CERES in Proof Schemata

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Introduction and Preliminaries

Aim

- Proof mining.
- Extraction of explicit information from proofs.
- Via cut-elimination: the removal of lemmas in proofs.
  - Reductive methods.
  - The CERES method.

Reductive Methods

- Proof rewriting system is defined which is terminating and its normal form is a cut-free proof.
- Example:

$$\dfrac{\dfrac{\begin{matrix}\phi_l\\ \Gamma_1 \vdash \Delta_1, A(x/\alpha)\end{matrix}}{\Gamma_1 \vdash \Delta_1, (\forall x)A}\ \forall : r \qquad \dfrac{\begin{matrix}\phi_r\\ A(x/t), \Gamma_2 \vdash \Delta_2\end{matrix}}{(\forall x)A, \Gamma_2 \vdash \Delta_2}\ \forall : l}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

transforms to

$$\dfrac{\begin{matrix}\phi_l(\alpha/t)\\ \Gamma_1 \vdash \Delta_1, A(x/t)\end{matrix} \qquad \begin{matrix}\phi_r\\ A(x/t), \Gamma_2 \vdash \Delta_2\end{matrix}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

The CERES Method

- CERES is a cut-elimination method by resolution.
- Method consists of the following steps:
  1. Skolemization of the proof (if it is not already skolemized).
  2. Computation of the characteristic clause set.
  3. Refutation of the characteristic clause set.
  4. Computation of the Projections and construction of the Atomic Cut Normal Form.

The CERES Method (ctd.)

- if ρ is an axiom of the form $\Gamma_C, \Gamma \vdash \Delta_C, \Delta$, then
$$\mathrm{CL}_\rho(\psi) = \{\Gamma_C \vdash \Delta_C\}\,.$$
- if ρ is an unary rule with immediate predecessor ρ', then
$$\mathrm{CL}_\rho(\psi) = \mathrm{CL}_{\rho'}(\psi).$$
- if ρ is a binary rule with immediate predecessors ρ_1, ρ_2, then either
$$\mathrm{CL}_\rho(\psi) = \mathrm{CL}_{\rho_1}(\psi) \cup \mathrm{CL}_{\rho_2}(\psi)$$
or
$$\mathrm{CL}_\rho(\psi) = \mathrm{CL}_{\rho_1}(\psi) \times \mathrm{CL}_{\rho_2}(\psi).$$
- $\mathrm{CL}(\psi) = \mathrm{CL}_{\rho_0}(\psi)$.

An Example

$$\varphi_1 = \dfrac{\dfrac{\dfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u) \Rightarrow Q(u) \vdash (\exists y)(P(u) \Rightarrow Q(y))}\ \exists : r, \Rightarrow : r, \Rightarrow : l}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(u) \Rightarrow Q(y))}\ \forall : l}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\forall x)(\exists y)(P(x) \Rightarrow Q(y))}\ \forall : r$$

$$\varphi_2 = \dfrac{\dfrac{\dfrac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{P(a) \Rightarrow Q(v) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ \exists : r, \Rightarrow : r, \Rightarrow : l}{(\exists y)(P(a) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ \exists : l}{(\forall x)(\exists y)(P(x) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ \forall : l$$

$$\varphi = \dfrac{\varphi_1 \qquad \varphi_2}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ cut$$

An Example

$$\varphi_1 = \cfrac{\cfrac{\cfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u) \Rightarrow Q(u) \vdash (\exists y)(P(u) \Rightarrow Q(y))}\ \exists : r, \Rightarrow : r, \Rightarrow : l}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(u) \Rightarrow Q(y))}\ \forall : l}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\forall x)(\exists y)(P(x) \Rightarrow Q(y))}\ \forall : r$$

$$\varphi_2 = \cfrac{\cfrac{\cfrac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{P(a) \Rightarrow Q(v) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ \exists : r, \Rightarrow : r, \Rightarrow : l}{(\exists y)(P(a) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ \exists : l}{(\forall x)(\exists y)(P(x) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ \forall : l$$

$$\varphi = \cfrac{\varphi_1 \qquad \varphi_2}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ cut$$

An Example

$$\varphi_1 = \cfrac{\cfrac{\cfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u) \Rightarrow Q(u) \vdash (\exists y)(P(u) \Rightarrow Q(y))}\ \exists : r, \Rightarrow : r, \Rightarrow : l}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(u) \Rightarrow Q(y))}\ \forall : l}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\forall x)(\exists y)(P(x) \Rightarrow Q(y))}\ \forall : r$$

$$\varphi_2 = \cfrac{\cfrac{\cfrac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{P(a) \Rightarrow Q(v) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ \exists : r, \Rightarrow : r, \Rightarrow : l}{(\exists y)(P(a) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ \exists : l}{(\forall x)(\exists y)(P(x) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ \forall : l$$

$$\varphi = \cfrac{\varphi_1 \qquad \varphi_2}{\vdash}\ cut$$

An Example

$$\varphi_1 = \dfrac{\dfrac{\dfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u) \Rightarrow Q(u) \vdash (\exists y)(P(u) \Rightarrow Q(y))}\ \exists : r, \Rightarrow : r, \Rightarrow : l}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(u) \Rightarrow Q(y))}\ \forall : l}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\forall x)(\exists y)(P(x) \Rightarrow Q(y))}\ \forall : r$$

$$\varphi_2 = \dfrac{\dfrac{\dfrac{\vdash P(a) \quad Q(v) \vdash}{P(a) \Rightarrow Q(v) \vdash}\ \Rightarrow : l}{(\exists y)(P(a) \Rightarrow Q(y)) \vdash}\ \exists : l}{(\forall x)(\exists y)(P(x) \Rightarrow Q(y)) \vdash}\ \forall : l$$

$$\varphi = \dfrac{\varphi_1 \qquad \varphi_2}{\vdash}\ cut$$

An Example

$$\varphi_1 = \cfrac{\cfrac{\cfrac{P(u) \vdash Q(u)}{\vdash (\exists y)(P(u) \Rightarrow Q(y))}\ \exists : r, \Rightarrow : r}{\vdash (\exists y)(P(u) \Rightarrow Q(y))}}{\vdash (\forall x)(\exists y)(P(x) \Rightarrow Q(y))}\ \forall : r$$

$$\varphi_2 = \cfrac{\cfrac{\cfrac{\vdash P(a) \quad Q(v) \vdash}{P(a) \Rightarrow Q(v) \vdash}\ \Rightarrow : l}{(\exists y)(P(a) \Rightarrow Q(y)) \vdash}\ \exists : l}{(\forall x)(\exists y)(P(x) \Rightarrow Q(y)) \vdash}\ \forall : l$$

$$\varphi = \cfrac{\varphi_1 \qquad \varphi_2}{\vdash}\ cut$$

An Example

$$\mathrm{CL}(\varphi) = \{P(u) \vdash Q(u);\ \vdash P(a);\ Q(v) \vdash\}$$

refutation:

$$\dfrac{\dfrac{\vdash P(a) \quad P(u) \vdash Q(u)}{\vdash Q(a)}\ R \qquad Q(v) \vdash}{\vdash}\ R$$

$\sigma = \{u \leftarrow a, v \leftarrow a\}$
ground refutation:

$$\dfrac{\dfrac{\vdash P(a) \quad P(a) \vdash Q(a)}{\vdash Q(a)}\ R \qquad Q(a) \vdash}{\vdash}\ R$$

