Proof Theoretical Reasoning – Lecture 4
Constructing Calculi and Outlook

Björn Lellmann

TU Wien

TRS Reasoning School 2015

Outline

An IndoLogical Problem

Imagine...

- You are an indologist and study texts of the Mīmāṃsā school of Indian Philosophy, concerned with analysing prescriptions contained in the Vedas, the sacred texts of Hinduism.

यत्र तूत्पत्त्यादयो न विध्यन्तरसिद्धास् तत्र स्वयमेव
स्वसम्बन्धिनामुत्पत्त्यादिचतुष्टयं करोति

An IndoLogical Problem

Imagine...

- You are an indologist and study texts of the Mīmāṃsā school of Indian Philosophy, concerned with analysing prescriptions contained in the Vedas, the sacred texts of Hinduism.
- You happen to meet an established proof theorist.
- In a lively discussion the two of you come up with the idea to use proof-theoretic reasoning to analyse different Mīmāṃsā authors by
  - extracting their modes of reasoning into (modal) logics;
  - constructing cut-free calculi for these logics;
  - comparing the different authors' interpretations using the corresponding calculi.

An IndoLogical Problem

Imagine further...

- In long, laborious work the two of you have managed to extract several modal logics from the texts.

  (In fact, you even extracted several modal logics for each author and are not sure which ones are best.)

So the only thing left to do is to analyse the logics using their proof theory. However, for this you need cut-free calculi for these logics...

How to construct sequent calculi for a given modal logic?

Reminder: Modal Logics

The formulae of modal logic are given by ($\mathcal{V}$ is a set of variables):

$$\mathcal{F} ::= \mathcal{V} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \to \mathcal{F} \mid \neg \mathcal{F} \mid \Box \mathcal{F}$$

with $\Diamond A$ abbreviating the formula $\neg \Box \neg A$.

A Kripke frame consists of a set W of worlds and an accessibility relation $R \subseteq W \times W$.

A Kripke model is a Kripke frame with a valuation $V : \mathcal{V} \to \mathcal{P}(W)$.

Truth at a world w in a model $\mathfrak{M}$ is defined via:

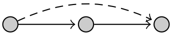
$$\begin{aligned} \mathfrak{M}, w \Vdash p &\text{ iff } w \in V(p) \\ \mathfrak{M}, w \Vdash \Box A &\text{ iff } \forall v \in W : wRv \implies \mathfrak{M}, v \Vdash A \\ \mathfrak{M}, w \Vdash \Diamond A &\text{ iff } \exists v \in W : wRv \;\&\; \mathfrak{M}, v \Vdash A \end{aligned}$$

Modal logics given by frame conditions

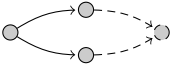
One way of specifying your favorite modal logic is by giving a frame condition: a first-order formula in the frame language characterising the class of Kripke frames which gives the logic.

Examples

- KT is given by reflexivity: $\forall x\, xRx$ 
- K4 is given by transitivity: $\forall x, y, z\, (xRy \wedge yRz \rightarrow xRz)$



- KB is given by symmetry: $\forall x, y\, (xRy \rightarrow yRx)$ 
- S5 is given by reflexivity, transitivity and symmetry.
- S4.2 is given by reflexivity, transitivity and directedness: $\forall x, y, z\, (xRy \wedge xRz \rightarrow \exists x\, (yRw \wedge zRw))$



Labelled Sequents

A very general method for constructing sequent calculi from frame conditions was developed e.g. in [Negri:'05, Negri, van Plato:'11].

Main idea: Explicitly mention the Kripke semantics in the calculus

Definition

Let $\mathcal{L}$ be a countably infinite set of labels.

- A labelled modal formula has the form $w : A$ for a label w and a modal formula A.
- A relational atom has the form wRv for labels w, v.
- A labelled sequent is a sequent consisting of labelled modal formulae and relational atoms.

