CERES for Propositional Proof Schemata

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Introduction
Overview

- **Schemata** are very useful in mathematical proofs (avoids explicit use of the induction).

- **Schemata** are used on meta-level.

- Many problems can be expressed in *propositional schema language*, like:
  - Circuit verification,
  - Graph coloring,
  - Pigeonhole principle, etc.
Propositional Schema Language

- Set of **index variables** is a set of variables over natural numbers.

- **Linear arithmetic expression** is as usual built on the signature $0, s, +, -$ and on a set of index variables.

- **Indexed proposition** is an expression of the form $p_a$, where $a$ is a linear arithmetic expression.

- **Propositional variable** is an indexed proposition $p_a$, where $a \in \mathbb{N}$. 
Syntax

- **Formula schema** is defined inductively:
  - Indexed proposition is a formula schema.
  - If \( \phi_1 \) and \( \phi_2 \) are formula schemata, then so are \( \phi_1 \lor \phi_2 \), \( \phi_1 \land \phi_2 \) and \( \neg \phi_1 \).
  - If \( \phi \) is a formula schema, \( a, b \) are linear arithmetic expressions and \( i \) is an index variable, then \( \bigwedge_{i=a}^{b} \phi \) and \( \bigvee_{i=a}^{b} \phi \) are formula schemata, called iterations.
Schematic Characteristic Clause Set

Semantics

- **Interpretation** is a pair of functions, $I = (\mathcal{I}, \mathcal{I}_p)$, s.t. $\mathcal{I}$ maps index variables to natural numbers and $\mathcal{I}_p$ maps propositional variables to truth values.

- **Truth value** $\llbracket \phi \rrbracket_I$ of a formula schema $\phi$ in an interpretation $I$ is defined inductively:
  
  - $\llbracket p_a \rrbracket_I = \mathcal{I}_p(p_{\mathcal{I}(a)})$.
  - $\llbracket \neg \phi \rrbracket_I = T$ iff $\llbracket \phi \rrbracket_I = F$.
  - $\llbracket \phi_1 \land (\lor) \phi_2 \rrbracket_I = T$ iff $\llbracket \phi_1 \rrbracket_I = T$ and (or) $\llbracket \phi_2 \rrbracket_I = T$.
  - $\llbracket \bigwedge_{i=a}^b \bigvee_{i=a}^b \phi \rrbracket_I = T$ iff for every (there is an) integer $\alpha$ s.t. $\mathcal{I}(a) \leq \alpha \leq \mathcal{I}(b)$, $\llbracket \phi \rrbracket_I[\alpha/i] = T$. 
Cut-Elimination on Proof Schemata

**Aim:** describe syntactically sequence of cut-free proofs \((\chi_n)_{n \in \mathbb{N}}\) obtained by cut-elimination on proof sequences \((\varphi_n)_{n \in \mathbb{N}}\).

- Cut-free proofs of schema typically are described in meta-language.
- Find object language to define sequence \((\chi_n)_{n \in \mathbb{N}}\).
Which cut-elimination method?

- Reductive cut-elimination.
  - Efficient.
  - Strong methods of redundancy-elimination.
  - Atomic cut-normal form is constructed via parts of the original proof.
- CERES.
The CERES Method

- **CERES** is a cut-elimination method by resolution.

- Method consists of the following steps:
  1. Skolemization of the proof (if it is not already skolemized).
  2. Computation of the characteristic clause set.
  3. Refutation of the characteristic clause set.
Schematic LK
Basic Notions

- **Sequent Schema** is an expression of the form $\Gamma \vdash \Delta$, where $\Gamma$ and $\Delta$ are multisets of formula schemata.

- **Initial Sequent Schema** is an expression of the form $A \vdash A$, where $A$ is an indexed proposition.

- **Proof Link** is a tuple $(\varphi, t)$, where $\varphi$ is a proof name and $t$ is a linear arithmetic expression.
Calculus LKS

- **Axioms:** initial sequent schemata or proof links.

- **Rules:**
Calculus LKS

- **Axioms**: initial sequent schemata or proof links.

