Proof Theoretical Reasoning – Lecture 3
Modal Logic S5 and Hypersequents

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Outline

Reminder: Modal Logics

The formulae of modal logic are given by ($\mathcal{V}$ is a set of variables):

$$\mathcal{F} \ ::= \ \mathcal{V} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \to \mathcal{F} \mid \neg\mathcal{F} \mid \Box\mathcal{F}$$

with $\Diamond A$ abbreviating the formula $\neg \Box \neg A$.

A Kripke frame consists of a set W of worlds and an accessibility relation $R \subseteq W \times W$.

A Kripke model is a Kripke frame with a valuation $V : \mathcal{V} \to \mathcal{P}(W)$.

Truth at a world w in a model $\mathfrak{M}$ is defined via:

$$\begin{aligned} \mathfrak{M}, w \Vdash p & \ \text{ iff } \ w \in V(p) \\ \mathfrak{M}, w \Vdash \Box A & \ \text{ iff } \ \forall v \in W : wRv \Rightarrow \mathfrak{M}, v \Vdash A \\ \mathfrak{M}, w \Vdash \Diamond A & \ \text{ iff } \ \exists v \in W : wRv \ \& \ \mathfrak{M}, v \Vdash A \end{aligned}$$

Modal Logic S5

Definition

Modal logic S5 is the logic given by the class of Kripke frames with universal accessibility relation, i.e., frames (W, R) with:

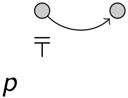
$$\forall x, y \in W : \ xRy \ .$$

Thus S5-theorems are those modal formulae which are true in every world of every Kripke model with universal accessibility relation.

Modal Logic S5

Example

The formulae $p \to \Box\Diamond p$ are theorems of S5:



Modal Logic S5

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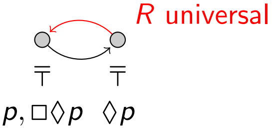

Modal Logic S5

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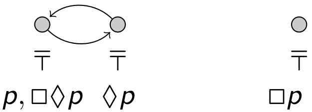

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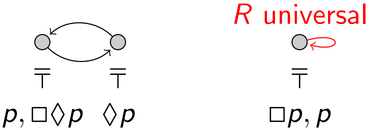

Modal Logic S5

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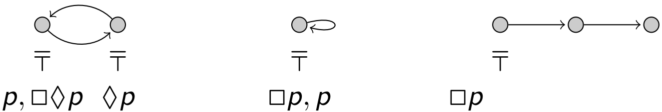

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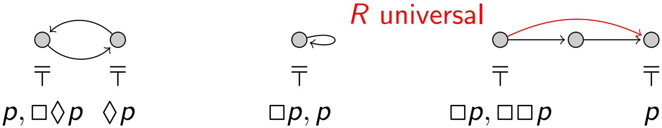

Modal Logic S5

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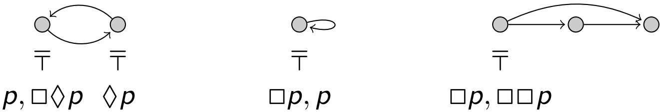

Hilbert-style Definition: S5 is given by closing the axioms

$$\Box(p \to q) \to (\Box p \to \Box q) \qquad p \to \Box\Diamond p \qquad \Box p \to p \qquad \Box p \to \Box\Box p$$

and propositional axioms under uniform substitution and the rules

$$\frac{A \quad A \to B}{B} \text{ modus ponens, MP} \qquad \frac{A}{\Box A} \text{ necessitation, nec}$$

A Sequent Calculus for S5

Definition (Takano 1992)

The sequent calculus LS5* contains the standard propositional rules and

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\ \mathsf{T} \qquad\qquad \frac{\Box\Gamma \Rightarrow A, \Box\Delta}{\Box\Gamma \Rightarrow \Box A, \Box\Delta}\ 45$$

Theorem

LS5* *is sound and complete (with* cut*) for* S5.

Proof.