Intuitive reading of a labelled formula $w : A$ is: $w \Vdash A$

The calculus G3K

The modal rules of the labelled sequent calculus G3K for modal logic K are

$$\frac{\Gamma, wRv \Rightarrow \Delta, v : A}{\Gamma \Rightarrow \Delta, w : \Box A}\ R\Box \qquad \frac{\Gamma, v : A, w : \Box A, wRv \Rightarrow \Delta}{\Gamma, w : \Box A, wRv \Rightarrow \Delta}\ L\Box$$

(v does not occur in Γ, Δ)

Intuition behind the rules:

- $R\Box$ is equivalent to the condition

$$\forall v. (wRv \implies v : A) \implies w : \Box A$$

- $L\Box$ is equivalent to the condition

$$w : \Box A \,\&\, wRv \implies v : A$$

The calculus G3K - propositional part

The propositional rules of G3K are essentially the standard ones extended with labels:

$$\frac{}{\Gamma, w : \bot \Rightarrow \Delta}\ L\bot$$

$$\frac{}{\Gamma, w : p \Rightarrow w : p, \Delta} \qquad \frac{}{\Gamma, wRv \Rightarrow wRv, \Delta}$$

$$\frac{\Gamma, w : A, w : B \Rightarrow \Delta}{\Gamma, w : A \wedge B \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow w : A, \Delta \quad \Gamma \Rightarrow w : B, \Delta}{\Gamma \Rightarrow w : A \wedge B, \Delta}\ R\wedge$$

$$\frac{\Gamma, w : A \Rightarrow \Delta \quad \Gamma, w : B \Rightarrow \Delta}{\Gamma, w : A \vee B \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow w : A, w : B\Delta}{\Gamma \Rightarrow w : A \vee B\Delta}\ R\vee$$

$$\frac{\Gamma, w : B \rightarrow \Delta \quad \Gamma \Rightarrow w : A, \Delta}{\Gamma, w : A \rightarrow B \Rightarrow \Delta}\ L\rightarrow \qquad \frac{\Gamma, w : A \rightarrow w : B, \Delta}{\Gamma \Rightarrow w : A \rightarrow B, \Delta}\ R\rightarrow$$

The calculus G3K

Example

The axiom $\Box(p \to q) \to (\Box p \to \Box q)$ is derived as follows:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\Gamma, v : q, v : p \Rightarrow v : q}\ \text{init}
      \qquad
      \dfrac{}{\Gamma, v : p \Rightarrow v : p, v : q}\ \text{init}
    }{w : \Box(p \to q), w : \Box p, wRv, v : p \to q, v : p \Rightarrow v : q}\ L\to
  }{w : \Box(p \to q), w : \Box p, wRv \Rightarrow v : q}\ L\Box
}{
  \dfrac{w : \Box(p \to q), w : \Box p \Rightarrow w : \Box q}{w : \Box(p \to q) \to (\Box p \to \Box q)}\ R\to
}\ R\Box
$$

The calculus G3K - useful properties

Proposition

The following properties can all be established by standard methods (mostly induction on the depth of the derivation):

- The sequent $\Gamma, w : A \Rightarrow w : A, \Delta$ is derivable for every A
- Substitution of labels $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma(v/w) \Rightarrow \Delta(v/w)}$ is depth-preserving admissible.
- Weakening is depth-preserving admissible.
- The labelled necessitation rule $\dfrac{\Rightarrow w : A}{\Rightarrow w : \Box A}$ is derivable.
- The rules of G3K are depth-preserving invertible.
- Contraction is depth-preserving admissible.

Soundness and completeness

The cut rule in the labelled sequent framework, written cut_ℓ, comes in two shapes, depending on the shape of the cut formula:

$$\frac{\Gamma \Rightarrow \Delta, w : A \quad w : A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \qquad \frac{\Gamma \Rightarrow \Delta, wRv \quad wRv, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}$$

Theorem

The calculus G3Kcut_ℓ *is sound and complete for modal logic* K*, i.e., for every formula A:*

$$A \text{ is a theorem of K} \quad \textit{iff} \quad \Rightarrow w : A \text{ is derivable in } \text{G3Kcut}_\ell\,.$$

Sketch of proof.

Since the labelled necessitation rule is admissible, deriving the axioms of K and simulating modus ponens using cut_ℓ is enough.

Cut Elimination for G3K

The cut elimination proof is essentially the standard one, using a double induction on the size of the cut formula and the height of the cut (the sum of the depths of the derivations of its premisses).

