of the propositions asserted by his opponent; if the associated experiment tests false, he is paid 1 €.

$\longrightarrow$ Gödel logic

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Variants for Other Logics

Changing Dialogue Rules

- just changing the evaluation scheme does not suffice
- introduction of the flag ¶ signalizing that in order to win the game, my risk has to be strictly negative
- dialogue rule for implication has to be adjusted
- loss of uniformity of rules for both players

Implication (by you)

If you assert $A \rightarrow B$ then, whenever I choose to attack this statement by asserting A, you have the following choice: either you assert B in reply or you challenge my attack on $A \rightarrow B$ by replacing the current game with a new one in which the flag ¶ is raised and I assert A while you assert B.

- also other ways to change the implication rule

Adequateness of Giles's Game for G

For a fixed assignment of probability values to atomic propositions and a corresponding interpretation, I have a winning strategy when asserting a formula A, if and only if A is true in Gödel Logic.

Adequateness of Giles's Game for Π

For a fixed assignment of probability values to atomic propositions and a corresponding interpretation, I have a winning strategy when asserting a formula A, if and only if A is true in Product Logic.

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Adequateness of Giles's Game for Π

For a fixed assignment of probability values to atomic propositions and a corresponding interpretation, I have a winning strategy when asserting a formula A, if and only if A is true in Product Logic.

Other Topics

Other topics the thesis deals with:

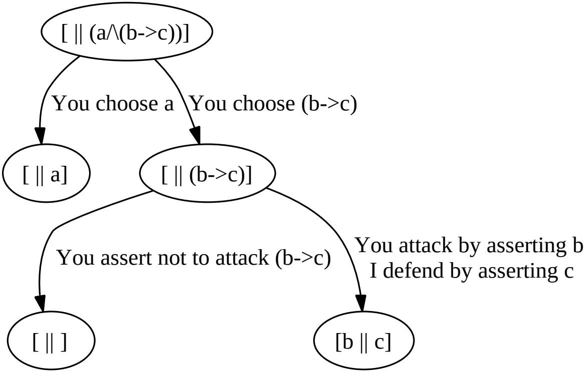
- Proofs using relational hypersequents
- Truth comparison games
- Giles's Game for first order logic
- Devising rules for other connectives
- Using games to prove equivalences
- . . .

Implementation

Giles Games

A small utility to visualize game trees.

Example: `$> giles "a/\(b->c)"` produces:



Implementation

Hypersequential Proofs

Similarly, a tool to visualize proofs in the r-hypersequential calculus *rH*.

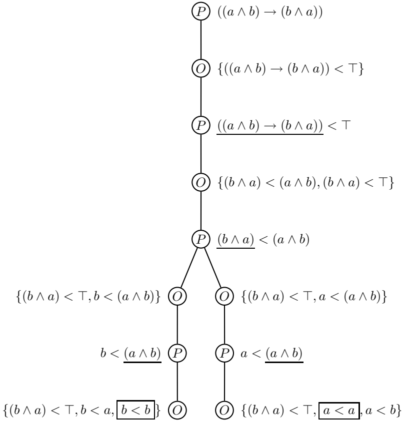
Example: `$> hypseq "a/\(b->c)"` produces:

$$\dfrac{\dfrac{(Atomic)}{\leq a} \qquad \dfrac{\dfrac{(Atomic)}{\leq} \qquad \dfrac{(Atomic)}{b \leq c \mid b \leq c}}{\leq b \to c}\,(\to, \leq, r)}{\leq a \wedge (b \to c)}\,(\wedge, \leq, r)$$

Implementation

Truth Comparison Games

A utility to find winning strategies for the proponent for a truth comparison game Example: `$> tcgame "(a /\ b) -> (b /\ a)"` produces:



Implementation

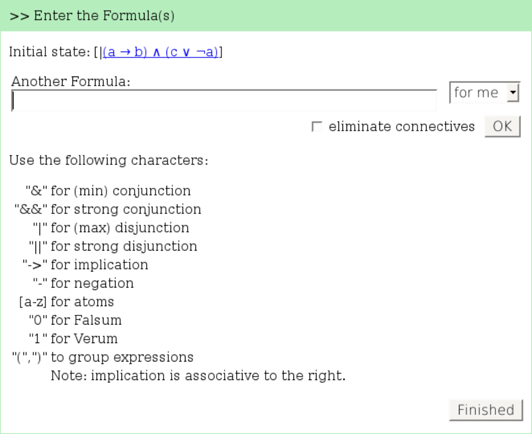
Webgame

A web page where you can actually play Giles style dialogue games.
Features:

- multiple undo and redo
- includes variants for Product and Gödel Logic
- elimination of connectives
- simulation of dispersive evaluation
- online at `http://www.logic.at/staff/roschger/thesis/webgame/`
- ...

