CERES for Propositional Proof Schemata

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Workshop on Structural and Computational Proof Theory,
Innsbruck, Austria.

October 27, 2011.

Introduction

Overview

- Schemata are very useful in mathematical proofs (avoids explicit use of the induction).
- Schemata are used on meta-level.
- Many problems can be expressed in propositional schema language, like:
  - Circuit verification,
  - Graph coloring,
  - Pigeonhole principle, etc.

Propositional Schema Language

- Set of index variables is a set of variables over natural numbers.
- Linear arithmetic expression is as usual built on the signature $0, s, +, -$ and on a set of index variables.
- Indexed proposition is an expression of the form p_a, where a is a linear arithmetic expression.
- Propositional variable is an indexed proposition p_a, where $a \in \mathbb{N}$.

Syntax

- Formula schema is defined inductively:
  - Indexed proposition is a formula schema.
  - If ϕ_1 and ϕ_2 are formula schemata, then so are $\phi_1 \vee \phi_2$, $\phi_1 \wedge \phi_2$ and $\neg\phi_1$.
  - If ϕ is a formula schema, a, b are linear arithmetic expressions and i is an index variable, then $\bigwedge_{i=a}^{b} \phi$ and $\bigvee_{i=a}^{b} \phi$ are formula schemata, called iterations.

Semantics

- Interpretation is a pair of functions, $I = (\mathcal{I}, \mathcal{I}_p)$, s.t. $\mathcal{I}$ maps index variables to natural numbers and $\mathcal{I}_p$ maps propositional variables to truth values.
- Truth value $[\![\phi]\!]_I$ of a formula schema ϕ in an interpretation I is defined inductively:
  - $[\![p_a]\!]_I = \mathcal{I}_p(p_{\mathcal{I}(a)})$.
  - $[\![\neg\phi]\!]_I = \mathbf{T}$ iff $[\![\phi]\!]_I = \mathbf{F}$.
  - $[\![\phi_1 \wedge (\vee) \phi_2]\!]_I = \mathbf{T}$ iff $[\![\phi_1]\!]_I = \mathbf{T}$ and (or) $[\![\phi_2]\!]_I = \mathbf{T}$.
  - $\left[\!\!\left[\bigwedge_{i=a}^{b} \left(\bigvee_{i=a}^{b}\right) \phi\right]\!\!\right]_I = \mathbf{T}$ iff for every (there is an) integer α s.t. $\mathcal{I}(a) \leq \alpha \leq \mathcal{I}(b)$, $[\![\phi]\!]_{I[\alpha/i]} = \mathbf{T}$.

Cut-Elimination on Proof Schemata

Aim: describe syntactically sequence of cut-free proofs $(\chi_n)_{n\in\mathbb{N}}$ obtained by cut-elimination on proof sequences $(\varphi_n)_{n\in\mathbb{N}}$.

- Cut-free proofs of schema typically are described in meta-language.
- Find object language to define sequence $(\chi_n)_{n\in\mathbb{N}}$.

Which cut-elimination method?

- Reductive cut-elimination.
- CERES.
    - Efficient.
    - Strong methods of redundancy-elimination.
    - Atomic cut-normal form is constructed via parts of the original proof.

The CERES Method

- CERES is a cut-elimination method by resolution.
- Method consists of the following steps:
  1. Skolemization of the proof (if it is not already skolemized).
  2. Computation of the characteristic clause set.
  3. Refutation of the characteristic clause set.
  4. Computation of the Projections and construction of the Atomic Cut Normal Form.

Schematic LK

Basic Notions

- Sequent Schema is an expression of the form $\Gamma \vdash \Delta$, where Γ and Δ are multisets of formula schemata.
- Initial Sequent Schema is an expression of the form $A \vdash A$, where A is an indexed proposition.
- Proof Link is a tuple (φ, t), where φ is a proof name and t is a linear arithmetic expression.