The interesting case:

$$\dfrac{\dfrac{\Gamma, wRx \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, w : \Box A}\,R\Box \qquad \dfrac{w : \Box A, wRv, v : A, \Sigma \Rightarrow \Pi}{w : \Box A, wRv, \Sigma \Rightarrow \Pi}\,L\Box}{\Gamma, wRv, \Sigma \Rightarrow \Delta, \Pi}\,\text{cut}_\ell$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\Gamma, wRx \Rightarrow \Delta, x : A}{\Gamma, wRv \Rightarrow \Delta, v : A}\,sb \qquad \dfrac{\dfrac{\Gamma, wRx \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, w : \Box A}\,R\Box \quad w : \Box A, wRv, v : A, \Sigma \Rightarrow \Pi}{\Gamma, v : A, wRv, \Sigma \Rightarrow \Delta, \Pi}}{\Gamma, wRv, \Gamma, wRv, \Sigma \Rightarrow \Delta, \Delta, \Pi}\,\text{cut}_\ell}{\Gamma, wRv, \Sigma \Rightarrow \Delta, \Pi}\,\text{Con}$$

Cut Elimination for G3K

The cut elimination proof is essentially the standard one, using a double induction on the size of the cut formula and the height of the cut (the sum of the depths of the derivations of its premisses).

Theorem

The labelled cut rule is admissible in G3K*. Hence the calculus* G3K *is cut-free complete for modal logic* K*, i.e.:*

$$\textit{If } A \textit{ is a theorem of } \mathsf{K} \textit{ then } \Rightarrow w : A \textit{ is derivable in } \mathsf{G3K}\,.$$

Converting frame conditions into rules

Definition

A geometric axiom is a formula of the form

$$\forall \vec{x}(P \rightarrow \exists y_1 M_1 \vee \cdots \vee \exists y_n M_n)$$

where

- the M_j and P are conjunctions of relational atoms
- the variables y_j are not free in P.

Examples

- $\forall x\, xRx$ for reflexivity
- $\forall x, y, z\, (xRy \wedge yRz \rightarrow xRz)$ for transitivity
- $\forall x, y\, (xRy \rightarrow yRx)$ for symmetry
- $\forall x, y, z\, (xRy \wedge xRz \rightarrow \exists w\, (yRw \wedge zRw))$ for directedness

Converting frame conditions into rules

Definition

A geometric axiom is a formula of the form

$$\forall \vec{x}(P \to \exists y_1 M_1 \vee \cdots \vee \exists y_n M_n)$$

where

- the M_j and P are conjunctions of relational atoms
- the variables y_j are not free in P.

Theorem

The geometric axiom above is equivalent to the geometric rule

$$\frac{\Gamma, \bar{P}, \bar{M}_1(z_1/y_1) \Rightarrow \Delta \quad \ldots \quad \Gamma, \bar{P}, \bar{M}_n(z_n/y_n) \Rightarrow \Delta}{\Gamma, \bar{P} \Rightarrow \Delta}$$

with $\bar{M}_i$ and $\bar{P}$ the multisets of relational atoms in M_i resp. P, and $z_1, \ldots, z_n$ not in the conclusion.

Converting frame conditions into rules: Examples

- Reflexivity $\forall x\, xRx$ is converted to

$$\frac{\Gamma, yRy \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

- Transitivity $\forall x, y, z\, (xRy \wedge yRz \rightarrow xRz)$ is converted to

$$\frac{\Gamma, xRy, yRz, xRz \Rightarrow \Delta}{\Gamma, xRy, yRz \Rightarrow \Delta}$$

- Symmetry $\forall x, y\, (xRy \rightarrow yRx)$ is converted to

$$\frac{\Gamma, xRy, yRz \Rightarrow \Delta}{\Gamma, xRy \Rightarrow \Delta}$$

- Directedness $\forall x, y, z\, (xRy \wedge xRz \rightarrow \exists w\, (yRw \wedge zRw))$ gives

$$\frac{\Gamma, xRy, xRz, yRv, zRv \Rightarrow \Delta}{\Gamma, xRy, xRz \Rightarrow \Delta}\ v \text{ not in conclusion}$$

Converting frame conditions into rules: Contraction

To obtain the nice structural properties for extensions of G3K with geometric rules we need to close the rule set under contraction:

Definition

A geometric rule set satisfies the closure condition if for every rule

$$\frac{\Gamma, \bar{P}, Q, R, \bar{M}_1(z_1/y_1) \Rightarrow \Delta \quad \dots \quad \Gamma, \bar{P}, Q, R, \bar{M}_n(z_n/y_n) \Rightarrow \Delta}{\Gamma, \bar{P}, Q, R \Rightarrow \Delta}$$

and injective renaming σ with $Q\sigma = R\sigma = Q$ it also includes

$$\frac{\Gamma, \bar{P}\sigma, Q, \bar{M}_1\sigma(z_1/y_1\sigma) \Rightarrow \Delta \quad \dots \quad \Gamma, \bar{P}\sigma, Q, \bar{M}_n\sigma(z_n/y_n\sigma) \Rightarrow \Delta}{\Gamma, \bar{P}\sigma, Q \Rightarrow \Delta}$$

Lemma

Contraction is admissible in extensions of G3K *with geometric rules satisfying the closure condition.*

Converting frame conditions into rules: Contraction

To obtain the nice structural properties for extensions of G3K with geometric rules we need to close the rule set under contraction:

Example

For directedness

$$\frac{\Gamma, xRy, xRz, yRv, zRv \Rightarrow \Delta}{\Gamma, xRy, xRz \Rightarrow \Delta} \; v \text{ not in conclusion}$$

we need to add the rule which identifies y and z and contracts the two occurrences of xRy:

$$\frac{\Gamma, xRy, yRv, yRv \Rightarrow \Delta}{\Gamma, xRy \Rightarrow \Delta} \; v \text{ not in conclusion}$$

Remark: Closing a rule set under contraction only demands the addition of finitely many rules and thus is unproblematic!

Cut elimination for extended calculi

The so constructed geometric rules

$$\frac{\Gamma, \bar{P}, \bar{M}_1(z_1/y_1) \Rightarrow \Delta \quad \ldots \quad \Gamma, \bar{P}, \bar{M}_n(z_n/y_n) \Rightarrow \Delta}{\Gamma, \bar{P} \Rightarrow \Delta}$$

have nice properties: all their active parts

- occur on the left hand side only
- consist of relational atoms only
- occur in the premisses if they occur in the conclusion.

Hence we can add them to G3K without harming cut elimination!

Cut elimination for extended calculi

Theorem

If G3K* *is an extension of* G3K *by finitely many geometric rules satisfying the closure condition, then* cut$_\ell$ *is admissible in* G3K*.*

Proof.

As for G3K, possibly renaming variables. E.g. for directedness:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, v : A}{\Gamma \Rightarrow \Delta, w : \Box A}\, R\Box \qquad \dfrac{w : \Box A, \Sigma, xRy, xRz, yRv, zRv \Rightarrow \Pi}{w : \Box A, \Sigma, xRy, xRz \Rightarrow \Pi}\, dir}{\Gamma, \Sigma, xRy, xRz \Rightarrow \Delta, \Pi}\, \text{cut}_\ell$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, v : A}{\Gamma \Rightarrow \Delta, w : \Box A}\, R\Box \qquad \dfrac{w : \Box A, \Sigma, xRy, xRz, yRv, zRv \Rightarrow \Pi}{w : \Box A, \Sigma, xRy, xRz, yRu, zRu \Rightarrow \Pi}\, sub}{\Gamma, \Sigma, xRy, xRz, yRu, zRu \Rightarrow \Delta, \Pi}\, \text{cut}_\ell}{\Gamma, \Sigma, xRy, xRz \Rightarrow \Delta, \Pi}\, dir$$

where u does not occur in $\Gamma, \Sigma, xRy, xRz \Rightarrow \Delta, \Pi$. □

Where's the catch?

So, labelled sequent calculi seem ideal to treat modal logics.

However, there are some issues:

- Decidability results need to be shown for every single logic.
- since the method is based heavily on Kripke semantics, the modification for non-normal modal logics is not immediately clear (see however [Gilbert, Maffezioli:'15] and recent work by S. Negri).
- The calculi are not fully internal: there is no formula translation of a labelled sequent.