Derive axioms and rules of the Hilbert-system. E.g., for $p \to \Box\Diamond p$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Box\neg p \Rightarrow \Box\neg p}\ \text{init}}{\Rightarrow \neg\Box\neg p, \Box\neg p}\ \neg_L}{\Rightarrow \Box\neg\Box\neg p, \Box\neg p}\ 45 \qquad \dfrac{\dfrac{\dfrac{}{p \Rightarrow p}\ \text{init}}{\neg p, p \Rightarrow}\ \neg_L}{\Box\neg p, p \Rightarrow}\ \mathsf{T}}{\dfrac{p \Rightarrow \Box\neg\Box\neg p}{\Rightarrow p \to \Box\Diamond p}\ \to_R}\ \text{cut}$$

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Proof.

E.g. the modus ponens rule $\frac{A \quad A \to B}{B}$ is simulated by:

$$\dfrac{\Rightarrow A \qquad \dfrac{\Rightarrow A \to B \qquad \dfrac{A, B \Rightarrow B \quad A \Rightarrow A, B}{A, A \to B \Rightarrow B}\ \to_L}{A \Rightarrow B}\ \text{cut}}{\Rightarrow B}\ \text{cut}$$

□

What about cut-free completeness?

Our standard proof of cut elimination fails:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \neg\Box\neg A, \Box\neg A}}{\Rightarrow \Box\neg\Box\neg A, \Box\neg A}\;45 \qquad \dfrac{\dfrac{\vdots}{\neg A, A \Rightarrow}}{\Box\neg A, A \Rightarrow}\;T}{A \Rightarrow \Box\neg\Box\neg A}\;\text{cut}$$

would need to reduce to:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow \neg\Box\neg A, \Box\neg A} \qquad \dfrac{\dfrac{\vdots}{\neg A, A \Rightarrow}}{\Box\neg A, A \Rightarrow}\;T}{A \Rightarrow \neg\Box\neg A}\;\text{cut}}{}\;??$$

But we can't apply rule 45 anymore since A is not boxed!

What about cut-free completeness?

But could there be a different derivation?
No! In fact we have:

Theorem

The sequent $p \Rightarrow \Box\Diamond p$ is not cut-free derivable in $\mathsf{LS5}^*$.

Proof.

The only rules that can be applied in a cut-free derivation ending in $p \Rightarrow \Box\Diamond p$ are weakening and contraction. Hence, such a derivation can only contain sequents of the form

$$\overbrace{p, \ldots, p}^{m\text{-times}} \Rightarrow \overbrace{\Box\neg\Box\neg p, \ldots, \Box\neg\Box\neg p}^{n\text{-times}}$$

with $m, n \geq 0$. Thus it cannot contain an initial sequent. □

Is there a cut-free sequent calculus for S5?

Trivial answer: Of course!
Take the rules $\{ \frac{}{\Rightarrow A} \mid A \text{ valid in S5}\}$.

Non-trivial answer: That depends on the shape of the rules!

Strategy for showing certain rule shapes cannot capture S5 even with cut:

- translate the rules into Hilbert-axioms of specific form
- connect Hilbert-style axiomatisability with frame definability
- show that the translations of the rules cannot define S5-frames.

(The translation involves cut, so this shows a stronger statement.)

What Is a Rule?

Let us call a sequent rule modal if it has the shape:

$$\frac{\Gamma_1, \Sigma_1 \Rightarrow \Pi_1, \Delta_1 \quad \dots \quad \Gamma_n, \Sigma_n \Rightarrow \Pi_n, \Delta_n}{\Gamma, \Box\Sigma \Rightarrow \Box\Pi, \Delta}$$

where (writing $\Gamma^\Box$ for the restriction of Γ to modal formulae)

- $\Sigma_i \subseteq \Sigma$, $\Pi_i \subseteq \Pi$
- Γ_i is one of $\emptyset, \Gamma, \Gamma^\Box$
- Δ_i is one of $\emptyset, \Delta, \Delta^\Box$

Example

$$\frac{\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}\,K \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\,T \quad \frac{\Gamma^\Box, \Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}\,4 \quad \frac{\Gamma^\Box \Rightarrow A, \Delta^\Box}{\Gamma \Rightarrow \Box A, \Delta}\,45$$

are all modal rules (and equivalent to the rules considered earlier).

What Is a Rule?