Calculus **LKS**

- Axioms: initial sequent schemata or proof links.
- Rules:

Calculus **LKS**

- Axioms: initial sequent schemata or proof links.
- Rules: $\wedge$ introduction:

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge : l1 \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge : l2$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Pi \vdash \Lambda, B}{\Gamma, \Pi \vdash \Delta, \Lambda, A \wedge B} \wedge : r$$

Equivalences: $A_0 \equiv \bigwedge_{i=0}^{0} A_i$ and $(\bigwedge_{i=0}^{n} A_i) \wedge A_{n+1} \equiv \bigwedge_{i=0}^{n+1} A_i$

Calculus **LKS**

- Axioms: initial sequent schemata or proof links.
- Rules: $\vee$ introduction:

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Pi \vdash \Lambda}{A \vee B, \Gamma, \Pi \vdash \Delta, \Lambda} \vee : l$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \vee : r1 \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B} \vee : r2$$

Equivalences: $A_0 \equiv \bigvee_{i=0}^{0} A_i$ and $(\bigvee_{i=0}^{n} A_i) \vee A_{n+1} \equiv \bigvee_{i=0}^{n+1} A_i$

Calculus **LKS**

- Axioms: initial sequent schemata or proof links.
- Rules: $\neg$ introduction:

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\neg : l \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\neg : r$$

Calculus **LKS**

- Axioms: initial sequent schemata or proof links.
- Rules: Weakening rules:

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\, w : l \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\, w : r$$

Calculus **LKS**

- Axioms: initial sequent schemata or proof links.
- Rules: Contraction rules:

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ c\colon l \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ c\colon r$$

Calculus **LKS**

- Axioms: initial sequent schemata or proof links.
- Rules: Cut rule:

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ cut$$

LKS-proof

- Derivation is a directed tree with nodes as sequences and edges as rules.
- **LKS**-proof of the sequence S is a derivation of S with axioms as leaf nodes.
- An **LKS**-proof is called ground if it does not contain free parameters, index variables, or proof links.

Proof Schemata

- Proof schema ψ is a tuple of pairs $\langle(\psi^1_{\text{base}}, \psi^1_{\text{step}}), \ldots, (\psi^m_{\text{base}}, \psi^m_{\text{step}})\rangle$ such that:
  - $\psi^1 \prec \psi^2 \prec \cdots \prec \psi^m$,
  - ψ^i_{base} is a ground **LKS**-proof of $S^i\,\{n \leftarrow 0\}$, for $i \in \{1, \ldots, m\}$,
  - ψ^i_{step} is an **LKS**-proof of $S^i\,\{n \leftarrow k+1\}$, where k is an index variable, and ψ^i_{step} contains proof links of the form (for $i \prec j$):

$$\frac{(\psi^i, k)}{S^i\,\{n \leftarrow k\}} \quad \text{or} \quad \frac{(\psi^j, k^j)}{S^j\,\{n \leftarrow k^j\}}$$

- From now on $m = 1$.

Proof Evaluation

- An evaluation of a proof schema ψ is a ground **LKS**-proof $eval(\psi, k)$, defined inductively:
  - $eval(\psi, 0) = \psi_{\text{base}}$, and
  - $eval(\psi, i + 1)$ is defined as ψ_{step} with end-sequent $S\{k \leftarrow i\}$ and every proof link to (ψ, k) in ψ_{step} are replaced by $eval(\psi, i)$.