An IndoLogical Problem revisited

Imagine again that you are the indologist from the beginning of the tutorial. You extracted the logics from the texts by interpreting principles like

यत्र तूत्पत्त्यादयो न विध्यन्तरसिद्धास् तत्र स्वयमेव
स्वसम्बन्धिनामुत्पत्त्यादिचतुष्टयं करोति

(I.e., "When, on the other hand, coming into existence [of something needed], etc., are not realised by another prescription, [the principal prescription] itself begets the four [stages] of coming into being, etc., [of the prescriptions] connected to itself.")

as Hilbert-style axioms, e.g. (with $\mathcal{O}$ for "ought to"):

$$\Box(A \to B) \to (\mathcal{O}A \to \mathcal{O}B)$$

An IndoLogical Problem revisited

Moreover, imagine that unfortunately you have not found evidence that the Mīmāṃsā logics for the modality $\mathcal{O}$ have a Kripke semantics.

This means that:

- You cannot use the labelled sequent systems based on Kripke semantics.
- Even if your logics had Kripke semantics, to construct labelled systems you would need to convert Hilbert-axioms into frame conditions (which can be tricky / impossible).

This problem leads to the obvious question...

How to construct sequent calculi for non-normal modal logics from Hilbert-axioms?

Non-normal Modal Logics

Definition

Classical modal logic E is given Hilbert-style by closing axioms for propositional logic under the rules

$$\frac{A \quad A \to B}{B} \text{ modus ponens, MP} \qquad \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B} \text{ congruence, Cg}$$

A classical modal logic is given by extending the Hilbert-system for E with further axioms.

Examples

The standard non-normal modal logics extend E with axioms from

$$\text{(m) } \Box(A \wedge B) \to \Box A \qquad \text{(c) } \Box A \wedge \Box B \to \Box(A \wedge B) \qquad \text{(n) } \Box \top$$

E.g., logic EC adds axiom (c), logic ECN adds (c), (n), etc. Logic EM is called monotone logic M. Note that MCN is modal logic K.

A Sequent Calculus for Classical Modal Logic

We need a base calculus for logic E which we can extend with rules.

Definition

The sequent calculus GCg contains the standard propositional rules and the modal sequent rule

$$\frac{A \Rightarrow B \quad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B, \Delta}\ \mathsf{Cg}$$

Theorem ([Lavendhomme, Lucas:'00])

GCg *is sound and cut-free complete for* E*.*

Sketch of proof.

For completeness: simulate the Hilbert-system using cut and show cut elimination.

A Sequent Calculus for Classical Modal Logic

The cut elimination proof is essentially the standard one.
The only interesting case is:

$$\dfrac{\dfrac{A \Rightarrow B \quad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B, \Delta}\,\text{Cg} \qquad \dfrac{B \Rightarrow C \quad C \Rightarrow B}{\Sigma, \Box B \Rightarrow \Box C, \Pi}\,\text{Cg}}{\Gamma, \Sigma, \Box A \Rightarrow \Box C, \Delta, \Pi}\,\text{cut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}\,\text{cut} \qquad \dfrac{C \Rightarrow B \quad B \Rightarrow A}{C \Rightarrow A}\,\text{cut}}{\Gamma, \Sigma, \Box A \Rightarrow \Box C, \Delta, \Pi}\,\text{Cg}$$

$\Box$

Constructing sequent calculi from axioms

How do we construct calculi from modal axioms, then?

Strategy:

- Convert axioms to logical sequent rules.
  (The resulting system is usually not cut-free!)
- Massage (or saturate) the rules set so that it has cut elimination.

Since the initially constructed rules are not cut-free we need:

Key ingredients:

- A general cut elimination theorem specifying sufficient conditions.
- A general method for saturating rule sets so that they satisfy these conditions.

Constructing sequent calculi from axioms

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Key ingredients:

- A general cut elimination theorem specifying sufficient conditions.
- A general method for saturating rule sets so that they satisfy these conditions.
- Bonus: A general decidability and complexity theorem.

Rank-1 axioms

We consider the ideas in a slightly simpler setting with axioms of a restricted form. (They can be generalised, of course.)

Definition

A rank-1 axiom is an axiom where every occurrence of a variable is under exactly one modality.

Examples

- The following axioms are rank-1 axioms:

$$(m)\ \Box(A \wedge B) \to \Box A \qquad (c)\ \Box A \wedge \Box B \to \Box(A \wedge B) \qquad (n)\ \Box\top$$

- The reflexivity axiom $\Box A \to A$ is not a rank-1 axiom.
- The transitivity axiom $\Box A \to \Box\Box A$ is not a rank-1 axiom.