An Example

$$\varphi_1 = \dfrac{\dfrac{\dfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u) \Rightarrow Q(u) \vdash (\exists y)(P(u) \Rightarrow Q(y))}\ \exists: r, \Rightarrow: r, \Rightarrow: l}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(u) \Rightarrow Q(y))}\ \forall: l}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\forall x)(\exists y)(P(x) \Rightarrow Q(y))}\ \forall: r$$

$$\varphi_2 = \dfrac{\dfrac{\dfrac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{P(a) \Rightarrow Q(v) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ \exists: r, \Rightarrow: r, \Rightarrow: l}{(\exists y)(P(a) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ \exists: l}{(\forall x)(\exists y)(P(x) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ \forall: l$$

$$\varphi = \dfrac{\varphi_1 \qquad \varphi_2}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ cut$$

An Example

$$\dfrac{\dfrac{\dfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u) \Rightarrow Q(u), P(u) \vdash Q(u)} \Rightarrow: l}{(\forall x)(P(x) \Rightarrow Q(x)), P(u) \vdash Q(u)} \forall : l}{\varphi_1 = (\forall x)(P(x) \Rightarrow Q(x)), P(u) \vdash Q(u)}$$

$$\dfrac{\dfrac{\dfrac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{P(a) \Rightarrow Q(v) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \exists : r, \Rightarrow: r, \Rightarrow: l}{(\exists y)(P(a) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \exists : l}{\varphi_2 = (\forall x)(\exists y)(P(x) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \forall : l$$

$$\dfrac{\varphi_1 \qquad \varphi_2}{\varphi = (\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \, cut$$

An Example

$$\varphi_1 = \dfrac{\dfrac{\dfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u) \Rightarrow Q(u), P(u) \vdash Q(u)} \Rightarrow: l}{(\forall x)(P(x) \Rightarrow Q(x)), P(u) \vdash Q(u)} \forall : l}{(\forall x)(P(x) \Rightarrow Q(x)), P(u) \vdash Q(u)}$$

$$\varphi_2 = \dfrac{\dfrac{\dfrac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{P(a) \Rightarrow Q(v) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \exists : r, \Rightarrow: r, \Rightarrow: l}{(\exists y)(P(a) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \exists : l}{(\forall x)(\exists y)(P(x) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \forall : l$$

$$\varphi = \dfrac{\varphi_1 \qquad \varphi_2}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \; cut$$

An Example

$$\varphi_1 = \dfrac{\dfrac{\dfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u) \Rightarrow Q(u), P(u) \vdash Q(u)} \Rightarrow: l}{(\forall x)(P(x) \Rightarrow Q(x)), P(u) \vdash Q(u)} \forall : l}{(\forall x)(P(x) \Rightarrow Q(x)), P(u) \vdash Q(u)}$$

$$\varphi_2 = \dfrac{\dfrac{\dfrac{P(a) \vdash P(a)}{\vdash P(a), (\exists y)(P(a) \Rightarrow Q(y))} \exists : r, \Rightarrow: r, w : r}{\vdash P(a), (\exists y)(P(a) \Rightarrow Q(y))}}{\vdash P(a), (\exists y)(P(a) \Rightarrow Q(y))}$$

$$\varphi = \dfrac{\varphi_1 \qquad \varphi_2}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \; cut$$

An Example

$$\varphi_1 = \cfrac{\cfrac{\cfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u) \Rightarrow Q(u), P(u) \vdash Q(u)} \Rightarrow: l}{(\forall x)(P(x) \Rightarrow Q(x)), P(u) \vdash Q(u)} \forall : l}{(\forall x)(P(x) \Rightarrow Q(x)), P(u) \vdash Q(u)}$$

$$\varphi_2 = \cfrac{\cfrac{\cfrac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{P(a) \Rightarrow Q(v) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \exists : r, \Rightarrow: r, \Rightarrow: l}{(\exists y)(P(a) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \exists : l}{(\forall x)(\exists y)(P(x) \Rightarrow Q(y)) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \forall : l$$

$$\varphi = \cfrac{\varphi_1 \qquad \varphi_2}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \; cut$$

An Example

$$\varphi_1 = \dfrac{\dfrac{\dfrac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u) \Rightarrow Q(u), P(u) \vdash Q(u)} \Rightarrow: l}{(\forall x)(P(x) \Rightarrow Q(x)), P(u) \vdash Q(u)} \forall: l}{(\forall x)(P(x) \Rightarrow Q(x)), P(u) \vdash Q(u)}$$

$$\varphi_2 = \dfrac{\dfrac{\dfrac{Q(v) \vdash Q(v)}{Q(v) \vdash (\exists y)(P(a) \Rightarrow Q(y))}}{Q(v) \vdash (\exists y)(P(a) \Rightarrow Q(y))}}{Q(v) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \quad \exists: r, \Rightarrow: r, w: l$$

$$\varphi = \dfrac{\varphi_1 \qquad \varphi_2}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(a) \Rightarrow Q(y))} \; cut$$

An Example

$$\varphi(P(a) \vdash Q(a)) =$$

$$\dfrac{\dfrac{\dfrac{P(a) \vdash P(a) \quad Q(a) \vdash Q(a)}{P(a), P(a) \Rightarrow Q(a) \vdash Q(a)} \Rightarrow: l}{P(a), (\forall x)(P(x) \Rightarrow Q(x)) \vdash Q(a)} \forall : l}{P(a), (\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(a) \Rightarrow Q(y)), Q(a)} w : r$$

$$\varphi(\vdash P(a)) =$$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \vdash P(a)}{P(a) \vdash P(a), Q(v)} w : r}{\vdash P(a) \Rightarrow Q(v), P(a)} \Rightarrow: r}{\vdash (\exists y)(P(a) \Rightarrow Q(y)), P(a)} \exists : r}{(\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(a) \Rightarrow Q(y)), P(a)} w : l$$

An Example

$$\varphi(Q(a) \vdash) =$$

$$\dfrac{\dfrac{\dfrac{\dfrac{Q(a) \vdash Q(a)}{P(a), Q(a) \vdash Q(a)}\ w:l}{Q(a) \vdash P(a) \Rightarrow Q(a)}\ \Rightarrow: r}{Q(a) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ \exists: r}{Q(a), (\forall x)(P(x) \Rightarrow Q(x)) \vdash (\exists y)(P(a) \Rightarrow Q(y))}\ w:l$$

An Example

$\varphi(\gamma) =$

$$\dfrac{\dfrac{\dfrac{\varphi(\vdash P(a))}{B \vdash C, P(a)} \quad \dfrac{\varphi(P(a) \vdash Q(a))}{P(a), B \vdash C, Q(a)}}{B, B \vdash C, C, Q(a)}\ cut \quad \dfrac{\varphi(\vdash Q(a))}{Q(a), B \vdash C}}{\dfrac{B, B, B \vdash C, C, C}{B \vdash C}\ c : l, r*}\ cut$$

where $B = (\forall x)(P(x) \Rightarrow Q(x))$, $C = (\exists y)(P(a) \Rightarrow Q(y))$.