An Example

- ψ_{base}:

$$\dfrac{\dfrac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash}\ \neg : l \qquad A_1 \vdash A_1}{A_0, \neg A_0 \vee A_1 \vdash A_1}\ \vee : l$$

- ψ_{step}:

$$\dfrac{\dfrac{\dfrac{(\psi, k)}{A_0, \bigwedge_{i=0}^{k}(\neg A_i \vee A_{i+1}) \vdash A_{k+1}} \qquad \dfrac{\dfrac{A_{k+1} \vdash A_{k+1}}{\neg A_{k+1}, A_{k+1} \vdash}\ \neg : l \qquad A_{k+2} \vdash A_{k+2}}{A_{k+1}, \neg A_{k+1} \vee A_{k+2} \vdash A_{k+2}}\ \vee : l}{A_0, \bigwedge_{i=0}^{k}(\neg A_i \vee A_{i+1}), \neg A_{k+1} \vee A_{k+2} \vdash A_{k+2}}\ cut}{A_0, \bigwedge_{i=0}^{k+1}(\neg A_i \vee A_{i+1}) \vdash A_{k+2}}\ \wedge : l$$

An Example (ctd.)

- $eval(\psi, 0)$:

$$\dfrac{\dfrac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash}\ \neg : l \qquad A_1 \vdash \mathbf{A_1}}{A_0, \neg A_0 \vee A_1 \vdash \mathbf{A_1}}\ \vee : l$$

- $eval(\psi, 1)$:

$$\dfrac{\dfrac{\dfrac{(eval(\psi,0))}{A_0, \bigwedge_{i=0}^{0}(\neg A_i \vee A_{i+1}) \vdash \mathbf{A_1}} \qquad \dfrac{\dfrac{\mathbf{A_1} \vdash A_1}{\neg A_1, \mathbf{A_1} \vdash}\ \neg : l \qquad A_2 \vdash A_2}{\mathbf{A_1}, \neg A_1 \vee A_2 \vdash A_2}\ \vee : l}{A_0, \bigwedge_{i=0}^{0}(\neg A_i \vee A_{i+1}), \neg A_1 \vee A_2 \vdash A_2}\ cut}{A_0, \bigwedge_{i=0}^{1}(\neg A_i \vee A_{i+1}) \vdash A_2}\ \wedge : l$$

Schematic Characteristic Clause Set

Basic Notions

- Cut-configuration Ω of ψ is a set of formula occurrences from the end-sequent of ψ.
- $\mathrm{cl}_k^{\Omega,\psi}$ is an unique indexed proposition symbol for all cut-configurations Ω of ψ.
- The intended semantics of $\mathrm{cl}_k^{\Omega,\psi}$ will be "the characteristic clause set of $eval(\psi, k)$, with the cut-configuration Ω".

Characteristic Clause Set

$\mathrm{CL}_\rho(\psi, \Omega)$ is defined inductively:

- if ρ is an axiom of the form $\Gamma_\Omega, \Gamma_C, \Gamma \vdash \Delta_\Omega, \Delta_C, \Delta$, then

$$\mathrm{CL}_\rho(\psi, \Omega) = \{\Gamma_\Omega, \Gamma_C \vdash \Delta_\Omega, \Delta_C\}\,.$$

- if ρ is a proof link of the form

$$\frac{(\psi, t)}{\Gamma_\Omega, \Gamma_C, \Gamma \vdash \Delta_\Omega, \Delta_C, \Delta}$$

then

$$\mathrm{CL}_\rho(\psi, \Omega) = \{\vdash \mathrm{cl}_t^{\Omega', \psi}\}.$$

Characteristic Clause Set (ctd.)

- if ρ is an unary rule with immediate predecessor ρ', then

$$\mathrm{CL}_{\rho}(\psi, \Omega) = \mathrm{CL}_{\rho'}(\psi, \Omega).$$

- if ρ is a binary rule with immediate predecessors ρ_1, ρ_2, then either

$$\mathrm{CL}_{\rho}(\psi, \Omega) = \mathrm{CL}_{\rho_1}(\psi, \Omega) \cup \mathrm{CL}_{\rho_2}(\psi, \Omega)$$

or

$$\mathrm{CL}_{\rho}(\psi, \Omega) = \mathrm{CL}_{\rho_1}(\psi, \Omega) \otimes \mathrm{CL}_{\rho_2}(\psi, \Omega).$$

Characteristic Clause Set (ctd.)