Rank-1 axioms

We consider the ideas in a slightly simpler setting with axioms of a restricted form. (They can be generalised, of course.)

Definition

A rank-1 axiom is an axiom where every occurrence of a variable is under exactly one modality.

Fact

Every shallow axiom is equivalent to a conjunction of rank-1 clauses of the form

$$\Box L_1 \wedge \cdots \wedge \Box L_n \to \Box R_1 \vee \cdots \vee \Box R_k$$

where the L_i and the R_j are purely propositional formulae.

Step 1: Axioms to Rules

To convert a rank-1 axiom, break it into rank-1 clauses.

Then, e.g., for the rank-1 clause

$$(c) \quad \frac{}{\Rightarrow \Box A \wedge \Box B \to \Box(A \wedge B)}$$

- invert the propositional rules

$$\frac{}{\Box A, \Box B \Rightarrow \Box(A \wedge B)}$$

- replace propositional formulae under modalities with variables

$$\frac{A \Rightarrow r \quad r \Rightarrow A \quad B \Rightarrow s \quad s \Rightarrow B \quad A \wedge B \Rightarrow t \quad t \Rightarrow A \wedge B}{\Box r, \Box s \Rightarrow \Box t}$$

- invert the propositional rules in the premisses

$$\frac{A \Rightarrow r \quad r \Rightarrow A \quad B \Rightarrow s \quad s \Rightarrow B \quad A, B \Rightarrow t \quad t \Rightarrow A \quad t \Rightarrow B}{\Box r, \Box s \Rightarrow \Box t}$$

- cut out superfluous formulae from the premisses (here: A, B)

$$\frac{r, s \Rightarrow t \quad t \Rightarrow r \quad t \Rightarrow s}{\Box r, \Box s \Rightarrow \Box t} \; \mathsf{C}$$

Step 1: Axioms to Rules

To convert a rank-1 axiom, break it into rank-1 clauses.

Then, e.g., for the rank-1 clause

$$(\mathrm{m}) \quad \frac{}{\Rightarrow \Box(A \land B) \to \Box A}$$

- invert the propositional rules

$$\frac{}{\Box(A \land B) \Rightarrow \Box A}$$

- replace propositional formulae under modalities with variables

$$\frac{r \Rightarrow A \land B \quad A \land B \Rightarrow s \quad A \Rightarrow s \quad s \Rightarrow A}{\Box r \Rightarrow \Box s}$$

- invert the propositional rules in the premisses

$$\frac{r \Rightarrow A \quad r \Rightarrow B \quad A, B \Rightarrow s \quad A \Rightarrow s \quad s \Rightarrow A}{\Box r \Rightarrow \Box s}$$

- cut out superfluous formulae from the premisses (here: A, B)

$$\frac{r \Rightarrow s}{\Box r \Rightarrow \Box s}\ \mathrm{M}$$

Step 1: Axioms to Rules

To convert a rank-1 axiom, break it into rank-1 clauses.

Then, e.g., for the rank-1 clause

$$(\mathrm{n}) \quad \frac{}{\Rightarrow \Box\top}$$

- invert the propositional rules

$$\frac{}{\Rightarrow \Box\top}$$

- replace propositional formulae under modalities with variables

$$\frac{\top \Rightarrow r \quad r \Rightarrow \top}{\Rightarrow \Box r}$$

- invert the propositional rules in the premisses

$$\frac{\Rightarrow r}{\Rightarrow \Box r}$$

- cut out superfluous formulae from the premisses

$$\frac{\Rightarrow r}{\Rightarrow \Box r}\ \mathrm{N}$$

The crucial lemma for the cutting step

Lemma (Soundness of Cuts)

The rules below are interderivable in GCgcut *(all p shown):*

$$\frac{\Omega \Rightarrow \Theta, p \quad p, \Sigma_1 \Rightarrow \Pi_1 \quad p, \Sigma_2 \Rightarrow \Pi_2}{\Gamma \Rightarrow \Delta} \qquad \frac{\Omega, \Sigma_1 \Rightarrow \Theta, \Pi_1 \quad \Omega, \Sigma_2 \Rightarrow \Theta, \Pi_2}{\Gamma \Rightarrow \Delta}$$

Proof.