Some Results

Proposition (Unsatisfiability)

*Let π be an **LK**-proof. Then $\mathrm{CL}(\pi)$ is unsatisfiable.*

Proposition (Soundness of projection sets)

*Let π be an **LK**-proof with end-sequent S, then for all clauses $C \in \mathrm{CL}(\pi)$, there exists an **LK**-proof $\pi \in PR(\pi)$ with end-sequent $S \circ C$.*

Generality of CERES

- Universal tool for different logics:
  - CERES^{ω}: CERES for simple type theory.
  - CERES_m: CERES for finitely valued logics.
  - CERES_G: CERES for Gödel logic.
  - iCERES: CERES for intuitionistic logic.
- Limitation: cannot handle induction!

Cut-Elimination and Induction

- Induction: an infinitary modus ponens rule.
- Cut-elimination in the presence of induction: not possible.
- A solution: avoid induction using schemata.

Fürstenberg's Proof of the Infinity of Primes

- Formalized as infinite sequence of proofs.
- Euclid's argument is extracted.
- Problems:
  - Not fully automated (weakness of first-order provers).
  - Clause set and its refutation is described on the meta-level.
- Then how we profit from schemata?

Extension of **LK**

- Induction rule:

$$\frac{\Gamma \vdash \Delta, A(\bar{0}) \qquad \Pi, A(\alpha) \vdash \Lambda, A(s(\alpha))}{\Gamma, \Pi \vdash \Delta, \Lambda, A(t)}\, ind$$

- Equational rule:

$$\frac{S[t]}{S[t']}\, \mathcal{E}$$

with the condition that an equational theory $\mathcal{E} \models t = t'$.

A Motivating Example

- $\mathcal{E} = \{\hat{f}(0,x) = x,\ \hat{f}(s(n),x) = f(\hat{f}(n,x))\}$.
- $\mathcal{E} \models \hat{f}(n,x) = f^n(x)$.
- We prove S:

$$(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall n)((P(\hat{f}(n,c)) \Rightarrow P(g(n,c))) \Rightarrow (P(c) \Rightarrow P(g(n,c))))$$

A Motivating Example (ctd.)

- φ is:

$$\dfrac{\overset{(\psi)}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash C} \qquad \overset{(1)}{C \vdash (\forall n)((P(\hat{f}(n,c)) \Rightarrow P(g(n,c))) \Rightarrow (P(c) \Rightarrow P(g(n,c))))}}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall n)((P(\hat{f}(n,c)) \Rightarrow P(g(n,c))) \Rightarrow (P(c) \Rightarrow P(g(n,c))))}\ cut$$

- $C = (\forall n)(\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))$ and (1) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad \dfrac{P(\hat{f}(\beta,c)) \vdash P(\hat{f}(\beta,c)) \qquad P(g(\beta,c)) \vdash P(g(\beta,c))}{P(\hat{f}(\beta,c)) \Rightarrow P(g(\beta,c)), P(\hat{f}(\beta,c)) \vdash P(g(\beta,c))}\Rightarrow: l}{P(c), P(\hat{f}(\beta,c)) \Rightarrow P(g(\beta,c)), P(c) \Rightarrow P(\hat{f}(\beta,c)) \vdash P(g(\beta,c))}\Rightarrow: l}{P(\hat{f}(\beta,c)) \Rightarrow P(g(\beta,c)), P(c) \Rightarrow P(\hat{f}(\beta,c)) \vdash P(c) \Rightarrow P(g(\beta,c))}\Rightarrow: r}{P(c) \Rightarrow P(\hat{f}(\beta,c)) \vdash (P(\hat{f}(\beta,c)) \Rightarrow P(g(\beta,c))) \Rightarrow (P(c) \Rightarrow P(g(\beta,c)))}\Rightarrow: r}{(\forall n)(\forall x)(P(x) \Rightarrow P(\hat{f}(n,x))) \vdash (P(\hat{f}(\beta,c)) \Rightarrow P(g(\beta,c))) \Rightarrow (P(c) \Rightarrow P(g(\beta,c)))}\forall: l*}{(\forall n)(\forall x)(P(x) \Rightarrow P(\hat{f}(n,x))) \vdash (\forall n)((P(\hat{f}(n,c)) \Rightarrow P(g(n,c))) \Rightarrow (P(c) \Rightarrow P(g(n,c))))}\forall: r$$

A Motivating Example (ctd.)

- ψ is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(\hat{f}(\bar{0},u)) \vdash P(\hat{f}(\bar{0},u))}{P(u) \vdash P(\hat{f}(\bar{0},u))}\,\mathcal{E}}{\vdash P(u) \Rightarrow P(\hat{f}(\bar{0},u))}\,\Rightarrow: r}{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(\bar{0},x)))}\,\forall: r \qquad \begin{array}{c}(2)\\ {\color{blue}A}, (\forall x)(P(x) \Rightarrow P(\hat{f}(\alpha,x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(s(\alpha),x)))\end{array}}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(\gamma,x)))}\,ind}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall n)(\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))}\,\forall: r$$

- ${\color{blue}A = (\forall x)(P(x) \Rightarrow P(f(x)))}$ and (2) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(u) \vdash P(u) \qquad \dfrac{\dfrac{P(\hat{f}(\alpha,u)) \vdash P(\hat{f}(\alpha,u)) \qquad \dfrac{P(\hat{f}(s(\alpha),u)) \vdash P(\hat{f}(s(\alpha),u))}{P(f(\hat{f}(\alpha,u))) \vdash P(\hat{f}(s(\alpha),u))}\,\mathcal{E}}{P(\hat{f}(\alpha,u)) \Rightarrow P(f(\hat{f}(\alpha,u))), P(\hat{f}(\alpha,u)) \vdash P(\hat{f}(s(\alpha),u))}\,\Rightarrow: l}{(\forall x)(P(x) \Rightarrow P(f(x))), P(\hat{f}(\alpha,u)) \vdash P(\hat{f}(s(\alpha),u))}\,\forall: l}{P(u), (\forall x)(P(x) \Rightarrow P(f(x))), P(u) \Rightarrow P(\hat{f}(\alpha,u)) \vdash P(\hat{f}(s(\alpha),u))}\,\Rightarrow: l}{(\forall x)(P(x) \Rightarrow P(f(x))), P(u) \Rightarrow P(\hat{f}(\alpha,u)) \vdash P(u) \Rightarrow P(\hat{f}(s(\alpha),u))}\,\Rightarrow: r}{(\forall x)(P(x) \Rightarrow P(f(x))), (\forall x)(P(x) \Rightarrow P(\hat{f}(\alpha,x))) \vdash P(u) \Rightarrow P(\hat{f}(s(\alpha),u)))}\,\forall: l}{(\forall x)(P(x) \Rightarrow P(f(x))), (\forall x)(P(x) \Rightarrow P(\hat{f}(\alpha,x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(s(\alpha),x)))}\,\forall: r$$

A Motivating Example (ctd.)

- After some reduction steps:

$$\dfrac{\dfrac{\dfrac{(\psi)}{A \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(\beta,x)))}\,ind \qquad \dfrac{(1')}{(\forall x)(P(x) \Rightarrow P(\hat{f}(\beta,x))) \vdash B}}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(\hat{f}(\beta,c)) \Rightarrow P(g(\beta,c))) \Rightarrow (P(c) \Rightarrow P(g(\beta,c)))}\,cut}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall n)((P(\hat{f}(n,c)) \Rightarrow P(g(n,c))) \Rightarrow (P(c) \Rightarrow P(g(n,c))))}\,\forall : r$$

- Cannot proceed!