- $\mathrm{CL}(\psi, \Omega) = \mathrm{CL}_\rho(\psi, \Omega)$, where ρ is the last inference of ψ.
- $\mathrm{CL}(\varphi) = \mathrm{CL}(\varphi, \emptyset)$, where φ is a ground **LKS**-proof.
- $\mathrm{CL}_{\mathrm{base}} = \bigcup_\Omega(\{\mathrm{cl}_0^{\Omega,\psi} \vdash\} \otimes \mathrm{CL}(\psi_{\mathrm{base}}, \Omega))$.
- $\mathrm{CL}_{\mathrm{step}} = \bigcup_\Omega(\{\mathrm{cl}_{k+1}^{\Omega,\psi} \vdash\} \otimes \mathrm{CL}(\psi_{\mathrm{step}}, \Omega))$, for $0 \leq k \leq n$.
- $\mathrm{CL}_{\mathbf{s}}(\psi) = \{\vdash \mathrm{cl}_n^{\emptyset,\psi}\} \cup \mathrm{CL}_{\mathrm{base}} \cup \mathrm{CL}_{\mathrm{step}}$.

Unsatisfiability of $\mathrm{CL}_{\mathbf{s}}(\psi)$

Lemma (2.1)

Let C be a clause and $\mathcal{C}$ be a clause set. Then an interpretation $I \models \{C\} \otimes \mathcal{C}$ iff $I \models C$ or $I \models \mathcal{C}$.

Lemma (2.2)

Let ψ be a proof schema and $\mathrm{CL}(\psi, \Omega)$ be a characteristic clause set as defined above. Assume that for all cut-configurations Ω, $I \models \mathrm{cl}_i^{\Omega,\psi}$ implies $I \models \mathrm{CL}(eval(\psi, i), \Omega)$. Then $I \models \mathrm{CL}(\psi_{\mathrm{step}}\, \{k \leftarrow i\}\, , \Omega)$ implies $I \models \mathrm{CL}(eval(\psi, i+1), \Omega)$.

Unsatisfiability of $\mathrm{CL}_\mathbf{s}(\psi)$ (ctd.)

Proposition (2.1)

Let φ be a ground **LKS***-proof. Then* $\mathrm{CL}(\varphi)$ *is unsatisfiable.*

Proposition (2.2)

If $I \models \mathrm{CL}_\mathbf{s}(\psi)$ *then* $I \models \mathrm{CL}(eval(\psi, I(n)))$.

Corollary (2.1)

Let ψ be a proof schema and $\mathrm{CL}_\mathbf{s}(\psi)$ *its characteristic clause set. Then* $\mathrm{CL}_\mathbf{s}(\psi)$ *is unsatisfiable.*

An Example

- ψ_{base}:

$$\dfrac{\dfrac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash}\neg : l \qquad A_1 \vdash A_1}{A_0, \neg A_0 \vee A_1 \vdash A_1}\vee : l$$

- ψ_{step}:

$$\dfrac{\dfrac{\dfrac{(\psi, k)}{A_0, \bigwedge_{i=0}^{k}(\neg A_i \vee A_{i+1}) \vdash A_{k+1}} \qquad \dfrac{\dfrac{A_{k+1} \vdash A_{k+1}}{\neg A_{k+1}, A_{k+1} \vdash}\neg : l \qquad A_{k+2} \vdash A_{k+2}}{A_{k+1}, \neg A_{k+1} \vee A_{k+2} \vdash A_{k+2}}\vee : l}{A_0, \bigwedge_{i=0}^{k}(\neg A_i \vee A_{i+1}), \neg A_{k+1} \vee A_{k+2} \vdash A_{k+2}}cut}{A_0, \bigwedge_{i=0}^{k+1}(\neg A_i \vee A_{i+1}) \vdash A_{k+2}}\wedge : l$$

An Example (ctd.)