- **Rules**: \( \land \) introduction:

\[
A, \Gamma \vdash \Delta \\
\frac{A \land B, \Gamma \vdash \Delta}{A \land B, \Gamma \vdash \Delta} \land : l1
\]

\[
B, \Gamma \vdash \Delta \\
\frac{A \land B, \Gamma \vdash \Delta}{A \land B, \Gamma \vdash \Delta} \land : l2
\]

\[
\Gamma \vdash \Delta, A \\
\Pi \vdash \Lambda, B \\
\frac{\Gamma, \Pi \vdash \Delta, \Lambda, A \land B}{\Gamma, \Pi \vdash \Delta, \Lambda, A \land B} \land : r
\]

**Equivalences**: \( A_0 \equiv \land_{i=0}^{0} A_i \) and \( (\land_{i=0}^{n} A_i) \land A_{n+1} \equiv \land_{i=0}^{n+1} A_i \)
Calculus LKS

- **Axioms:** initial sequent schemata or proof links.

- **Rules:** $\lor$ introduction:
  
  \[
  \frac{A, \Gamma \vdash \Delta \quad B, \Pi \vdash \Lambda}{A \lor B, \Gamma, \Pi \vdash \Delta, \Lambda} \quad \lor : l
  \]
  
  \[
  \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \lor B} \quad \lor : r1
  \]
  
  \[
  \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \lor B} \quad \lor : r2
  \]

**Equivalences:** $A_0 \equiv \bigvee_{i=0}^{0} A_i$ and $(\bigvee_{i=0}^{n} A_i) \lor A_{n+1} \equiv \bigvee_{i=0}^{n+1} A_i$
Calculus LKS

- **Axioms:** initial sequent schemata or proof links.

- **Rules:** \( \neg \) introduction:

\[
\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \quad \vdash l \\
\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \quad \vdash r
\]
Calculus LKS

- **Axioms**: initial sequent schemata or proof links.

- **Rules**: Weakening rules:
  
  \[
  \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad w: l \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \quad w: r
  \]
Axioms: initial sequent schemata or proof links.

Rules: Contraction rules:

\[
\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad c : l \\
\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} \quad c : r
\]
Calculus LKS

- **Axioms**: initial sequent schemata or proof links.

- **Rules**: Cut rule:

\[
\frac{\Gamma \vdash \Delta, A \quad A, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda} \quad \text{cut}
\]
LKS-proof

- **Derivation** is a directed tree with nodes as sequences and edges as rules.

- **LKS-proof** of the sequence $S$ is a derivation of $S$ with axioms as leaf nodes.

- An **LKS-proof** is called **ground** if it does not contain free parameters, index variables, or proof links.
Proof Schemata

- **Proof schema** $\psi$ is a tuple of pairs $\langle (\psi_1^{\text{base}}, \psi_1^{\text{step}}), \ldots, (\psi_m^{\text{base}}, \psi_m^{\text{step}}) \rangle$ such that:

  - $\psi^1 \prec \psi^2 \prec \cdots \prec \psi^m$,
  - $\psi_i^{\text{base}}$ is a ground LKS-proof of $S^i \{ n \leftarrow 0 \}$, for $i \in \{1, \ldots, m\}$,
  - $\psi_i^{\text{step}}$ is an LKS-proof of $S^i \{ n \leftarrow k + 1 \}$, where $k$ is an index variable, and $\psi_i^{\text{step}}$ contains proof links of the form (for $i \prec j$):
    \[
    S^i \{ n \leftarrow k \} \text{ or } S^j \{ n \leftarrow k^j \}
    \]
  - From now on $m = 1$. 


- ▶ From now on $m = 1$. 

Proof Evaluation

An **evaluation** of a proof schema $\psi$ is a ground LKS-proof $eval(\psi, k)$, defined inductively:

- $eval(\psi, 0) = \psi_{\text{base}}$, and

- $eval(\psi, i + 1)$ is defined as $\psi_{\text{step}}$ with end-sequent $S \{ k \leftarrow i \}$ and every proof link to $(\psi, k)$ in $\psi_{\text{step}}$ are replaced by $eval(\psi, i)$. 