- In fact, there is no cut-free proof of S, induction on $(\forall n)((P(\hat{f}(n,c)) \Rightarrow P(g(n,c))) \Rightarrow (P(c) \Rightarrow P(g(n,c))))$ fails.

A Motivating Example (ctd.)

- The sequents S_n:

$$(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(\hat{f}(\bar{n}, c)) \Rightarrow P(g(\bar{n}, c))) \Rightarrow (P(c) \Rightarrow P(g(\bar{n}, c)))$$

  do have cut-free proofs for all $\bar{n}$.

- Uniform description of the sequence of cut-free proofs is needed.
- Develop machinery to obtain such a description.

Schematic Proof Systems

Language

- Consider two sorts ω, ι.
- Our language consists of:
  - arithmetical variables $i, j, k, n \colon \omega$,
  - first-order variables $x, y, z \colon \iota$,
  - schematic variables $u, v \colon \omega \to \iota$,
  - constant function symbols $f, g \colon \tau_1 \times \cdots \times \tau_n \to \tau$,
  - defined function symbols $\hat{f}, \hat{g} \colon \omega \times \tau_1 \times \cdots \times \tau_n \to \tau$,
  - predicate symbols P, Q and the logical connectives $\neg, \wedge, \vee, \Rightarrow, \forall, \exists, \bigwedge, \bigvee$.

Language (ctd.)

- Terms are defined in usual inductive fashion using variables and constant function symbols.
- Arithmetical terms are subset of terms constructed using $0 : \omega, s : \omega \to \omega, + : \omega \times \omega \to \omega$ and arithmetical variables.
- Formulas are defined in usual inductive fashion using predicate symbols and connectives $\neg, \wedge, \vee, \Rightarrow, \forall, \exists$ (quantification is allowed only on first-order variables).

Language (ctd.)

- Term schemata: terms and primitive recursion on terms using defined function symbols, i.e. for every $\hat{f}$:

$$\begin{array}{rcl} \hat{f}(0, x_1, \ldots, x_n) & \rightarrow & s, \\ \hat{f}(k+1, x_1, \ldots, x_n) & \rightarrow & t[\hat{f}(k, x_1, \ldots, x_n)] \end{array}$$

s.t. $V(s) \cup V(t) = \{x_1, \ldots, x_n\}$ and s, t are terms.

- Example: $\hat{f}(n, x)$ defining $f^n(x)$:

$$\begin{array}{rcl} \hat{f}(0, x) & \rightarrow & x, \\ \hat{f}(k+1, x) & \rightarrow & f(\hat{f}(k, x)). \end{array}$$

Language (ctd.)

- Formula schemata: formulas are formula schemata and if A is a formula schema, then $\bigwedge_{i=a}^{b} A$ and $\bigvee_{i=a}^{b} A$ are formula schemata as well.
- Example: $(\exists y)(\bigvee_{i=0}^{n}(\forall x)A(i,x,y))$ defining $(\exists y)((\forall x)A(0,x,y) \vee \cdots \vee (\forall x)A(n,x,y))$ which is equivalent to $(\exists y)((\forall x_0)A(0,x_0,y) \vee \cdots \vee (\forall x_n)A(n,x_n,y))$.

Calculus $\mathbf{LK}_s$

- Sequent: expression $S(x_1, \ldots, x_\alpha) : \Gamma \vdash \Delta$.
- Proof link: expression $\dfrac{(\varphi(a_1, \ldots, a_\alpha))}{S(a_1, \ldots, a_\alpha)}$
- Axioms: proof links or $A \vdash A$.
- Usual **LK** rules operating on formula schemata and the $\mathcal{E}$ rule.

Proof Schema

- Tuple of pairs of $\mathbf{LK}_s$-proofs.
- Each pair is associated with one proof symbol.
- Each pair corresponds to a base and step case of inductive definition.
- The proof symbols are ordered.

An Example

- Proof schema $\Psi = \langle \psi \rangle$ of $\bigvee_{i=0}^{n} (\forall x) P(i,x) \vdash \bigvee_{i=0}^{n} (\forall x) P(i,x)$, where ψ is associated to the pair:

$$\dfrac{\dfrac{P(0,u(0)) \vdash P(0,u(0))}{(\forall x)P(0,x) \vdash P(0,u(0))}\ \forall : l}{(\forall x)P(0,x) \vdash (\forall x)P(0,x)}\ \forall : r$$

and

$$\dfrac{\dfrac{\dfrac{(\psi(k))}{\bigvee_{i=0}^{k}(\forall x)P(i,x) \vdash \bigvee_{i=0}^{k}(\forall x)P(i,x)} \quad \dfrac{\dfrac{P(k+1,u(k+1)) \vdash P(k+1,u(k+1))}{(\forall x)P(k+1,x) \vdash P(k+1,u(k+1))}\ \forall : l}{(\forall x)P(k+1,x) \vdash (\forall x)P(k+1,x)}\ \forall : r}{\bigvee_{i=0}^{k+1}(\forall x)P(i,x) \vdash \bigvee_{i=0}^{k}(\forall x)P(i,x), (\forall x)P(k+1,x)}\ \vee : l}{\bigvee_{i=0}^{k+1}(\forall x)P(i,x) \vdash \bigvee_{i=0}^{k+1}(\forall x)P(i,x)}\ \vee : r$$

An Example (ctd.)

- $\Psi\downarrow_0$:

$$\dfrac{\dfrac{P(0,u_0) \vdash P(0,u_0)}{(\forall x)P(0,x) \vdash P(0,u_0)}\ \forall: l}{(\forall x)P(0,x) \vdash (\forall x)P(0,x)}\ \forall: r$$

An Example (ctd.)

- $\Psi\downarrow_1$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(0,u_0) \vdash P(0,u_0)}{(\forall x)P(0,x) \vdash P(0,u_0)}\ \forall: l}{(\forall x)P(0,x) \vdash (\forall x)P(0,x)}\ \forall: r \qquad \dfrac{\dfrac{P(1,u_1) \vdash P(1,u_1)}{(\forall x)P(1,x) \vdash P(1,u_1)}\ \forall: l}{(\forall x)P(1,x) \vdash (\forall x)P(1,x)}\ \forall: r}{(\forall x)P(0,x) \vee (\forall x)P(1,x) \vdash (\forall x)P(0,x), (\forall x)P(1,x)}\ \vee: l}{(\forall x)P(0,x) \vee (\forall x)P(1,x) \vdash (\forall x)P(0,x) \vee (\forall x)P(1,x)}\ \vee: r
$$

An Example (ctd.)