- The characteristic clause set schema of ψ is:

(1) $\vdash \mathrm{cl}_n^{\emptyset,\psi}$

(2) $\mathrm{cl}_0^{\emptyset,\psi} \vdash$

(3) $\mathrm{cl}_0^{\{A_{k'+1}\},\psi} \vdash A_1$

(4) $\mathrm{cl}_{k+1}^{\{A_{k'+1}\},\psi} \vdash \mathrm{cl}_k^{\{A_{k'+1}\},\psi}$

(5) $\mathrm{cl}_{k+1}^{\{A_{k'+1}\},\psi}, A_{k+1} \vdash A_{k+2}$

(6) $\mathrm{cl}_{k+1}^{\emptyset,\psi} \vdash \mathrm{cl}_k^{\{A_{k'+1}\},\psi}$

(7) $\mathrm{cl}_{k+1}^{\emptyset,\psi}, A_{k+1} \vdash$

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(5) $\mathrm{cl}_{k+1}^{\{A_{k'+1}\},\psi}, A_{k+1} \vdash A_{k+2}$

(6) $\mathrm{cl}_{k+1}^{\emptyset,\psi} \vdash \mathrm{cl}_k^{\{A_{k'+1}\},\psi}$

(7) $\mathrm{cl}_{k+1}^{\emptyset,\psi}, A_{k+1} \vdash$

Schematic Projections

Basic Notions

- Let ρ be an unary and σ a binary rule. Let ϕ, ψ be **LKS**-proofs, then $\rho(\phi)$ is the **LKS**-proof obtained from the ϕ by applying ρ, and $\sigma(\phi, \psi)$ is the proof obtained from the proofs ϕ and ψ by applying σ.

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$$\phi = A_0 \vdash A_0$$

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$$\neg(\phi) = \frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash} \neg : l$$

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Basic Notions

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$$\vee(\neg(\phi), \psi) = \cfrac{\cfrac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash}\ \neg : l \qquad A_1 \vdash A_1}{A_0, \neg A_0 \vee A_1 \vdash A_1}\ \vee : l$$

Basic Notions (ctd.)

- $P^{\Gamma \vdash \Delta} = \{\psi^{\Gamma \vdash \Delta} \mid \psi \in P\}$, where $\psi^{\Gamma \vdash \Delta}$ is ψ followed by weakenings adding $\Gamma \vdash \Delta$.

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$$\psi = \quad \dfrac{\dfrac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash}\ \neg : l \qquad A_1 \vdash A_1}{A_0, \neg A_0 \vee A_1 \vdash A_1}\ \vee : l$$

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$$\psi^{\Gamma \vdash \Delta} = \cfrac{\cfrac{\cfrac{\cfrac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash}\ \neg: l \qquad A_1 \vdash A_1}{A_0, \neg A_0 \vee A_1 \vdash A_1}\ \vee: l}{A_0, \neg A_0 \vee A_1, \Gamma \vdash A_1}\ w: l*}{A_0, \neg A_0 \vee A_1, \Gamma \vdash \Delta, A_1}\ w: r*$$

Basic Notions (ctd.)

- $P \times_\sigma Q = \{\sigma(\phi, \psi) \mid \phi \in P, \psi \in Q\}$.

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$$P = \left\{ \frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash} \neg : l \quad , \quad \frac{B_0 \vdash B_0}{\neg A_0, B_0 \vdash B_0} w : l \right\}$$

$$Q = \left\{ A_1 \vdash A_1 \quad , \quad \frac{B_1 \vdash B_1}{A_1, B_1 \vdash B_1} w : l \right\}$$

Basic Notions (ctd.)