Let us call a sequent rule modal if it has the shape:

$$\frac{\Gamma_1, \Sigma_1 \Rightarrow \Pi_1, \Delta_1 \quad \ldots \quad \Gamma_n, \Sigma_n \Rightarrow \Pi_n, \Delta_n}{\Gamma, \Box\Sigma \Rightarrow \Box\Pi, \Delta}$$

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- Γ_i is one of $\emptyset, \Gamma, \Gamma^{\Box}$
- Δ_i is one of $\emptyset, \Delta, \Delta^{\Box}$

Example

$$\frac{\Gamma^{\Box}, \Sigma, \Box A \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}\ \text{GLR}$$

is not a modal rule (because the $\Box A$ changes sides).

Mixed-cut-closed Rule Sets

LS5* has modal rules in this sense, so we need something more.

Definition

A set of modal rules is mixed-cut-closed if principal-context cuts can be permuted up in the context.

Example

The set with modal rule $\dfrac{\Gamma^{\Box}, \Sigma \Rightarrow A}{\Gamma \Box \Sigma \Rightarrow \Box A, \Delta}\;4$ is mixed-cut-closed:

E.g.:

$$\dfrac{\dfrac{\Gamma^{\Box}, \Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Box A, \Delta}\;4 \qquad \dfrac{\Box A, \Omega^{\Box}, \Theta \Rightarrow B}{\Box A, \Omega, \Box\Theta \Rightarrow \Box B, \Xi}\;4}{\Gamma, \Box\Sigma, \Omega, \Box\Theta \Rightarrow \Delta, \Box B, \Xi}\;\text{cut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma^{\Box}, \Sigma \Rightarrow A}{\Gamma^{\Box}, \Box\Sigma \Rightarrow \Box A, \Delta}\;4 \qquad \Box A, \Omega^{\Box}, \Theta \Rightarrow B}{\Gamma^{\Box}, \Sigma, \Omega^{\Box}, \Theta \Rightarrow B}\;\text{cut}}{\Gamma, \Box\Sigma, \Omega, \Box\Theta \Rightarrow \Delta, \Box B, \Xi}\;4$$

Mixed-cut-closed Rule Sets

LS5* has modal rules in this sense, so we need something more.

Definition

A set of modal rules is mixed-cut-closed if principal-context cuts can be permuted up in the context.

Example

The set LS5* is not mixed-cut-closed: the principal-context cut

$$\dfrac{\dfrac{\Gamma^{\Box} \Rightarrow B, \Delta^{\Box}, \Box A}{\Gamma \Rightarrow \Box B, \Delta, \Box A}\,45 \qquad \dfrac{\Sigma, A \Rightarrow \Pi}{\Sigma, \Box A \Rightarrow \Pi}\,\mathsf{T}}{\Gamma, \Sigma \Rightarrow \Box B, \Delta, \Pi}\,\mathsf{cut}$$

cannot be permuted up in the context since Σ, Π are not boxed (see above).

Mixed-cut-closed Rule Sets Are Nice.

Lemma

If $\mathcal{R}$ is a mixed-cut-closed rule set for S5, then the contexts in all the premisses of the modal rules have one of the forms

$$\Rightarrow \qquad \text{or} \qquad \Gamma \Rightarrow \Delta \qquad \text{or} \qquad \Gamma^{\Box} \Rightarrow \quad .$$

Idea of proof.

Show that every such rule set for S5 must include a rule similar to

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\ \mathsf{T}$$

Use this rule and mixed-cut-closure to replace contexts $\Gamma^{\Box} \Rightarrow \Delta^{\Box}$ with $\Gamma \Rightarrow \Delta$. □

Strategy for Translating Rules to Axioms

- We consider all the representative instances of a modal rule

$$\frac{\Gamma_1, \Sigma_1 \Rightarrow \Pi_1, \Delta_1 \quad \dots \quad \Gamma_n, \Sigma_n \Rightarrow \Pi_n, \Delta_n}{\Gamma, \Box\Sigma \Rightarrow \Box\Pi, \Delta}$$

  i.e., instances of the modal rule where
  - Σ, Π consists of variables only
  - Γ, Δ consists of variables and boxed variables only
  - every variable occurs at most once in $\Gamma, \Delta, \Sigma, \Pi$.
- Premisses and conclusion of these are turned into the formulae

$$\text{prem} = \bigwedge_{i=1}^{n}(\bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \to \bigvee \Pi_i \bigvee \Delta_i)$$
$$\text{conc} = \bigwedge \Gamma \wedge \bigwedge \Box\Sigma \to \bigvee \Box\Pi \vee \bigvee \Delta$$