- $\Psi\downarrow_2$:

$$
\dfrac{
  \dfrac{
    \dfrac{\dfrac{P(0,u_0) \vdash P(0,u_0)}{(\forall x)P(0,x) \vdash P(0,u_0)}\ \forall : l}{(\forall x)P(0,x) \vdash (\forall x)P(0,x)}\ \forall : r
    \qquad
    \dfrac{\dfrac{P(1,u_1) \vdash P(1,u_1)}{(\forall x)P(1,x) \vdash P(1,u_1)}\ \forall : l}{(\forall x)P(1,x) \vdash (\forall x)P(1,x)}\ \forall : r
  }{
    \dfrac{(\forall x)P(0,x) \lor (\forall x)P(1,x) \vdash (\forall x)P(0,x), (\forall x)P(1,x)}{(\forall x)P(0,x) \lor (\forall x)P(1,x) \vdash (\forall x)P(0,x) \lor (\forall x)P(1,x)}\ \lor : r
  }\ \lor : l
  \qquad
  \dfrac{\dfrac{P(2,u_2) \vdash P(2,u_2)}{(\forall x)P(2,x) \vdash P(2,u_2)}\ \forall : l}{(\forall x)P(2,x) \vdash (\forall x)P(2,x)}\ \forall : r
}{
  \dfrac{(\forall x)P(0,x) \lor (\forall x)P(1,x) \lor (\forall x)P(2,x) \vdash (\forall x)P(0,x) \lor (\forall x)P(1,x), (\forall x)P(2,x)}{(\forall x)P(0,x) \lor (\forall x)P(1,x) \lor (\forall x)P(2,x) \vdash (\forall x)P(0,x) \lor (\forall x)P(1,x) \lor (\forall x)P(2,x)}\ \lor : r
}\ \lor : l
$$

Resolution Calculus $\mathcal{R}_s$

- Clauses and clause schemata.
- Clause-set terms and clause set schemata.
- Resolution terms and resolution proof schemata.
- Substitution schema.

Some Results

Proposition (Soundness)

$\mathbf{LK}_s$ *and* $\mathcal{R}_s$ *are sound.*

Proof.

Trivial by soundness of the inferences. □

Proposition

Unification problem is undecidable for term schemata.

The $CERES_s$ Method

Basic Notions

- Configuration Ω of ψ is a set of formula occurrences from the end-sequent of ψ.
- $\mathrm{cl}^{\psi,\Omega}$ is a unique symbol, called *clause-set symbol*.
- $\mathrm{pr}^{\psi,\Omega}$ is a unique symbol, called *projection symbol*.

An Example

- Let $\hat{f} : \omega \times \iota \to \iota$ and define $\hat{f}(n, x)$ as:

$$\begin{array}{rcl} \hat{f}(0, x) & \to & x, \\ \hat{f}(k+1, x) & \to & f(\hat{f}(k, x)) \end{array}$$

- Let $\Psi = \langle(\pi_1, \nu_1(k)), (\pi_2, \nu_2(k))\rangle$ be a proof schema of

$$(\forall x)(P(x) \Rightarrow P(f(x))) \vdash$$
$$(P(\hat{f}(n, c)) \Rightarrow P(g(n, c))) \Rightarrow (P(c) \Rightarrow P(g(n, c)))$$

An Example (ctd.)

- φ, associated with $(\pi_1, \nu_1(k))$, is:

$$\dfrac{\dfrac{(\psi(k+1))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))} \qquad (1)}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\ cut$$

where (1) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad \dfrac{P(\hat{f}(k+1,c)) \vdash P(\hat{f}(k+1,c)) \qquad P(g(k+1,c)) \vdash P(g(k+1,c))}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}\ \Rightarrow: l}{P(c), P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}\ \Rightarrow: l}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(c) \Rightarrow P(g(k+1,c))}\ \Rightarrow: r}{P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\ \Rightarrow: r}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\ \forall: l$$

An Example (ctd.)

- ψ, associated with $(\pi_2, \nu_2(k))$, is: π_2 (induction basis):

$$\cfrac{\cfrac{\cfrac{\cfrac{P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))}{P(u(0)) \vdash P(\hat{f}(0,u(0)))}\ \mathcal{E}}{\vdash P(u(0)) \Rightarrow P(\hat{f}(0,u(0)))}\ \Rightarrow: r}{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(0,x)))}\ \forall: r}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(0,x)))}\ w: l$$

- and $\nu_2(k)$ is:

$$\cfrac{\cfrac{(\psi(k))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k,x)))} \qquad (2)}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))}\ cut, c: l$$

An Example (ctd.)

where (2) is (induction step):

$$\dfrac{\dfrac{P(u(sk)) \vdash P(u(sk)) \quad \dfrac{\dfrac{P(\hat{f}(k,u(sk))) \vdash P(\hat{f}(k,u(sk))) \quad \dfrac{P(\hat{f}(sk,u(sk))) \vdash P(\hat{f}(sk,u(sk)))}{P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}\ \mathcal{E}}{P(\hat{f}(k,u(sk))), P(\hat{f}(k,u(sk))) \Rightarrow P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}\ \Rightarrow: l}{P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}\ \forall: l}{\dfrac{\dfrac{P(u(sk)), P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}{P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}\ \Rightarrow: r}{\dfrac{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(sk,x)))}\ \forall: r}\ \forall: l}\ \Rightarrow: l$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- π_2:

$$
\cfrac{\cfrac{\cfrac{\cfrac{P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))}{P(u(0)) \vdash P(\hat{f}(0,u(0)))}\ \mathcal{E}}{\vdash P(u(0)) \Rightarrow P(\hat{f}(0,u(0)))}\ \Rightarrow: r}{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(0,x)))}\ \forall: r}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(0,x)))}\ w: l
$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- π_2:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))}{P(u(0)) \vdash P(\hat{f}(0,u(0)))}\ \mathcal{E}}{\vdash P(u(0)) \Rightarrow P(\hat{f}(0,u(0)))}\ \Rightarrow: r}{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(0,x)))}\ \forall: r}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(0,x)))}\ w: l$$

$\mathrm{cl}^{\psi,\Omega}(0) = \{P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))\}$ and

$\mathrm{pr}^{\psi,\Omega}(0) = w_l(P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0))))$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.
- $\nu_2(k)$:

$$\dfrac{\dfrac{(\psi(k))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k,x)))} \quad (2)}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))}\ cut, c: l$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.
- $\nu_2(k)$:

$$\dfrac{\dfrac{(\psi(k))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k,x)))} \quad (2)}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))}\ cut, c: l$$

$$\mathrm{cl}^{\psi,\Omega}(k+1) = \mathrm{cl}^{\psi,\Omega}(k) \oplus$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- (2) is:

$$P(\hat{f}(sk,u(sk))) \vdash P(\hat{f}(sk,u(sk))) \quad \overline{}\ \mathcal{E}$$
$$P(\hat{f}(k,u(sk))) \vdash P(\hat{f}(k,u(sk))) \qquad P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk))) \quad \overline{}\ \Rightarrow: l$$
$$P(u(sk)) \vdash P(u(sk)) \qquad P(\hat{f}(k,u(sk))), P(\hat{f}(k,u(sk))) \Rightarrow P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk))) \quad \overline{}\ \forall: l$$
$$P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk))) \quad \overline{}\ \Rightarrow: l$$
$$P(u(sk)), P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk))) \quad \overline{}\ \Rightarrow: r$$
$$P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk))) \quad \overline{}\ \forall: l$$
$$(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk))) \quad \overline{}\ \forall: r$$
$$(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(sk,x)))$$

$$\mathrm{cl}^{\psi,\Omega}(k+1) = \mathrm{cl}^{\psi,\Omega}(k) \oplus$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- (2) is:

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(u(sk)) \vdash P(u(sk)) \quad \dfrac{\dfrac{P(\hat{f}(k,u(sk))) \vdash P(\hat{f}(k,u(sk))) \quad \dfrac{P(\hat{f}(sk,u(sk))) \vdash P(\hat{f}(sk,u(sk)))}{P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}\,\mathcal{E}}{P(\hat{f}(k,u(sk))), P(\hat{f}(k,u(sk))) \Rightarrow P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}\Rightarrow: l}{P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}\forall: l}{P(u(sk)), P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}\Rightarrow: l}{P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}\Rightarrow: r}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}\forall: l}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(sk,x)))}\forall: r
$$

$$
\mathrm{cl}^{\psi,\Omega}(k+1) = \mathrm{cl}^{\psi,\Omega}(k) \oplus \{P(u(sk)) \vdash P(u(sk))\} \oplus (\{P(\hat{f}(k,u(sk))) \vdash\} \otimes \{\vdash P(\hat{f}(sk,u(sk)))\})
$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- $\nu_2(k)$:

$$\dfrac{\dfrac{(\psi(k))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k,x)))} \quad (2)}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))}\ cut, c: l$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n, x)))\}$ for ψ.