$$P \times_\vee Q = \left\{ \frac{\dfrac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash}\,\neg : l \qquad A_1 \vdash A_1}{A_0, \neg A_0 \vee A_1 \vdash A_1}\,\vee : l \quad , \right.$$

$$\frac{\dfrac{B_0 \vdash B_0}{\neg A_0, B_0 \vdash B_0}\,w : l \qquad A_1 \vdash A_1}{B_0, \neg A_0 \vee A_1 \vdash B_0, A_1}\,\vee : l \quad ,$$

$$\frac{\dfrac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash}\,\neg : l \qquad \dfrac{B_1 \vdash B_1}{A_1, B_1 \vdash B_1}\,w : l}{A_0, B_1, \neg A_0 \vee A_1 \vdash B_1}\,\vee : l \quad ,$$

$$\left. \frac{\dfrac{B_0 \vdash B_0}{\neg A_0, B_0 \vdash B_0}\,w : l \qquad \dfrac{B_1 \vdash B_1}{A_1, B_1 \vdash B_1}\,w : l}{B_0, B_1, \neg A_0 \vee A_1 \vdash B_0, B_1}\,\vee : l \right\}$$

Projections

$PR(\psi, \rho, \Omega)$ is defined inductively:

- if ρ is an axiom S, then $PR(\psi, \rho, \Omega) = \{S\}$.
- if ρ is a proof link of the form

$$\frac{(\psi, t)}{\Gamma_\Omega, \Gamma_C, \Gamma \vdash \Delta_\Omega, \Delta_C, \Delta}$$

then $PR(\psi, \rho, \Omega)$ is:

$$\frac{(pr^{\Omega',\psi}, t)}{\Gamma \vdash \Delta, \mathrm{cl}_t^{\Omega',\psi}}$$

Projections (ctd.)

- If ρ is an unary inference with immediate predecessor ρ' and

$$PR(\psi, \rho', \Omega) = \{\phi_1, \ldots, \phi_n\},$$

then either

$$PR(\psi, \rho, \Omega) = PR(\psi, \rho', \Omega)$$

or

$$PR(\psi, \rho, \Omega) = \{\rho(\phi_1), \ldots, \rho(\phi_n)\}.$$

Projections (ctd.)

- If ρ is a binary inference with immediate predecessors ρ_1 and ρ_2, then either

$$PR(\psi, \rho, \Omega) = PR(\psi, \rho_1, \Omega)^{\Gamma_2 \vdash \Delta_2} \cup PR(\psi, \rho_2, \Omega)^{\Gamma_1 \vdash \Delta_1}$$

or

$$PR(\psi, \rho, \Omega) = PR(\psi, \rho_1, \Omega) \times_\rho PR(\psi, \rho_2, \Omega)$$

Projections (ctd.)

- The set of projections of ψ is defined as follows:

$$PR(\psi) = \bigcup_{\Omega}(PR(\psi_{\text{base}}, \rho_{\text{base}}, \Omega) \cup PR(\psi_{\text{step}}, \rho_{\text{step}}, \Omega)).$$

An Example

- ψ_{base}:

$$\dfrac{\dfrac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash}\neg : l \qquad A_1 \vdash A_1}{A_0, \neg A_0 \vee A_1 \vdash A_1}\vee : l$$

- ψ_{step}:

$$\dfrac{\dfrac{\dfrac{(\psi, k)}{A_0, \bigwedge_{i=0}^{k}(\neg A_i \vee A_{i+1}) \vdash A_{k+1}} \qquad \dfrac{\dfrac{A_{k+1} \vdash A_{k+1}}{\neg A_{k+1}, A_{k+1} \vdash}\neg : l \qquad A_{k+2} \vdash A_{k+2}}{A_{k+1}, \neg A_{k+1} \vee A_{k+2} \vdash A_{k+2}}\vee : l}{A_0, \bigwedge_{i=0}^{k}(\neg A_i \vee A_{i+1}), \neg A_{k+1} \vee A_{k+2} \vdash A_{k+2}}cut}{A_0, \bigwedge_{i=0}^{k+1}(\neg A_i \vee A_{i+1}) \vdash A_{k+2}}\wedge : l$$

An Example (ctd.)