An Example

► \( \psi_{\text{base}} \):

\[
\begin{align*}
A_0 & \vdash A_0 \\
\neg A_0, A_0 & \vdash \neg \vdash \vdash A_{1} \\
A_0, \neg A_0 \lor A_1 & \vdash A_1 \lor \vdash l
\end{align*}
\]

► \( \psi_{\text{step}} \):

\[
\begin{align*}
\bigwedge_{i=0}^{k} (\neg A_i \lor A_{i+1}) & \vdash A_{k+1} \\
A_{k+1} & \vdash A_{k+1} \\
\neg A_{k+1}, A_{k+1} & \vdash \neg \vdash \vdash A_{k+2} \\
A_{k+1}, \neg A_{k+1} \lor A_{k+2} & \vdash A_{k+2} \lor \vdash l \\
\bigwedge_{i=0}^{k} (\neg A_i \lor A_{i+1}) \land A_{k+2} & \vdash A_{k+2} \land \vdash l
\end{align*}
\]
An Example (ctd.)

- **eval(ψ, 0):**

\[
\begin{align*}
A_0 & \vdash A_0 \\
\neg A_0, A_0 & \vdash \vdash: l \\
A_1 & \vdash A_1 \\
A_0, \neg A_0 \lor A_1 & \vdash A_1 \\
\end{align*}
\]

- **eval(ψ, 1):**

\[
\begin{align*}
\text{(eval(ψ, 0))} & \\
A_0, \bigwedge_{i=0}^0 (\neg A_i \lor A_{i+1}) & \vdash A_1 \\
\text{A_1 \vdash A_1} & \\
\neg A_1, A_1 & \vdash \vdash: l \\
A_2 & \vdash A_2 \\
A_1, \neg A_1 \lor A_2 & \vdash A_2 \\
\text{cut} & \\
A_0, \bigwedge_{i=0}^0 (\neg A_i \lor A_{i+1}), \neg A_1 \lor A_2 & \vdash A_2 \\
\text{A_0, \bigwedge_{i=0}^1 (\neg A_i \lor A_{i+1}) \vdash A_2} & \vdash: l \\
\end{align*}
\]
Schematic Characteristic Clause Set
Basic Notions

- **Cut-configuration** $\Omega$ of $\psi$ is a set of formula occurrences from the end-sequent of $\psi$.

- $cl^\Omega_{\psi,k}$ is an unique indexed proposition symbol for all cut-configurations $\Omega$ of $\psi$.

- The intended semantics of $cl^\Omega_{k,\psi}$ will be “the characteristic clause set of $eval(\psi, k)$, with the cut-configuration $\Omega$”.
Characteristic Clause Set

$CL_\rho(\psi, \Omega)$ is defined inductively:

- if $\rho$ is an axiom of the form $\Gamma_\Omega, \Gamma_C, \Gamma \vdash \Delta_\Omega, \Delta_C, \Delta$, then
  
  $$CL_\rho(\psi, \Omega) = \{\Gamma_\Omega, \Gamma_C \vdash \Delta_\Omega, \Delta_C\}.$$

- if $\rho$ is a proof link of the form

  $$\begin{array}{c}
  \vdash (\psi, t) \\
  \Gamma_\Omega, \Gamma_C, \Gamma \vdash \Delta_\Omega, \Delta_C, \Delta
  \end{array}$$

  then

  $$CL_\rho(\psi, \Omega) = \{\vdash cl_t^{\Omega', \psi}\}.$$
Characteristic Clause Set (ctd.)

- If $\rho$ is an unary rule with immediate predecessor $\rho'$, then
  \[ CL_\rho(\psi, \Omega) = CL_{\rho'}(\psi, \Omega). \]

- If $\rho$ is a binary rule with immediate predecessors $\rho_1, \rho_2$, then either
  \[ CL_\rho(\psi, \Omega) = CL_{\rho_1}(\psi, \Omega) \cup CL_{\rho_2}(\psi, \Omega) \]
  or
  \[ CL_\rho(\psi, \Omega) = CL_{\rho_1}(\psi, \Omega) \otimes CL_{\rho_2}(\psi, \Omega). \]
Characteristic Clause Set (ctd.)

- \( \text{CL}(\psi, \Omega) = \text{CL}_\rho(\psi, \Omega) \), where \( \rho \) is the last inference of \( \psi \).

- \( \text{CL}(\varphi) = \text{CL}(\varphi, \emptyset) \), where \( \varphi \) is a ground LKS-proof.

- \( \text{CL}_{\text{base}} = \bigcup_{\Omega} (\{ \text{cl}_{0}^{\Omega, \psi} \vdash \} \otimes \text{CL}(\psi_{\text{base}}, \Omega)) \).

