Dialogue Games for Fuzzy Logics

Giles’s Game

Overview & Motivation

- dialogue game introduced by Robin Giles in the 1970s
- models reasoning in physical theories
- asserting a proposition means committing oneself to pay a certain amount of money if the associated experiment(s) fail(s)
- separates evaluation of atomic formulas from decomposing compound formulas

Betraying Positive Results:

- each atomic proposition is associated with a binary (yes/no) experiment
- experiments may be probabilistic, i.e. show dispersion
- for each assertion of an atomic proposition an experiment is made
- each player places bets on positive outcomes of experiments corresponding to his claims

Decomposing Compound Formulas:

- arguments about complex formulas are systematically reduced to arguments about less complex formulas
- dialogue rules have already been introduced by Lorenzen for Intuitionistic Logic
- these rules characterize the meaning of logical connectives, independently of the underlying betting scheme

An Example Dialogue

I assert \( a \rightarrow (b \lor c) \).

You say: “I challenge your assertion by claiming \( a \) myself. Do you still claim that \( b \lor c \)?”

I say: “Yes, and it suffices for me to assert \( c \) to defend \( b \lor c \).

Then we end up with my claim of \( a \) against your claim of \( c \). Let’s make corresponding experiments \( t \) and \( e \).”

Rules

Atomic Evaluation: Let \( a \) be an atomic proposition. He who asserts \( a \) agrees to pay his opponent \( e \) if a trial of the experiment associated with \( a \) yields the outcome “no”.

Implication: He who asserts \( A \rightarrow B \) agrees to assert \( A \) if his opponent will assert \( B \) where \( A \) is associated with an experiment that always evaluates to “no”.

Negation: He who asserts \( \neg A \) agrees to assert \( A \) if his opponent will assert \( A \) where \( A \) is associated with an experiment that always evaluates to “no”.

Disjunction: He who asserts \( A \lor B \) commits himself to assert either \( A \) or \( B \) at his own choice.

Conjunction: He who asserts \( A \land B \) commits himself to assert both \( A \) and \( B \) at his opponent’s choice.

Strong conjunction: He who asserts \( A \\bar{\land} B \) commits himself either to assert both \( A \) and \( B \) or to admit falsity by asserting \( \bot \).

After being attacked, a formula is being deleted from the game.

Łukasiewicz Logic

T-Norm Based Fuzzy Logics

- many valued logics: \( \mathcal{O} \) stands for absolute falsity, \( \top \) for truth, but infinitely many intermediate degrees of truth between \( \mathcal{O} \) and \( \top \)
- truth function for (strong) conjunction \( \land \) is a continuous t-norm
- a t-norm is a commutative, associative function \( x \ast y = \min(x, y) \) with unit 1 which is order preserving
- truth function for implication – is the residuum of a t-norm
- the residuum \( \Rightarrow \) of a t-norm \( \ast \) is determined by \( x \Rightarrow y = \max(0, 1 - x \ast y) \)
- other connectives \( \lor, \rightarrow \) are derived from \( \Rightarrow \)

Łukasiewicz Logic

- one of three fundamental t-norm based fuzzy logics
- originally \( \mathcal{L} \) Łukasiewicz defined a three-valued logic for modelling future contingents, which has later been extended to infinitely many truth values
- \( \mathcal{L} \) Łukasiewicz t-norm: \( x \ast y = \max(0, 1 - x + y) \)
- associated residuum: \( x \Rightarrow y = \max(0, 1 - x + y) \)
- the unique fuzzy logic where all truth functions are continuous
- all connectives can be derived from \( \land \) and \( \Rightarrow \)

Other Fuzzy Logics

Gödel Logic

- also known as Intuitionistic Fuzzy Logic
- based on the Gödel t-norm \( x \ast y = \min(x, y) \)
- associated residuum: \( x \Rightarrow y = \min(1, 1 - x + y) \)
- Product Logic

- introduced in 1996 by Hajek, Godo, and Esteva
- based on the Product t-norm \( x \ast y = x \cdot y \)
- associated residuum: \( x \Rightarrow y = x/y \) if \( x > y \), and is \( 1 \) otherwise

Adequateness of Giles’s Game

- Already proved by Giles in the 1970s:
  - A formula \( \mathcal{F} \) is valid in Łukasiewicz Logic iff I have a strategy to avoid risk (expected loss) in a game starting with me asserting \( \mathcal{F} \) for any assignment of probability values to experiments.
  - Moreover: given a fixed interpretation, my expected loss of money from asserting a formula in the game directly corresponds to a valuation in Łukasiewicz Logic.

For Łukasiewicz Logic

- Variants of Giles’s Game presented by Fermüller recently
- alternative betting schemes: selecting representatives (Gödel Logic) and joint bets (Product Logic)
- dialogue rule for implication has to be extended as well
- dialogue rules correspond to the logical rules of an analytic proof system based on relational hypersequents.

For Gödel & Product Logic

- Presented in this thesis
- another way to adapt the dialogue rule for implication for Gödel Logic and Product Logic.
- game gets simpler compared to the other approach
- connection to the hypersequent calculus is lost.

Alternative Dialogue Rules

- Small Haskell program to display game trees of Giles’s game
- given a formula, computes a game tree of the corresponding game and outputs the tree as a .dot-Graph specification

Accompanying Implementation

Webgame

- Web-based application which allows playing Giles’s Game interactively
- simulates evaluation by dispersive experiments
- see http://logic.us/people/roschger/theses/ogamegame

Giles

- Small Haskell program to display game trees of Giles’s game
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Hypseq

- Utility to find derivations of hypersequents in the relational hypersequent calculus \( \mathcal{H} \), computes all possible derivations and outputs the one with the smallest height.

TCGame

- Utility to find a winning strategy for the propositional \( P \) in a Truth Comparison Game
- for Gödel Logic
- winning strategy for P can be seen as a proof of the starting formula.

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