- $\bigcup_{\Omega \in \{\emptyset, \{A_{k'+1}\}\}} PR(\psi_{\text{base}}, \rho_{\text{base}}, \Omega)$ is:

$$\dfrac{\dfrac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash}\ \neg : l \qquad A_1 \vdash A_1}{A_0, \neg A_0 \vee A_1 \vdash A_1}\ \vee : l$$

- $\bigcup_{\Omega \in \{\emptyset, \{A_{k'+1}\}\}} PR(\psi_{\text{step}}, \rho_{\text{step}}, \Omega)$ is:

$$\dfrac{\dfrac{\dfrac{\dfrac{A_{k+1} \vdash A_{k+1}}{\neg A_{k+1}, A_{k+1} \vdash}\ \neg : l \qquad A_{k+2} \vdash A_{k+2}}{A_{k+1}, \neg A_{k+1} \vee A_{k+2} \vdash A_{k+2}}\ \vee : l}{A_{k+1}, A_0, \bigwedge_{i=0}^{k}(\neg A_i \vee A_{i+1}), \neg A_{k+1} \vee A_{k+2} \vdash A_{k+2}}\ w : l*}{A_{k+1}, A_0, \bigwedge_{i=0}^{k+1}(\neg A_i \vee A_{i+1}) \vdash A_{k+2}}\ \wedge : l$$

An Example (ctd.)

$$\dfrac{\dfrac{\dfrac{(pr^{\{A_{k'+1}\},\psi}, k)}{A_0, \bigwedge_{i=0}^{k}(\neg A_i \lor A_{i+1}) \vdash \mathrm{cl}_k^{\{A_{k+1}\},\psi}}}{A_0, \bigwedge_{i=0}^{k}(\neg A_i \lor A_{i+1}), \neg A_{k+1} \lor A_{k+2} \vdash \mathrm{cl}_k^{\{A_{k+1}\},\psi}}\ w: l}{A_0, \bigwedge_{i=0}^{k+1}(\neg A_i \lor A_{i+1}) \vdash \mathrm{cl}_k^{\{A_{k+1}\},\psi}}\ \land: l$$

and

$$\dfrac{\dfrac{\dfrac{(pr^{\{A_{k'+1}\},\psi}, k)}{A_0, \bigwedge_{i=0}^{k}(\neg A_i \lor A_{i+1}) \vdash \mathrm{cl}_k^{\{A_{k+1}\},\psi}}}{A_0, \bigwedge_{i=0}^{k}(\neg A_i \lor A_{i+1}), \neg A_{k+1} \lor A_{k+2} \vdash \mathrm{cl}_k^{\{A_{k+1}\},\psi}, A_{k+2}}\ w: l, r}{A_0, \bigwedge_{i=0}^{k+1}(\neg A_i \lor A_{i+1}) \vdash \mathrm{cl}_k^{\{A_{k+1}\},\psi}, A_{k+2}}\ \land: l$$

Ongoing and Future Work

Correctness of the definition of $PR(\psi)$

- Let ψ be a proof schema and $PR(\psi)$ the set of projections of ψ as defined above. Then by $Proj(\psi, k)$ we denote the set $\{eval(\phi, k) \mid \phi \in PR(\psi)\}$.
- Let $PR(eval(\psi, k), \Omega)$ be a set of projections for a ground **LKS**-proof $eval(\psi, k)$ with the cut-configuration Ω.

Correctness of the definition of $PR(\psi)$ (ctd.)

Lemma (3.1)

Let ψ be a proof schema and (ψ, k) an arbitrary proof link of ψ, then for all cut-configurations Ω, $(pr^{\Omega,\psi}, k)$ evaluates to the set $PR(eval(\psi, k), \Omega)$.

Proposition (3.1)

Let ψ be a proof schema, then $PR(eval(\psi, k), \emptyset) \subseteq Proj(\psi, k)$.

Future Work

- Given the schemata of refutations and projections construct the schema of ACNF.
- Extend these results for the first order proof schemata.
- Cut-elimination on proof schema for Fürstenberg's prime proof.

Questions?