- The information of the premisses is captured in a substitution σ_{prem} and injected into the conclusion by taking $\text{conc}\,\sigma_{\text{prem}}$

Constructing The Substitution σ_{prem}

We assume that our rule set includes the Monotonicity Rule

$$\frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B, \Delta}\ \text{Mon}$$

Definition (Adapted from [Ghilardi:'99])

A formula A is (S5-)projective via a substitution $\sigma : \mathcal{V} \to \mathcal{F}$ of variables by formulae if:

1. $\Rightarrow A\,\sigma$ is derivable in GcutMon
2. for every $B \in \mathcal{F}$ the rule $\dfrac{\Rightarrow A}{\Rightarrow B \leftrightarrow B\sigma}$ is derivable in GcutMon.

Remark

For 2 it is enough to show for every $p \in \mathcal{V}$ derivability of the rule

$$\frac{\Rightarrow A}{\Rightarrow p \leftrightarrow p\sigma}\ .$$

Constructing The Substitution σ_{prem}

Lemma

The formula $\text{prem} = \bigwedge_{i=1}^{n}(\bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \rightarrow \bigvee \Pi_i \vee \bigvee \Delta_i)$ *is projective via*

$$\sigma_{\text{prem}}(p) = \begin{cases} \text{prem} \wedge p, & p \in \Sigma \\ \text{prem} \rightarrow p, & p \in \Pi \\ p, & \textit{otherwise} \end{cases}$$

Proof.

- To see that $\vdash_{\text{GcutMon}} \Rightarrow \text{prem}\, \sigma_{\text{prem}}$:

For every clause $(\bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \rightarrow \bigvee \Pi_i \vee \bigvee \Delta_i)$ of prem we have:

$$\begin{aligned} &(\bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \rightarrow \bigvee \Pi_i \vee \bigvee \Delta_i)\sigma_{\text{prem}} \\ &\equiv \bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \sigma_{\text{prem}} \rightarrow \bigvee \Pi_i \sigma_{\text{prem}} \vee \bigvee \Delta_i \\ &\equiv \bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \wedge \text{prem} \rightarrow \bigvee \Pi_i \vee \bigvee \Delta_i \end{aligned}$$

Since $(\bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \rightarrow \bigvee \Pi_i \vee \bigvee \Delta_i)$ is a clause in prem this is derivable.

Constructing The Substitution σ_{prem}

Lemma

The formula $\text{prem} = \bigwedge_{i=1}^{n}(\bigwedge \Gamma_i \wedge \bigwedge \Sigma_i \rightarrow \bigvee \Pi_i \vee \bigvee \Delta_i)$ *is projective via*

$$\sigma_{\text{prem}}(p) = \begin{cases} \text{prem} \wedge p, & p \in \Sigma \\ \text{prem} \rightarrow p, & p \in \Pi \\ p, & \textit{otherwise} \end{cases}$$

Proof.

- To see that $\dfrac{\Rightarrow \text{prem}}{\Rightarrow p \leftrightarrow p\sigma_{\text{prem}}}$ is derivable is straightforward:

E.g., for $p \in \Pi$:

$$\dfrac{\dfrac{}{p \Rightarrow \text{prem} \rightarrow p}\text{prop} \qquad \dfrac{\Rightarrow \text{prem} \qquad \dfrac{}{\text{prem}, \text{prem} \rightarrow p \Rightarrow p}\text{prop}}{\text{prem} \rightarrow p \Rightarrow p}\text{cut}}{\Rightarrow p\sigma_{\text{prem}} \leftrightarrow p}\text{prop}$$

□

Theorem

A modal rule

$$\frac{\Gamma_1, \Sigma_1 \Rightarrow \Pi_1, \Delta_1 \quad \ldots \quad \Gamma_n, \Sigma_n \Rightarrow \Pi_n, \Delta_n}{\Gamma, \Box\Sigma \Rightarrow \Box\Pi, \Delta} \; R$$

is interderivable over GcutMon *with the axioms* $\mathsf{conc}\,\sigma_{\mathsf{prem}}$ *obtained from its representative instances.*

Proof.