- \( \text{CL}_{\text{step}} = \bigcup_{\Omega} (\{ \text{cl}_{k+1}^{\Omega, \psi} \vdash \} \otimes \text{CL}(\psi_{\text{step}}, \Omega)) \), for \( 0 \leq k \leq n \).

- \( \text{CL}_{s}(\psi) = \{ \vdash \text{cl}_{n}^{\emptyset, \psi} \} \cup \text{CL}_{\text{base}} \cup \text{CL}_{\text{step}} \).
Unsatisfiability of $\text{CL}_s(\psi)$

Lemma (2.1)

Let $C$ be a clause and $\mathcal{C}$ be a clause set. Then an interpretation $I \models \{C\} \otimes \mathcal{C}$ iff $I \models C$ or $I \models \mathcal{C}$.

Lemma (2.2)

Let $\psi$ be a proof schema and $\text{CL}(\psi, \Omega)$ be a characteristic clause set as defined above. Assume that for all cut-configurations $\Omega$, $I \models \text{cl}_i^{\Omega, \psi}$ implies $I \models \text{CL}(\text{eval}(\psi, i), \Omega)$. Then $I \models \text{CL}(\psi_{\text{step}} \{k \leftarrow i\}, \Omega)$ implies $I \models \text{CL}(\text{eval}(\psi, i + 1), \Omega)$. 
Unsatisfiability of $\text{CL}_s(\psi)$ (ctd.)

**Proposition (2.1)**

Let $\varphi$ be a ground LKS-proof. Then $\text{CL}(\varphi)$ is unsatisfiable.

**Proposition (2.2)**

If $I \models \text{CL}_s(\psi)$ then $I \models \text{CL}(\text{eval}(\psi, I(n)))$.

**Corollary (2.1)**

Let $\psi$ be a proof schema and $\text{CL}_s(\psi)$ its characteristic clause set. Then $\text{CL}_s(\psi)$ is unsatisfiable.
An Example

\[ \psi_{\text{base}}: \]

\[
\begin{aligned}
A_0, A_0 \vdash & A_1 \\
\neg A_0, A_0 \vdash & \neg: l \\
A_0, \neg A_0 \lor A_1 \vdash & A_1 \\
\end{aligned}
\]

\[ \psi_{\text{step}}: \]

\[
\begin{aligned}
A_0, \bigwedge_{i=0}^{k} \neg A_i \lor A_{i+1} \vdash & A_{k+1} \\
A_{k+1}, A_{k+1} \vdash & \neg: l \\
A_{k+1}, \neg A_{k+1} \lor A_{k+2} \vdash & A_{k+2} \\
A_0, \bigwedge_{i=0}^{k} \neg A_i \lor A_{i+1}, \neg A_{k+1} \lor A_{k+2} \vdash & A_{k+2} \\
\end{aligned}
\]

\[ \text{cut} \]

\[ A_0, \bigwedge_{i=0}^{k+1} \neg A_i \lor A_{i+1} \vdash A_{k+2} \]

\[ \land: l \]
The characteristic clause set schema of $\psi$ is:

1. $\vdash \text{cl}\emptyset,\psi$
2. $\text{cl}\emptyset,\psi \vdash$
3. $\text{cl}_0\{A_{k'+1}\},\psi \vdash A_1$
4. $\text{cl}_{k+1}\{A_{k'+1}\},\psi \vdash \text{cl}_k\{A_{k'+1}\},\psi$
5. $\text{cl}_{k+1}\{A_{k'+1}\},\psi, A_{k+1} \vdash A_{k+2}$
6. $\text{cl}\emptyset,\psi \vdash \text{cl}_k\{A_{k'+1}\},\psi$
7. $\text{cl}_{k+1}\emptyset,\psi, A_{k+1} \vdash$
The characteristic clause set schema of $\psi$ is:

1. $\vdash \text{cl}^\emptyset_n, \psi$
2. $\text{cl}^\emptyset_0, \psi \vdash$
3. $\text{cl}^\emptyset_0, \{A_{k'} + 1\}, \psi \vdash A_1$
4. $\text{cl}^\emptyset_{k+1}, \{A_{k'} + 1\}, \psi \vdash \text{cl}^\emptyset_k, \{A_{k'} + 1\}, \psi$
5. $\text{cl}^\emptyset_{k+1}, \{A_{k'} + 1\}, \psi, A_{k+1} \vdash A_{k+2}$
6. $\text{cl}^\emptyset_{k+1}, \psi \vdash \text{cl}^\emptyset_k, \{A_{k'} + 1\}, \psi$
7. $\text{cl}^\emptyset_{k+1}, A_{k+1} \vdash$
Schematic Projections
Let $\rho$ be an unary and $\sigma$ a binary rule. Let $\phi, \psi$ be LKS-proofs, then $\rho(\phi)$ is the LKS-proof obtained from the $\phi$ by applying $\rho$, and $\sigma(\phi, \psi)$ is the proof obtained from the proofs $\phi$ and $\psi$ by applying $\sigma$. 
Let $\rho$ be an unary and $\sigma$ a binary rule. Let $\phi, \psi$ be LKS-proofs, then $\rho(\phi)$ is the LKS-proof obtained from the $\phi$ by applying $\rho$, and $\sigma(\phi, \psi)$ is the proof obtained from the proofs $\phi$ and $\psi$ by applying $\sigma$.

$$\phi = A_0 \vdash A_0$$
Basic Notions

Let $\rho$ be an unary and $\sigma$ a binary rule. Let $\phi, \psi$ be $\text{LKS}$-proofs, then $\rho(\phi)$ is the $\text{LKS}$-proof obtained from the $\phi$ by applying $\rho$, and $\sigma(\phi, \psi)$ is the proof obtained from the proofs $\phi$ and $\psi$ by applying $\sigma$.

\[\neg(\phi) = \frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash \neg : l}\]
Let $\rho$ be an unary and $\sigma$ a binary rule. Let $\phi, \psi$ be LKS-proofs, then $\rho(\phi)$ is the LKS-proof obtained from the $\phi$ by applying $\rho$, and $\sigma(\phi, \psi)$ is the proof obtained from the proofs $\phi$ and $\psi$ by applying $\sigma$.

\[
\neg(\phi) = \frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash \neg: l} \quad \psi = A_1 \vdash A_1
\]
Basic Notions

Let $\rho$ be an unary and $\sigma$ a binary rule. Let $\phi, \psi$ be LKS-proofs, then $\rho(\phi)$ is the LKS-proof obtained from the $\phi$ by applying $\rho$, and $\sigma(\phi, \psi)$ is the proof obtained from the proofs $\phi$ and $\psi$ by applying $\sigma$.

$$
\forall (\neg(\phi), \psi) = \frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash \neg: l} \quad \frac{A_1 \vdash A_1}{A_0, \neg A_0 \lor A_1 \vdash A_1} \quad \lor: l
$$
Basic Notions (ctd.)

$P^{Γ \vdash Δ} = \{ ψ^{Γ \vdash Δ} \mid ψ ∈ P \}$, where $ψ^{Γ \vdash Δ}$ is $ψ$ followed by weakenings adding $Γ \vdash Δ$. 
Basic Notions (ctd.)

\[ P^{\Gamma \vdash \Delta} = \{ \psi^{\Gamma \vdash \Delta} \mid \psi \in P \} \text{, where } \psi^{\Gamma \vdash \Delta} \text{ is } \psi \text{ followed by weakenings adding } \Gamma \vdash \Delta. \]

\[
\psi = \frac{A_0 \vdash A_0 \quad \neg: l \quad A_1 \vdash A_1 \quad \lor: l}{\neg A_0, A_0 \vdash \lor: A_0, \neg A_0 \lor A_1 \vdash A_1}
\]
Basic Notions (ctd.)

\[ P^\Gamma \vdash \Delta = \{ \psi^\Gamma \vdash \Delta \mid \psi \in P \}, \text{ where } \psi^\Gamma \vdash \Delta \text{ is } \psi \text{ followed by weakenings adding } \Gamma \vdash \Delta. \]

\[
\psi^\Gamma \vdash \Delta = \quad
\begin{array}{c}
A_0 \vdash A_0 \\
\neg A_0, A_0 \vdash \\
A_0, \neg A_0 \lor A_1 \vdash A_1 \\
A_0, \neg A_0 \lor A_1, \Gamma \vdash A_1 \\
A_0, \neg A_0 \lor A_1, \Gamma \vdash \Delta, A_1
\end{array}
\]

\[ \vdash l \quad A_1 \vdash A_1 \quad \lor : l \\
\vdash l^* \quad w: l^* \\
w: r^* \]
Basic Notions (ctd.)