- $\nu_2(k)$:

$$\dfrac{\dfrac{(\psi(k))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k, x)))} \quad (2)}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1, x)))}\ cut, c: l$$

$$\mathrm{pr}^{\psi,\Omega}(k+1) = c_l(w^{A\vdash}(\mathrm{pr}^{\psi,\Omega}(k)) \oplus w^{A\vdash}(\qquad))$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.
- (2) is:

$$
\dfrac{P(u(sk)) \vdash P(u(sk)) \qquad \dfrac{\dfrac{P(\hat{f}(k,u(sk))) \vdash P(\hat{f}(k,u(sk))) \qquad \dfrac{P(\hat{f}(sk,u(sk))) \vdash P(\hat{f}(sk,u(sk)))}{P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}\,\mathcal{E}}{P(\hat{f}(k,u(sk))), P(\hat{f}(k,u(sk))) \Rightarrow P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}\Rightarrow: l}{P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}\forall: l}{\dfrac{P(u(sk)), P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}{\dfrac{P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}{\dfrac{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(sk,x)))}\forall: r}\forall: l}\Rightarrow: r}\Rightarrow: l
$$

$$
\mathrm{pr}^{\psi,\Omega}(k+1) = c_l(w^{A\vdash}(\mathrm{pr}^{\psi,\Omega}(k)) \oplus w^{A\vdash}(\qquad))
$$

An Example (ctd.)

- (2) is:

$$
\dfrac{P(u(sk)) \vdash P(u(sk)) \qquad \dfrac{\dfrac{P(\hat{f}(k,u(sk))) \vdash P(\hat{f}(k,u(sk))) \qquad \dfrac{P(\hat{f}(sk,u(sk))) \vdash P(\hat{f}(sk,u(sk)))}{P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}\ \mathcal{E}}{P(\hat{f}(k,u(sk))), P(\hat{f}(k,u(sk))) \Rightarrow P(f(\hat{f}(k,u(sk)))) \vdash P(\hat{f}(sk,u(sk)))}\ \Rightarrow: l}{P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}\ \forall: l}{\dfrac{\dfrac{\dfrac{P(u(sk)), P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(\hat{f}(sk,u(sk)))}{P(u(sk)) \Rightarrow P(\hat{f}(k,u(sk))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}\ \Rightarrow: r}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(u(sk)) \Rightarrow P(\hat{f}(sk,u(sk)))}\ \forall: l}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k,x))), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(sk,x)))}\ \forall: r}\ \Rightarrow: l
$$

$$
\begin{aligned}
\mathrm{pr}^{\psi,\Omega}(k+1) = c_l(w^{A\vdash}(\mathrm{pr}^{\psi,\Omega}(k)) \oplus w^{A\vdash}(&\\
w^{A\vdash}(P(u(sk)) \vdash P(u(sk)))\oplus&\\
w^{\vdash}(\forall_l(P(\hat{f}(k,u(sk))) \vdash P(\hat{f}(k,u(sk)))\otimes_{\Rightarrow_l}&\\
P(\hat{f}(sk,u(sk))) \vdash P(\hat{f}(sk,u(sk))))))&
\end{aligned}
$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- $\nu_1(k)$:

$$\dfrac{\dfrac{(\psi(k+1))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))} \quad (1)}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\ cut$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n, x)))\}$ for ψ.

- $\nu_1(k)$:

$$\dfrac{\dfrac{(\psi(k+1))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1, x)))} \quad (1)}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(\hat{f}(k+1, c)) \Rightarrow P(g(k+1, c))) \Rightarrow (P(c) \Rightarrow P(g(k+1, c)))}\ cut$$

$$\mathrm{cl}^{\varphi,\emptyset}(k+1) = \mathrm{cl}^{\psi,\Omega}(k+1) \oplus$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- (1) is:

$$
\dfrac{P(c) \vdash P(c) \qquad \dfrac{P(\hat{f}(k+1,c)) \vdash P(\hat{f}(k+1,c)) \qquad P(g(k+1,c)) \vdash P(g(k+1,c))}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}\Rightarrow: l}{\dfrac{\dfrac{\dfrac{P(c), P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(c) \Rightarrow P(g(k+1,c))}\Rightarrow: r}{P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\Rightarrow: r}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\forall: l}\Rightarrow: l
$$

$$
\mathrm{cl}^{\varphi,\emptyset}(k+1) = \mathrm{cl}^{\psi,\Omega}(k+1) \oplus
$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- (1) is:

$$
\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad \dfrac{P(\hat{f}(k+1,c)) \vdash P(\hat{f}(k+1,c)) \qquad P(g(k+1,c)) \vdash P(g(k+1,c))}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}\Rightarrow: l}{\dfrac{P(c), P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(c) \Rightarrow P(g(k+1,c))}\Rightarrow: r}\Rightarrow: l}{P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\Rightarrow: r}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\forall: l
$$

$$
\mathrm{cl}^{\varphi,\emptyset}(k+1) = \mathrm{cl}^{\psi,\Omega}(k+1) \oplus \{\vdash P(c)\} \oplus (\{P(\hat{f}(k+1,c) \vdash\} \otimes \{\vdash\})
$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- For π_1:

$$\mathrm{cl}^{\varphi,\emptyset}(0) = \{P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))\} \oplus \{\vdash P(c)\} \oplus (\{P(\hat{f}(0,c) \vdash\} \otimes \{\vdash\})$$

An Example (ctd.)

- Two configurations: $\emptyset$ for φ, and $\Omega = \{\vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(n,x)))\}$ for ψ.