Derive the rule from the axiom using:

$$\dfrac{\Rightarrow \mathsf{conc}\,\sigma_{\mathsf{prem}} \qquad \dfrac{\dfrac{\dfrac{\Gamma_1, \Sigma_1 \Rightarrow \Pi_1, \Delta_1 \quad \ldots \quad \Gamma_n, \Sigma_n \Rightarrow \Pi_n, \Delta_n}{\Rightarrow \mathsf{prem}}\;\mathsf{prop}}{\Rightarrow \mathsf{conc} \leftrightarrow \mathsf{conc}\,\sigma_{\mathsf{prem}}}\;\mathsf{projectivity}}{\mathsf{conc}\,\sigma_{\mathsf{prem}} \Rightarrow \mathsf{conc}}\;\mathsf{prop}}{\dfrac{\Rightarrow \mathsf{conc}}{\Gamma, \Box\Sigma \Rightarrow \Box\Pi, \Delta}\;\mathsf{prop}}\;\mathsf{cut}$$

Theorem

A modal rule

$$\frac{\Gamma_1, \Sigma_1 \Rightarrow \Pi_1, \Delta_1 \quad \ldots \quad \Gamma_n, \Sigma_n \Rightarrow \Pi_n, \Delta_n}{\Gamma, \Box\Sigma \Rightarrow \Box\Pi, \Delta} R$$

is interderivable over GcutMon *with the axioms* $\mathsf{conc}\,\sigma_{\mathsf{prem}}$ *obtained from its representative instances.*

Proof.

Derive the axiom from the rule by:

$$\dfrac{\dfrac{\dfrac{}{\Rightarrow \mathsf{prem}\,\sigma_{\mathsf{prem}}}\text{projectivity}}{(\Gamma_1, \Sigma_1 \Rightarrow \Pi_1, \Delta_i)\,\sigma_{\mathsf{prem}}}\text{prop} \quad \ldots \quad \dfrac{\dfrac{}{\Rightarrow \mathsf{prem}\,\sigma_{\mathsf{prem}}}\text{projectivity}}{(\Gamma_n, \Sigma_n \Rightarrow \Pi_n, \Delta_n)\,\sigma_{\mathsf{prem}}}\text{prop}}{\dfrac{(\Gamma, \Box\Sigma \Rightarrow \Box\Pi, \Delta)\,\sigma_{\mathsf{prem}}}{\Rightarrow \mathsf{conc}\,\sigma_{\mathsf{prem}}}\text{prop}} R$$

□

Example

The rule $\dfrac{\Gamma^{\Box} \Rightarrow A, \Delta^{\Box}}{\Gamma \Rightarrow \Box A, \Delta}$ 45 has representative instances

$$\dfrac{\Box p_1, \ldots, \Box p_n \Rightarrow q, \Box r_1, \ldots, \Box r_k}{\Box p_1, \ldots, \Box p_n \Rightarrow \Box q, \Box r_1, \ldots, \Box r_k}$$

The formulae and substitution are

$$\text{prem} = \bigwedge_{i=1}^{n} \Box p_i \to q \vee \bigvee_{j=1}^{k} \Box r_j \qquad \text{conc} = \bigwedge_{i=1}^{n} \Box p_i \to \Box q \vee \bigvee_{j=1}^{k} \Box r_j$$

$$\sigma_{\text{prem}}(q) = \text{prem} \to q \qquad \sigma_{\text{prem}}(s) = s \text{ for } s \neq q$$

E.g., for $n = 1$ and $k = 1$ the corresponding axiom is:

$$\text{conc}\,\sigma_{\text{prem}} = \Box p_1 \to \Box((\Box p_1 \to q \vee \Box r_1) \to q) \vee \Box r_1$$

Instantiating q with $\bot$ we have the instance

$$\Box p_1 \to \Box(\Box p_1 \wedge \neg \Box r_1) \vee \Box r_1 \quad \equiv \quad (\Box p_1 \to \Box \Box p_1) \wedge (\Diamond \Box r_1 \to \Box r_1)$$

What Do The Axioms Look Like?