\[ P \times_\sigma Q = \{ \sigma(\phi, \psi) \mid \phi \in P, \psi \in Q \}. \]
Basic Notions (ctd.)

\[
P \times \sigma Q = \{ \sigma(\phi, \psi) \mid \phi \in P, \psi \in Q \}.\]

\[
P = \left\{ \frac{A_0 \vdash A_0}{\neg A_0, A_0 \vdash \bot}, \quad \frac{B_0 \vdash B_0}{\neg A_0, B_0 \vdash B_0} \right\}
\]

\[
Q = \left\{ \frac{A_1 \vdash A_1}{}, \quad \frac{B_1 \vdash B_1}{A_1, B_1 \vdash B_1} \right\}
\]
Basic Notions (ctd.)

\[
P \times \lor Q = \begin{cases} 
A_0 \vdash A_0 & \rightarrow: l \\
A_1 \vdash A_1 & \lor: l \\
\neg A_0, A_0 \vdash & A_0, \neg A_0 \lor A_1 \vdash A_1 \\
\lor: l & \\

B_0 \vdash B_0 & w: l \\
A_1 \vdash A_1 & \lor: l \\
\neg A_0, B_0 \vdash & B_0, \neg A_0 \lor A_1 \vdash B_0, A_1 \\
\lor: l & \\

A_0 \vdash A_0 & \rightarrow: l \\
B_1 \vdash B_1 & w: l \\
\neg A_0, A_0 \vdash & A_1, B_1 \vdash B_1 \\
\lor: l & \\

B_0 \vdash B_0 & w: l \\
B_1 \vdash B_1 & w: l \\
\neg A_0, B_0 \vdash & A_1, B_1 \vdash B_0, B_1 \\
\lor: l & \\

\end{cases}
\]
Projections

$PR(\psi, \rho, \Omega)$ is defined inductively:

- if $\rho$ is an axiom $S$, then $PR(\psi, \rho, \Omega) = \{S\}$.

- if $\rho$ is a proof link of the form:

  $\frac{}{\Gamma_{\Omega}, \Gamma_{C}, \Gamma \vdash \Delta_{\Omega}, \Delta_{C}, \Delta}
  \quad (\psi, t)$

  then $PR(\psi, \rho, \Omega)$ is:

  $\frac{}{\Gamma \vdash \Delta, cl^{\Omega', \psi}_{t}}
  \quad (pr^{\Omega', \psi}, t)$
If $\rho$ is an unary inference with immediate predecessor $\rho'$ and

$$PR(\psi, \rho', \Omega) = \{\phi_1, \ldots, \phi_n\},$$

then either

$$PR(\psi, \rho, \Omega) = PR(\psi, \rho', \Omega)$$

or

$$PR(\psi, \rho, \Omega) = \{\rho(\phi_1), \ldots, \rho(\phi_n)\}.$$
If $\rho$ is a binary inference with immediate predecessors $\rho_1$ and $\rho_2$, then either

$$PR(\psi, \rho, \Omega) = PR(\psi, \rho_1, \Omega)^{\Gamma_2 \vdash \Delta_2} \cup PR(\psi, \rho_2, \Omega)^{\Gamma_1 \vdash \Delta_1}$$

or

$$PR(\psi, \rho, \Omega) = PR(\psi, \rho_1, \Omega) \times_\rho PR(\psi, \rho_2, \Omega)$$
The set of projections of $\psi$ is defined as follows:

\[
PR(\psi) = \bigcup_{\Omega} (PR(\psi_{\text{base}}, \rho_{\text{base}}, \Omega) \cup PR(\psi_{\text{step}}, \rho_{\text{step}}, \Omega)).
\]
An Example

▶ \( \psi_{\text{base}} \):

\[
\begin{align*}
A_0 &\vdash A_0 \\
\neg A_0, A_0 &\vdash \neg: l \\
A_1 &\vdash A_1 \\
A_0, \neg A_0 \lor A_1 &\vdash A_1 \lor: l
\end{align*}
\]