- $\nu_1(k)$:

$$\dfrac{\dfrac{(\psi(k+1))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))} \quad (1)}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))} \; cut$$

An Example (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$ and $B(k+1) = (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))$
- $\nu_1(k)$:

$$\dfrac{\dfrac{(\psi(k+1))}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x)))} \qquad (1)}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}\; cut$$

$$\mathrm{pr}^{\varphi,\emptyset}(k+1) = w^{\vdash B(k+1)}(\mathrm{pr}^{\psi,\Omega}(k+1)) \oplus w^{A\vdash}(\qquad)$$

An Example (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$ and $B(k+1) = (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))$
- (1) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \quad \dfrac{P(\hat{f}(k+1,c)) \vdash P(\hat{f}(k+1,c)) \quad P(g(k+1,c)) \vdash P(g(k+1,c))}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))} \Rightarrow: l}{P(c), P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))} \Rightarrow: l}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(c) \Rightarrow P(g(k+1,c))} \Rightarrow: r}{P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))} \Rightarrow: r$$

$$\dfrac{P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))} \forall: l$$

$$\mathrm{pr}^{\varphi,\emptyset}(k+1) = w^{\vdash B(k+1)}(\mathrm{pr}^{\psi,\Omega}(k+1)) \oplus w^{A\vdash}(\qquad)$$

An Example (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$ and $B(k+1) = (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))$
- (1) is:

$$\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad \dfrac{P(\hat{f}(k+1,c)) \vdash P(\hat{f}(k+1,c)) \qquad P(g(k+1,c)) \vdash P(g(k+1,c))}{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))} \Rightarrow: l}{P(c), P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(g(k+1,c))} \Rightarrow: l}{\dfrac{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)), P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash P(c) \Rightarrow P(g(k+1,c))}{\dfrac{P(c) \Rightarrow P(\hat{f}(k+1,c)) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))}{(\forall x)(P(x) \Rightarrow P(\hat{f}(k+1,x))) \vdash (P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c))) \Rightarrow (P(c) \Rightarrow P(g(k+1,c)))} \forall: l} \Rightarrow: r} \Rightarrow: r$$

$$\mathrm{pr}^{\varphi,\emptyset}(k+1) = w^{\vdash B(k+1)}(\mathrm{pr}^{\psi,\Omega}(k+1)) \oplus w^{A\vdash}($$
$$\Rightarrow_r (\Rightarrow_r (w^{P(\hat{f}(k+1,c)) \Rightarrow P(g(k+1,c)) \vdash P(g(k+1,c))}(P(c) \vdash P(c)) \oplus$$
$$w^{P(c)\vdash}(P(\hat{f}(k+1,c)) \vdash P(\hat{f}(k+1,c)) \otimes_{\Rightarrow_l}$$
$$P(g(k+1,c)) \vdash P(g(k+1,c))))))$$

An Example (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$
- Let $B(0) = (P(\hat{f}(0,c)) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))$
- For π_1:

$$\mathrm{pr}^{\varphi,\emptyset}(0) = w^{\vdash B(0)}(w_l(P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))))\oplus$$
$$w^{A\vdash}(\Rightarrow_r (\Rightarrow_r (w^{P(\hat{f}(0,c))\Rightarrow P(g(0,c))\vdash P(g(0,c))}(P(c) \vdash P(c))\oplus$$
$$w^{P(c)\vdash}(P(\hat{f}(0,c)) \vdash P(\hat{f}(0,c)) \otimes_{\Rightarrow_l} P(g(0,c)) \vdash P(g(0,c))))))$$

An Example (ctd.)

- $\mathrm{CL}(\Psi) = (\mathrm{cl}^{\varphi,\emptyset}, \mathrm{cl}^{\psi,\Omega})$ where

$$
\begin{aligned}
\mathrm{cl}^{\varphi,\emptyset}(0) &\rightarrow \{P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))\} \oplus \\
&\quad \{\vdash P(c)\} \oplus (\{P(\hat{f}(0,c) \vdash\} \otimes \{\vdash\}) \\
\mathrm{cl}^{\varphi,\emptyset}(k+1) &\rightarrow \mathrm{cl}^{\psi,\Omega}(k+1) \oplus \{\vdash P(c)\} \oplus \\
&\quad (\{P(\hat{f}(k+1,c) \vdash\} \otimes \{\vdash\}) \\
\mathrm{cl}^{\psi,\Omega}(0) &\rightarrow \{P(\hat{f}(0,u(0))) \vdash P(\hat{f}(0,u(0)))\} \\
\mathrm{cl}^{\psi,\Omega}(k+1) &\rightarrow \mathrm{cl}^{\psi,\Omega}(k) \oplus \{P(u(k+1)) \vdash P(u(k+1))\} \oplus \\
&\quad (\{P(\hat{f}(k,u(k+1))) \vdash\} \otimes \\
&\quad \{\vdash P(\hat{f}(k+1,u(k+1)))\})
\end{aligned}
$$

An Example (ctd.)

- The sequence of $\mathrm{CL}(\Psi)\downarrow_0, \mathrm{CL}(\Psi)\downarrow_1, \mathrm{CL}(\Psi)\downarrow_2, \ldots$ is:

$$\{P(u_0) \vdash P(u_0)\ ;\ \vdash P(c)\ ;\ P(c) \vdash\},$$

$$\{P(u_0) \vdash P(u_0)\ ;\ P(f(u_1)) \vdash P(f(u_1))\ ;\ P(u_1) \vdash P(f(u_1))\ ;\ \vdash P(c)\ ;\ P(f(c)) \vdash\},$$

$$\{P(u_0) \vdash P(u_0)\ ;\ P(f(u_1)) \vdash P(f(u_1))\ ;\ P(f(f(u_2))) \vdash P(f(f(u_2)))\ ;\ P(u_1) \vdash P(f(u_1))\ ;\ P(f(u_2)) \vdash P(f(f(u_2)))\ ;\ \vdash P(c)\ ;\ P(f(f(c))) \vdash\}, \ldots$$

- $\mathrm{CL}(\Psi)\downarrow_\gamma$ boils down to $\{P(u_1) \vdash P(f(u_1)); \vdash P(c); P(f^\gamma(c)) \vdash\}$.

An Example (ctd.)

- $PR(\Psi)\downarrow_0 : \Bigg\{$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c)}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(c), P(g(0,c))}\; w\colon l,r}{P(c) \Rightarrow P(g(0,c)) \vdash P(c), P(c) \Rightarrow P(g(0,c))}\; \Rightarrow\colon r}{\vdash P(c), (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))}\; \Rightarrow\colon r}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash P(c), (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))}\; w\colon l$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad P(g(0,c)) \vdash P(g(0,c))}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))}\; \Rightarrow\colon l}{P(c), P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))}\; w\colon l}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(c) \Rightarrow P(g(0,c))}\; \Rightarrow\colon r}{P(c) \vdash (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))}\; \Rightarrow\colon r}{P(c), (\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))}\; w\colon l$$

$$\dfrac{P(u_0) \vdash P(u_0)}{(\forall x)(P(x) \Rightarrow P(f(x))), P(u_0) \vdash P(u_0), (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))}\; w\colon l,r \quad \Bigg\}$$

Some Results

Proposition (Commutativity)

For all $\gamma \in \mathbb{N}$*:*

- $\mathrm{CL}(\Psi \downarrow_{\gamma}) = \mathrm{CL}(\Psi) \downarrow_{\gamma}$,
- $PR(\Psi \downarrow_{\gamma}) = PR(\Psi) \downarrow_{\gamma}$.

Proof.

By double induction. □

Some Results (ctd.)

Proposition (Unsatisfiability for schema)

$\mathrm{CL}(\Psi)\downarrow_\gamma$ *is unsatisfiable for all* $\gamma \in \mathbb{N}$ *(i.e.* $\mathrm{CL}(\Psi)$ *is unsatisfiable).*

Proof.

$\mathrm{CL}(\Psi)\downarrow_\alpha = \mathrm{CL}(\Psi\downarrow_\alpha)$. □

Some Results (ctd.)