An exemplary representative instance of a modal rule from a mixed-cut-closed rule set has the form

$$\frac{\Sigma_1 \Rightarrow \Pi_1 \qquad p, \Box q, \Sigma_2 \Rightarrow \Pi_2, r \qquad \Box q, \Sigma_3 \Rightarrow \Pi_3}{p, \Box q, \Box\Sigma \Rightarrow \Box\Pi, r}$$

The formula prem is

$$(\bigwedge \Sigma_1 \rightarrow \bigvee \Pi_1) \wedge (p, \Box q \wedge \bigwedge \Sigma_2 \rightarrow \bigvee \Pi_2 \vee r) \wedge (\Box q \wedge \bigwedge \Sigma_3 \rightarrow \bigwedge \Pi_3)$$

and the axiom is

$$A_{S5} \quad = \quad p \wedge \Box q \wedge \bigwedge_{s \in \Sigma} \Box(\text{prem} \wedge s) \rightarrow \bigvee_{t \in \Pi} \Box(\text{prem} \rightarrow t) \vee r$$

Such axioms cannot define S5.

Lemma

If A_{S5} is satisfiable in one of the frames $\mathfrak{F} = (\mathbb{N}, \mathbb{N} \times \mathbb{N})$ and $\mathfrak{F}' = (\mathbb{N}, \leq)$, then it is also satisfiable in the other.

Proof.

$$\neg A_{S5} \quad \equiv \quad p \wedge \Box q \wedge \bigwedge_{s \in \Sigma} \Box(\text{prem} \wedge s) \wedge \bigwedge_{t \in \Pi} \Diamond(\text{prem} \wedge \neg t) \wedge \neg t$$

E.g., if $\mathfrak{F}', V', 1 \Vdash \neg A$ for a valuation V', then $\mathfrak{F}, V, 0 \Vdash \neg A$ with

$$V(n) := V'(n+1)$$

(The only boxed formula in prem is $\Box q$!) $\Box$

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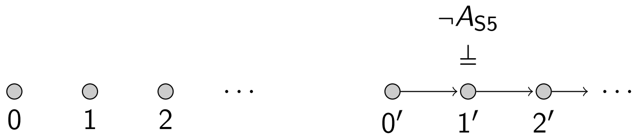

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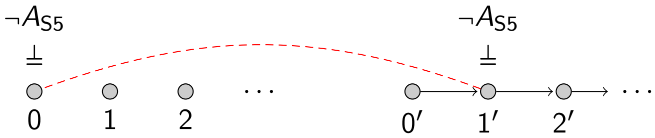

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(The only boxed formula in prem is $\Box q$!) □

No Mixed-cut-closed Rule Sets for S5

Theorem

No sequent calculus with mixed-cut-closed propositional and modal rules is sound and complete for S5 *(even with* cut*).*

Proof.

- The translations of such rules would have a shape like A_{S5} above.
- By the Lemma, such axioms are valid in the S5-frame $(\mathbb{N}, \mathbb{N} \times \mathbb{N})$ iff they are valid in $(\mathbb{N}, \leq)$
- So all axioms (and hence: theorems) of S5 would be valid in $(\mathbb{N}, \leq)$ – but e.g. $p \to \Box\Diamond p$ is not.

□

Can we extend the sequent framework to obtain a cut-free sequent-style calculus for logics like S5?

Hypersequent Calculi

Hypersequents

General idea

Consider several sequents in parallel, allowing for interaction!

Definition

A hypersequent is a multiset $\mathcal{G}$ of sequents, written as

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \,.$$

The sequents $\Gamma_i \Rightarrow \Delta_i$ are called the components of $\mathcal{G}$.

Hypersequent calculi for S5 were independently introduced in

[Mints:'74], [Pottinger:'83], [Avron:'96]

Hypersequents were also used to provide cut-free calculi for many other logics including modal, substructural and intermediate logics.

Hypersequents for S5

The (S5-)interpretation of $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ is

$$\iota(\mathcal{G}) \quad := \quad \Box(\bigwedge \Gamma_1 \rightarrow \bigvee \Delta_1) \vee \cdots \vee \Box(\bigwedge \Gamma_n \rightarrow \bigvee \Delta_n)$$

This interpretation suggests the external structural rules

$$\frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ \text{EW} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ \text{EC}$$

Hypersequent Rules for S5

The calculus HS5 for S5 contains the modal rules

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Box A}\ \Box_R \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \Box A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}\ \Box_L \qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{T}$$

the standard propositional rules in every component and the external structural rules [Restall:'07].