▶ \( \psi_{\text{step}} \):

\[
\begin{align*}
A_0, \bigwedge_{i=0}^{k} (\neg A_i \lor A_{i+1}) &\vdash A_{k+1} \\
A_{k+1} &\vdash A_{k+1} \\
\neg A_{k+1}, A_{k+1} &\vdash \neg: l \\
A_{k+2} &\vdash A_{k+2} \\
A_{k+1}, \neg A_{k+1} \lor A_{k+2} &\vdash A_{k+2} \\
cut
A_0, \bigwedge_{i=0}^{k} (\neg A_i \lor A_{i+1}), \neg A_{k+1} \lor A_{k+2} &\vdash A_{k+2} \\
\land: l
\end{align*}
\]
An Example (ctd.)

\[ \bigcup_{\Omega \in \{\emptyset, \{A_{k'+1}\}\}} PR(\psi_{\text{base}}, \rho_{\text{base}}, \Omega) \text{ is:} \]

\[
\begin{align*}
A_0 & \vdash A_0 \\
\neg A_0, A_0 & \vdash \neg: l \\
A_1 & \vdash A_1 \\
A_0, \neg A_0 \vee A_1 & \vdash A_1 \\
\end{align*}
\]

\[ \bigcup_{\Omega \in \{\emptyset, \{A_{k'+1}\}\}} PR(\psi_{\text{step}}, \rho_{\text{step}}, \Omega) \text{ is:} \]

\[
\begin{align*}
A_{k+1} & \vdash A_{k+1} \\
\neg A_{k+1}, A_{k+1} & \vdash \neg: l \\
A_{k+2} & \vdash A_{k+2} \\
A_{k+1}, \neg A_{k+1} \vee A_{k+2} & \vdash A_{k+2} \\
A_{k+1}, A_0, \bigwedge_{i=0}^{k} (\neg A_i \vee A_{i+1}), \neg A_{k+1} \vee A_{k+2} & \vdash A_{k+2} \\
\end{align*}
\]

\[ w: l^* \]

\[ \wedge: l \]
An Example (ctd.)

\[ (pr\{A_{k'}+1\},\psi,k) \]

\[
\begin{align*}
A_0, \land_{i=0}^{k} (\neg A_i \lor A_{i+1}) & \vdash \text{cl}_k\{A_{k+1}\},\psi \\
A_0, \land_{i=0}^{k} (\neg A_i \lor A_{i+1}), \neg A_{k+1} \lor A_{k+2} & \vdash \text{cl}_k\{A_{k+1}\},\psi
\end{align*}
\]

\[ w: l \]

\[ \land: l \]

and

\[ (pr\{A_{k'}+1\},\psi,k) \]

\[
\begin{align*}
A_0, \land_{i=0}^{k} (\neg A_i \lor A_{i+1}) & \vdash \text{cl}_k\{A_{k+1}\},\psi \\
A_0, \land_{i=0}^{k} (\neg A_i \lor A_{i+1}), \neg A_{k+1} \lor A_{k+2} & \vdash \text{cl}_k\{A_{k+1}\},\psi,A_{k+2}
\end{align*}
\]

\[ w: l, r \]

\[ \land: l \]
Ongoing and Future Work
Correctness of the definition of \( PR(\psi) \)

- Let \( \psi \) be a proof schema and \( PR(\psi) \) the set of projections of \( \psi \) as defined above. Then by \( Proj(\psi, k) \) we denote the set \( \{ eval(\phi, k) \mid \phi \in PR(\psi) \} \).

- Let \( PR(eval(\psi, k), \Omega) \) be a set of projections for a ground \( LKS \)-proof \( eval(\psi, k) \) with the cut-configuration \( \Omega \).
Correctness of the definition of \( PR(\psi) \) (ctd.)

**Lemma (3.1)**

Let \( \psi \) be a proof schema and \((\psi, k)\) an arbitrary proof link of \( \psi \), then for all cut-configurations \( \Omega \), \((pr^{\Omega}, \psi, k)\) evaluates to the set \( PR(\text{eval}(\psi, k), \Omega) \).

**Proposition (3.1)**

Let \( \psi \) be a proof schema, then \( PR(\text{eval}(\psi, k), \emptyset) \subseteq Proj(\psi, k) \).
Future Work

- Given the schemata of refutations and projections construct the schema of ACNF.

- Extend these results for the first order proof schemata.

- Cut-elimination on proof schema for Fürstenberg’s prime proof.
Questions?