The tricky bit is to derive the premisses of the left rule from those of the right rule. For this we construct a formula for p and do:

$$\frac{\Omega, \Sigma_1 \Rightarrow \Theta, \Pi_1 \quad \Omega, \Sigma_2 \Rightarrow \Theta, \Pi_2}{\Omega \Rightarrow \Theta, (\bigwedge \Sigma_1 \to \bigvee \Pi_1) \wedge (\bigwedge \Sigma_2 \to \bigvee \Pi_2)}\ \textit{prop}$$

$$\frac{\dfrac{}{\bigwedge \Sigma_1 \to \bigvee \Pi_1, \Sigma_1 \Rightarrow \Pi_1}\ \textit{prop}}{(\bigwedge \Sigma_1 \to \bigvee \Pi_1) \wedge (\bigwedge \Sigma_2 \to \bigvee \Pi_2), \Sigma_1 \Rightarrow \Pi_1}\ \textit{prop}$$

$\square$

Step 2: What about cut?

The rule sets obtained from this procedure generally are not cut-free. E.g. we cannot reduce the cut

$$\dfrac{\dfrac{A, B \Rightarrow C \quad C \Rightarrow A \quad C \Rightarrow B}{\Box A, \Box B \Rightarrow \Box C}\,\mathsf{C} \qquad \dfrac{C, D \Rightarrow E \quad E \Rightarrow C \quad E \Rightarrow D}{\Box C, \Box D \Rightarrow \Box E}\,\mathsf{C}}{\Box A, \Box B, \Box D \Rightarrow \Box E}\,\mathsf{cut}$$

The solution is to simply add the missing rule to the rule set:

$$\dfrac{A, B, D \Rightarrow E \quad E \Rightarrow A \quad E \Rightarrow B \quad E \Rightarrow D}{\Box A, \Box B, \Box D \Rightarrow \Box E}$$

Note that the premisses of this rule are obtained by cutting superfluous formulae from the premisses of the derivation above (seen as a "macro rule").

The previous lemma ensures that this rule is sound.

Step 2: What about cut?

Definition

A modal rule set is saturated if it is closed under the addition of the missing rules from the previous slide and the rules required to meet the closure condition (closure under contraction).

Theorem (Cut elimination)

In a saturated rule set contraction and cut are admissible.

Proof.

The standard ones, with the interesting case:

$$\dfrac{\dfrac{\mathcal{P}_R}{\Gamma \Rightarrow \Delta, \Box A}\,R \quad \dfrac{\mathcal{P}_Q}{\Box A, \Sigma \Rightarrow \Pi}\,Q}{\Gamma \Rightarrow \Delta}\,\text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathcal{P}_R \quad \mathcal{Q}_R}{(\mathcal{P}_R \cup \mathcal{P}_Q) \ominus A}}{\Gamma \Rightarrow \Delta}\,\text{cut}(R, Q)$$

(Where $(\mathcal{P}_R \cup \mathcal{P}_Q) \ominus A$ comes from $\mathcal{P}_R \cup \mathcal{P}_Q$ by cutting on A in all possible ways.) □

Examples

Constructing cut-free calculi by this method starting from

$$\text{(c)}\ \Box A \land \Box B \to \Box(A \land B)$$

for logic MC results first in the rules

$$\frac{A_1, \ldots, A_n \Rightarrow B \quad B \Rightarrow A_1 \quad \ldots \quad B \Rightarrow A_n}{\Box A_1, \ldots, \Box A_n \Rightarrow \Box B}\ \mathsf{C}_n$$

for $n \geq 1$. Adding (m) $\Box(A \land B) \to \Box A$ and saturating yields the rules

$$\frac{A_1, \ldots, A_n \Rightarrow B}{\Box A_1, \ldots, \Box A_n \Rightarrow \Box B}\ \mathsf{MC}_n$$

for logic MC. Finally, adding (n) $\Box \top$ gives the well-known rules

$$\frac{A_1, \ldots, A_n \Rightarrow B}{\Box A_1, \ldots, \Box A_n \Rightarrow \Box B}\ \mathsf{K}_n$$

$(n \geq 0)$ for logic MCN, i.e., modal logic K!