Example

The derivations of $p \Rightarrow \Box\Diamond p$ and $\Box p \Rightarrow \Box\Box p$ are as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \Rightarrow p \mid \Rightarrow}\ \text{init}}{p, \neg p \Rightarrow \mid \Rightarrow}\ \neg_L}{p \Rightarrow \mid \Box\neg p \Rightarrow}\ \Box_L}{p \Rightarrow \mid \Rightarrow \neg\Box\neg p}\ \neg_R}{p \Rightarrow \Box\neg\Box\neg p}\ \Box_R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Rightarrow \mid \Rightarrow \mid p \Rightarrow p}\ \text{init}}{\Box p \Rightarrow \mid \Rightarrow \mid \Rightarrow p}\ \Box_L}{\Box p \Rightarrow \mid \Rightarrow \Box;}\ \Box_R}{\Box p \Rightarrow \Box\Box p}\ \Box_R$$

Soundness of HS5

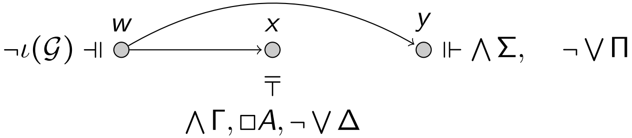
Theorem

The rules of HS5 *preserve validity under the* S5*-interpretation.*

Proof.

E.g., for $\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \Box A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}\ \Box_L$:

If $\mathfrak{M}, w \Vdash \neg\iota(\mathcal{G}) \wedge \Diamond(\bigwedge \Gamma \wedge \Box A \wedge \neg \bigvee \Delta) \wedge \Diamond(\bigwedge \Sigma \wedge \neg \bigvee \Pi)$ we have:



Soundness of HS5

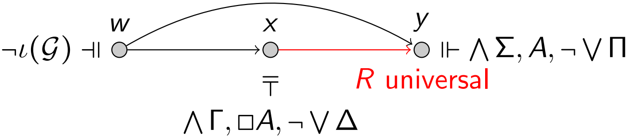
Theorem

The rules of HS5 *preserve validity under the* S5*-interpretation.*

Proof.

E.g., for $\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \Box A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \; \Box_L$:

If $\mathfrak{M}, w \Vdash \neg\iota(\mathcal{G}) \wedge \Diamond(\bigwedge \Gamma \wedge \Box A \wedge \neg \bigvee \Delta) \wedge \Diamond(\bigwedge \Sigma \wedge \neg \bigvee \Pi)$ we have:



So $\mathfrak{M}, w \Vdash \neg\iota(\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi)$. □

Soundness of HS5

Theorem

The rules of HS5 *preserve validity under the* S5*-interpretation.*

Corollary

If $\Rightarrow A$ *is derivable in* HS5, *then* A *is valid in* S5.

Proof.

By induction on the depth of the derivation, and using that the rule

$$\frac{\Box A}{A}$$

is admissible in S5. □

Completeness of HS5

We first show completeness with the hypersequent cut rule

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \qquad \mathcal{H} \mid A, \Sigma \Rightarrow \Pi}{\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ hcut}$$

Theorem

If A is S5*-valid, then $\Rightarrow A$ is derivable in* HS5 *with* hcut.

Proof.

Derive the axioms of S5 and simulate the rule of modus ponens by:

$$\dfrac{\begin{array}{c}\vdots\\ \Rightarrow A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \Rightarrow A \to B\end{array} \qquad \dfrac{\dfrac{}{B, A \Rightarrow B}\text{ init} \qquad \dfrac{}{A \Rightarrow A, B}\text{ init}}{A \to B, A \Rightarrow B}\to_L}{\Rightarrow A \to B}\text{ hcut}}{\Rightarrow B}\text{ hcut}$$

□

Hypersequent Cut Elimination - Complications

Cut elimination for hypersequents is complicated by the external structural rules, in particular by the rule of external contraction:

E.g. we might have the situation

$$\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \qquad \dfrac{\mathcal{H} \mid A, \Sigma \Rightarrow \Pi \mid A, \Sigma \Rightarrow \Pi}{\mathcal{H} \mid A, \Sigma \Rightarrow \Pi}\text{EC}}{\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}\text{hcut}$$

Permuting the cut upwards replaces it by two cuts of the same complexity:

$$\dfrac{\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \qquad \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \qquad \mathcal{H} \mid A, \Sigma \Rightarrow \Pi \mid A, \Sigma \Rightarrow \Pi}{\mathcal{G} \mid \mathcal{H} \mid A, \Sigma \Rightarrow \Pi \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}\text{hcut}}{\mathcal{G} \mid \mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}\text{hcut}}{\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}\text{EC}$$

Cut Elimination for HS5 - Outline

Several methods of cut elimination are possible.
Here we follow one which generalises rather well [Ciabattoni:'10, L.:'14].

