Second Order Cut-Elimination

Mikheil Rukhaia

Technische Universität Wien

Supervisor: Prof. Alexander Leitsch

Outline

Introduction
- What is cut-elimination?

Cut-elimination Methods
- Reductive methods
- CERES
- Extension to the Second Order Calculus

Implementation and Demonstration

Summary

Overview

- Cut-elimination is a proof transformation that removes all cut rules from a proof.
- The cut-elimination theorem was proved by Gerhard Gentzen in 1934.
- For the systems, that have a cut-elimination theorem, it is easy to prove consistency.
- Cut-elimination is nonelementary in general, i.e. there is no elementary bound on the size of cut-free proof w.r.t the original one.

Sequent Calculus **LK**

- A sequent is an expression of the form $\Gamma \vdash \Delta$, where Γ and Δ are lists of formulas.
- A rule is an inference of a lower sequent from an upper sequent(s).
- A derivation is a directed tree with nodes as sequences and edges as inferences.
- A proof of the sequence S is a derivation of S with axioms as leaf nodes.

Cut rule

- The cut rule:

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ cut$$

- The *cut* rule is the only rule such that its upper sequents may contain formulas that do not appear in the lower sesuents.
- The *cut* rule is the only rule that may produce an empty sequent $\vdash$ (inconsistency).
- The upper sequents of a *cut* rule corresponds to the lemmas into the proof.

Gentzen's method of cut-elimination

- Gentzen's method of cut-elimination is reductive, i.e. proof rewriting system is defined which is terminating and its normal form is a cut-free proof.
- Rewriting rules are divided into two parts: grade reduction and rank reduction rules.
- Grade of a cut rule is the number of logical symbols in the cut-formula.
- Rank of a cut rule is the number of sequents in the left cut-derivation, where cut-formula occurs in its succedent plus the number of sequents in the right cut-derivation, where the cut-formula occurs in its antecedent.

The method CERES

- CERES is a cut-elimination method by resolution.
- The CERES method radically differs from reductive methods.
- The CERES method consists of the following steps:
  1. The skolemization of the proof (if it is not already skolemized).
  2. The computation of the characteristic clause set.
  3. The refutation of the characteristic clause set.
  4. The computation of the proof projections and construction of the atomic cut normal form.

The system CERES

CERES system consists of the following parts:

HLK :
Program, that is used to formalize mathematical proofs and generate input for CERES.

CERES :
Program, that is used to transform formal proofs and extract relevant information.

ProofTool :
Program, that is used to visualize these formal proofs.

CERES home page: http://www.logic.at/ceres

Sequent Calculus **LKII**

The calculus **LKII** is defined as calculus **LK** plus following second order quantifier rules:

$$\frac{A(X/\lambda\bar{x}.F), \Gamma \vdash \Delta}{(\forall X)A, \Gamma \vdash \Delta}\ \forall{:}\mathrm{l} \qquad \text{and} \qquad \frac{\Gamma \vdash \Delta, A(X/\lambda\bar{x}.F)}{\Gamma \vdash \Delta, (\exists X)A}\ \exists{:}\mathrm{r}$$

$$\frac{A(X/Y), \Gamma \vdash \Delta}{(\exists X)A, \Gamma \vdash \Delta}\ \exists{:}\mathrm{l} \qquad \text{and} \qquad \frac{\Gamma \vdash \Delta, A(X/Y)}{\Gamma \vdash \Delta, (\forall X)A}\ \forall{:}\mathrm{r}$$

Where X is a second order variable, F is a first order formula with free variables not bound in A and bound variables of F not in A. Y is a second order eigenvariable of the same type as X.

Extension for **LKII**

Aim :

Extend CERES system to the second order calculus.

Problems :

* Second order clauses are not closed under substitution.
* Skolemization of the end-sequent is not enough, eigenvariable conditions can be still violated, as the active formulas of strong quantifier rules may be ancestors of formulas removed by weak second-order quantifier rules and therefore, the corresponding strong quantifiers will not be present in the end-sequent.

Extension for **LKII** (ctd.)

- There is on going work to solve these problems.
- Other solution was to extend Gentzen's method and implement it.
- Second order reduction rules:

$$\dfrac{\dfrac{\overset{\phi_l}{\Gamma_1 \vdash \Delta_1, A(X/Y)}}{\Gamma_1 \vdash \Delta_1, (\forall X)A}\ \forall{:}\mathrm{r} \qquad \dfrac{\overset{\phi_r}{A(X/\lambda\bar{x}.F), \Gamma_2 \vdash \Delta_2}}{(\forall X)A, \Gamma_2 \vdash \Delta_2}\ \forall{:}\mathrm{l}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \mathrm{cut}$$

transform to

$$\dfrac{\overset{\phi_l(Y/\lambda\bar{x}.F)}{\Gamma_1 \vdash \Delta_1, A(X/\lambda\bar{x}.F)} \qquad \overset{\phi_r}{A(X/\lambda\bar{x}.F), \Gamma_2 \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \mathrm{cut}$$

Implementation

- The system CERES is written in C++.
- Our algorithm is the following:
  - Select leftmost topmost cut.
  - Try to reduce grade in the following order: second order quantifiers, first order quantifiers, $\supset$, $\wedge$, $\vee$, $\neg$.
  - Try to reduce rank first on the left, then on the right cut-derivation in the following order: weakening rule cases, axiom rule cases, contraction rule cases, arbitrary unary and binary rule cases, permutation rule cases.
  - Repeat until all cuts are eliminated.

Demonstration

Let now run program with some short proofs.

Summary

- We extended Gentzen's method to the second order calculus.
- We extended CERES system to handle second order proofs.
- We can compare performance of the different methods of cut-elimination.

Thank you!