Implementation

Webgame - Screenshots



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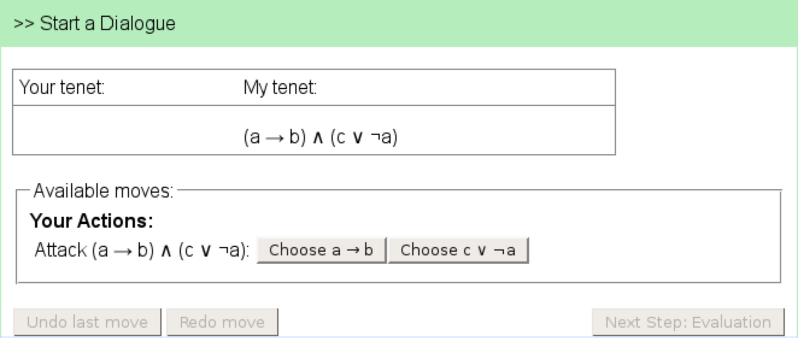

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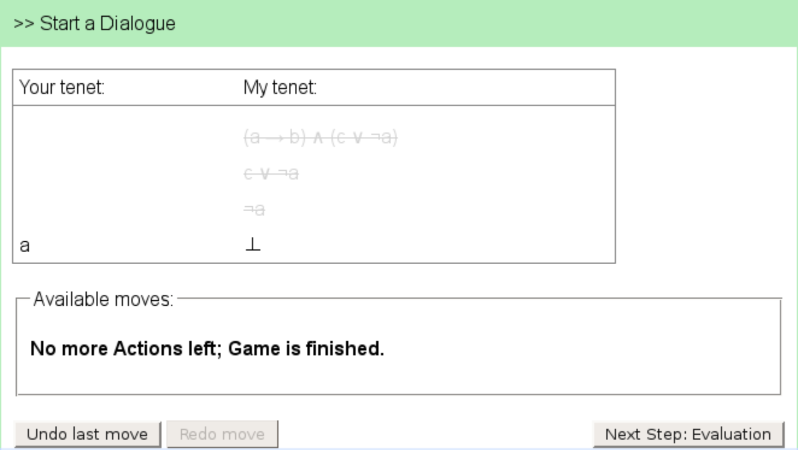

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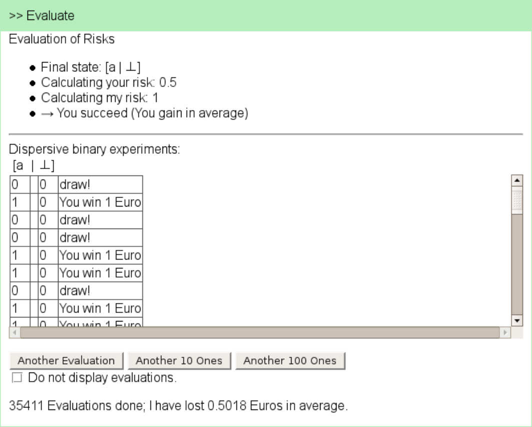

>> Evaluate

Evaluation of Risks

- Final state: [a | ⊥]
- Calculating your risk: 0.5
- Calculating my risk: 1
- → You succeed (You gain in average)

Dispersive binary experiments:

[a | ⊥]

0	0	draw!
1	0	You win 1 Euro
0	0	draw!
0	0	draw!
1	0	You win 1 Euro
1	0	You win 1 Euro
0	0	draw!
1	0	You win 1 Euro
1	0	You win 1 Euro

Another Evaluation | Another 10 Ones | Another 100 Ones

☐ Do not display evaluations.

35411 Evaluations done; I have lost 0.5018 Euros in average.

That's it

Thanks for your attention!

Any questions?