Proposition (Correctness)

Let $\gamma \in \mathbb{N}$, then for every clause $C \in \mathrm{CL}(\Psi){\downarrow_\gamma}$ there exists an $\mathbf{LK}_s$-proof $\pi \in PR(\Psi){\downarrow_\gamma}$ with end-sequent $C \circ S(\gamma)$.

Proof.

$\mathrm{PR}(\Psi) \downarrow_\alpha = \mathrm{PR}(\Psi \downarrow_\alpha)$. □

Resolution to $\mathbf{LK}_s$-skeleton Transformation

Let ϱ be a normalized resolution refutation. Then the transformation $TR(\varrho)$ is defined inductively:

- if $\varrho = C$ for a clause C, then $TR(\varrho) = C$,
- if $\varrho = r(\varrho_1; \varrho_2; P)$, then $TR(\varrho)$ is:

$$\dfrac{\dfrac{\dfrac{(TR(\varrho_1))}{\Gamma \vdash \Delta, P, \ldots, P}}{\Gamma \vdash \Delta, P}\, c\colon r* \qquad \dfrac{\dfrac{(TR(\varrho_2))}{P, \ldots, P, \Pi \vdash \Lambda}}{P, \Pi \vdash \Lambda}\, c\colon l*}{\Gamma, \Pi \vdash \Delta, \Lambda}\, cut$$

Atomic Cut Normal Form

- Substitute each clause at the leaf nodes of this skeleton by the corresponding projections.
- Propagate contexts downwards.
- Append necessary contractions at the end of the proof.

An Example (ctd.)

- The resolution refutation schema $R = (\varrho, \delta)$ where
  - $\varrho(0, u) \rightarrow r(\delta(0, u);\ P(\hat{f}(0, c)) \vdash;\ P(\hat{f}(0, c)))$,
  - $\varrho(k+1, u) \rightarrow r(\delta(k+1, u);\ P(\hat{f}(k+1, c)) \vdash;\ P(\hat{f}(k+1, c)))$,
  - $\delta(0, u) \rightarrow \vdash P(c)$,
  - $\delta(k{+}1, u) \rightarrow r(\delta(k, u);\ P(u(k{+}1)) \vdash P(f(u(k{+}1)));\ P(\hat{f}(k, c)))$.
- Let $\hat{pre} : \omega \rightarrow \omega$ be a defined function symbol, then define the function $\hat{pre}(n)$ as:
  - $\hat{pre}(0) \rightarrow 0$, and
  - $\hat{pre}(k+1) \rightarrow k$.
- $\theta = \{u \leftarrow \lambda k.\hat{f}(\hat{pre}(k), c)\}$.

An Example (ctd.)

- $\varrho(n,u)\theta\downarrow_\gamma$ is resolution refutation for all $\gamma \in \mathbb{N}$.
- $\varrho(n,u)\theta\downarrow_0 = r(\vdash P(c)\ ;\ P(c)\vdash\ ;\ P(c))$, and
- $TR(\varrho(n,u)\theta\downarrow_0)$ is:

$$\frac{\vdash P(c) \qquad P(c)\vdash}{\vdash}\ cut$$

An Example (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$ and $B = (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))$, then:

$$\dfrac{\vdash P(c) \qquad P(c) \vdash}{\vdash}\ cut$$

An Example (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$ and $B = (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))$, then:

$$
\dfrac{\vdash P(c) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad P(g(0,c)) \vdash P(g(0,c))}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))}\Rightarrow : l}{P(c), P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))}w : l}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(c) \Rightarrow P(g(0,c))}\Rightarrow : r}{P(c) \vdash B}\Rightarrow : r}{P(c), A \vdash B}w : l}{A \vdash B}cut
$$

An Example (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$ and
  $B = (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))$, then:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c)}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(c), P(g(0,c))}\, w: l, r}{P(c) \Rightarrow P(g(0,c)) \vdash P(c), P(c) \Rightarrow P(g(0,c))}\, \Rightarrow: r}{\vdash P(c), B}\, \Rightarrow: r}{A \vdash P(c), B}\, w: l
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad P(g(0,c)) \vdash P(g(0,c))}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))}\, \Rightarrow: l}{P(c), P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))}\, w: l}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(c) \Rightarrow P(g(0,c))}\, \Rightarrow: r}{P(c) \vdash B}\, \Rightarrow: r}{P(c), A \vdash B}\, w: l
}{A, A \vdash B, B}\, cut
$$

An Example (ctd.)

- Let $A = (\forall x)(P(x) \Rightarrow P(f(x)))$ and $B = (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))$, then:

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c)}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(c), P(g(0,c))}\ w\colon l, r}{P(c) \Rightarrow P(g(0,c)) \vdash P(c), P(c) \Rightarrow P(g(0,c))}\ \Rightarrow\colon r}{\vdash P(c), B}\ \Rightarrow\colon r}{A \vdash P(c), B}\ w\colon l
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad P(g(0,c)) \vdash P(g(0,c))}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))}\ \Rightarrow\colon l}{P(c), P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(g(0,c))}\ w\colon l}{P(c) \Rightarrow P(g(0,c)), P(c) \vdash P(c) \Rightarrow P(g(0,c))}\ \Rightarrow\colon r}{P(c) \vdash B}\ \Rightarrow\colon r}{P(c), A \vdash B}\ w\colon l
$$

$$
\dfrac{\dfrac{A \vdash P(c), B \qquad P(c), A \vdash B}{A, A \vdash B, B}\ cut}{(\forall x)(P(x) \Rightarrow P(f(x))) \vdash (P(c) \Rightarrow P(g(0,c))) \Rightarrow (P(c) \Rightarrow P(g(0,c)))}\ c\colon l, r
$$

Main Theorem

Theorem (ACNF)

Let Ψ be a proof schema with end-sequent $S(n)$, and let R be a resolution refutation schema of $\mathrm{CL}(\Psi)$. Then for all $\alpha \in \mathbb{N}$ there exists a normalized $\mathbf{LK}_s$-proof π of $S(\alpha)$ with at most atomic cuts such that its size $l(\pi)$ is polynomial in $l(R{\downarrow_\alpha}) \cdot l(PR(\Psi){\downarrow_\alpha})$.

- Drawback: the method is inherently incomplete.

Summary

Whole $CERES_s$ Procedure

- Phase 1 of $CERES_s$ (schematic construction):
  - compute $\mathrm{CL}(\Psi)$;
  - compute $\mathrm{PR}(\Psi)$;
  - construct a resolution refutation schema $\mathcal{R}$ of $\mathrm{CL}(\Psi)$ and a substitution schema ϑ.
  - then ACNF schema is $(\mathrm{PR}(\Psi), \mathcal{R}, \vartheta)$.
- Phase 2 of $CERES_s$ (evaluation, given a number α):
  - compute $\mathrm{PR}(\Psi)\downarrow_\alpha$;
  - compute $\mathcal{R}\vartheta\downarrow_\alpha$ and $T_\alpha : TR(\mathcal{R}\vartheta\downarrow_\alpha)$;
  - append the corresponding projections in $\mathrm{PR}(\Psi)\downarrow_\alpha$ to T_α, propagate the contexts down and append necessary contractions at the end of the proof.

Future Work

- Extract valuable information such as Herbrand sequent from the ACNF schema.
- Handle Fürstenberg's proof (equality is needed).
- Investigate the resolution calculus (decidable fragments, etc.).
- Extend proof schema systems and the method to multiple parameters.

Questions?