Bonus: Decidability and complexity

So, what can we do with the calculi?

Theorem

Derivability in a saturated rule set is decidable in polynomial space.

Proof.

By the standard backwards proof search algorithm:

On input $\Gamma \Rightarrow \Delta$:

- if $\Gamma \Rightarrow \Delta$ is initial sequent, then accept; otherwise
- existentially guess a rule with conclusion $\Gamma \Rightarrow \Delta$
- universally choose a premiss $\Sigma \Rightarrow \Pi$ of this rule
- recursively call the algorithm with input $\Sigma \Rightarrow \Pi$.

The complexity of the sequents strictly decreases from conclusion to premisses in every rule, so branches of the search tree have *polynomial length*. By complexity theory we get PSPACE. □

An IndoLogical problem revisited, again.

Constructing a Mīmāṃsā deontic logic

With these tools our indologist now can approach her problem.

A promising language might include

- a modality $\Box$ to model necessity
- a binary modality $\mathcal{O}(\cdot/\cdot)$ to model conditional obligation: a formula

$$\mathcal{O}(A/B)$$

  reads "under the conditions B it is obligatory that A".

(The methods above extend readily to this.)

As a starting point we take $\Box$ to be a S4-modality with the axioms

$$(\text{t}) \quad \Box A \to A \qquad (4) \quad \Box A \to \Box\Box A$$

Constructing a Mīmāṃsā deontic logic

The principle

यत्र तूत्पत्त्यादयो न विध्यन्तरसिद्धास् तत्र स्वयमेव
स्वसम्बन्धिनामुत्पत्त्यादिचतुष्टयं करोति

(I.e., "When, on the other hand, coming into existence [of something needed], etc., are not realised by another prescription, [the principal prescription] itself begets the four [stages] of coming into being, etc., [of the prescriptions] connected to itself.")

and two other principles could be formalised as the axioms

$$\Box(A \to B) \to (\mathcal{O}(A/C) \to \mathcal{O}(B/C))$$

$$\Box(B \to \neg A) \to \neg(\mathcal{O}(A/C) \wedge \mathcal{O}(B/C))$$

$$\Box(B \leftrightarrow C) \wedge \mathcal{O}(A/B) \to \mathcal{O}(A/C)$$

Constructing a Mīmāṃsā deontic logic

Conversion into rules and saturation with the standard S4-rules

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\ \mathsf{T} \qquad \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ 4$$

gives the rules

$$\frac{\Box\Gamma, A \Rightarrow C \quad \Box\Gamma, B \Rightarrow D \quad \Box\Gamma, D \Rightarrow B}{\Box\Gamma, \mathcal{O}(A/B) \Rightarrow \mathcal{O}(C/D)}\ \mathsf{Mon}$$

$$\frac{\Box\Gamma, A \Rightarrow}{\Box\Gamma, \mathcal{O}(A/B) \Rightarrow}\ \mathsf{D}_1 \qquad \frac{\Box\Gamma, A, C \Rightarrow \quad \Box\Gamma, B \Rightarrow D \quad \Box\Gamma, D \Rightarrow B}{\Box\Gamma, \mathcal{O}(A/B), \mathcal{O}(C/D) \Rightarrow}\ \mathsf{D}_2$$

Theorem

The calculus with the above modal rules has cut elimination and derivability is decidable in exponential time.

A Mīmāṃsā deontic logic

The question now might arise whether this is "the right" logic.

Minimal requirement: consistency with seemingly contradictory statements from the vedas, e.g., the problem of the Śyena:

- *You should not harm any living being*
- *If you desire to harm your enemy, you should perform the Śyena sacrifice*

The statement that this is contradictory could be formalised as

$$\Box(\mathsf{hrm\_e} \to \mathsf{hrm}), \Box(\mathsf{sy} \to \mathsf{hrm\_e}), \Box\mathcal{O}(\neg\mathsf{hrm}/\top), \Box\mathcal{O}(\mathsf{sy}/\mathsf{des\_hrm}) \Rightarrow \bot$$

Backwards proof search gives:

Theorem

The problem of the Śyena is not contradictory in Mīmāṃsā deontic logic, i.e., the above sequent is not derivable.

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