Strategy

- pick a top-most cut of maximal complexity
- shift up to the left until the cut formula is introduced ("Shift Left Lemma")
- shift up to the right until the cut formula is introduced ("Shift Right Lemma")
- reduce the complexity of the cut

Key Ingredient

Absorb contractions by considering a more general induction hypothesis, similar to a one-sided mix rule.

Cut Elimination for HS5 - Shift Right Lemma

Definition

The cut rank of a derivation in HS5hcut is the maximal complexity $|A|$ of a cut formula A in it.

Lemma (Shift Right Lemma)

If there are HS5hcut-*derivations*

$$\begin{array}{c} \vdots\ \mathcal{D} \\ \mathcal{G} \mid \Gamma \Rightarrow \Delta, A \end{array} \qquad \textit{and} \qquad \begin{array}{c} \vdots\ \mathcal{E} \\ \mathcal{H} \mid A^{k_1}, \Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid A^{k_n}, \Sigma_n \Rightarrow \Pi_n \end{array}$$

of cut rank $< |A|$ *with* A *principal in the last rule of* $\mathcal{D}$*, then there is a derivation of cut rank* $< |A|$ *of*

$$\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid \ldots \mid \Gamma, \Sigma_n \Rightarrow \Delta, \Pi_n \ .$$

Proof (Shift Right Lemma).

By induction on the depth of the derivation $\mathcal{E}$, distinguishing cases according to the last rule in $\mathcal{E}$. Some interesting cases:

- Last applied rule EC:

$$\dfrac{\vdots\ \mathcal{D}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A} \qquad \dfrac{\dfrac{\vdots\ \mathcal{E}'}{\mathcal{H} \mid A^{k_1}, \Sigma_1 \Rightarrow \Pi_1 \mid \dots \mid A^{k_n}, \Sigma_n \Rightarrow \Pi_n \mid A^{k_n}, \Sigma_n \Rightarrow \Pi_n}}{\mathcal{H} \mid A^{k_1}, \Sigma_1 \Rightarrow \Pi_1 \mid \dots \mid A^{k_n}, \Sigma_n \Rightarrow \Pi_n}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A} \quad \dfrac{\vdots\ \mathcal{E}'}{\mathcal{H} \mid A^{k_1}, \Sigma_1 \Rightarrow \Pi_1 \mid \dots \mid A^{k_n}, \Sigma_n \Rightarrow \Pi_n \mid A^{k_n}, \Sigma_n \Rightarrow \Pi_n}}{\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid \dots \mid \Gamma, \Sigma_n \Rightarrow \Delta, \Pi_n \mid \Gamma, \Sigma_n \Rightarrow \Delta, \Pi_n}\ IH}{\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid \dots \mid \Gamma, \Sigma_n \Rightarrow \Delta, \Pi_n}\ EC$$

Proof (Shift Right Lemma).

By induction on the depth of the derivation $\mathcal{E}$, distinguishing cases according to the last rule in $\mathcal{E}$. Some interesting cases:

- $A = \Box B$ and last applied rule $\Box_L$ with $\Box B$ principal (omitting side hypersequents and showing only two components):

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \Gamma \Rightarrow \Delta \mid\ \Rightarrow B\end{array}}{\Gamma \Rightarrow \Delta, \Box B}\ \Box_R \qquad \dfrac{\begin{array}{c}\vdots\ \mathcal{E}'\\ \Box B^{k_1-1}, \Sigma_1 \Rightarrow \Pi_1 \mid B, \Box B^{k_2}, \Sigma_2 \Rightarrow \Pi_2\end{array}}{\Box B^{k_1}, \Sigma_1 \Rightarrow \Pi_1 \mid \Box B^{k_2}, \Sigma_2 \Rightarrow \Pi_2}\ \Box_L$$

$$\rightsquigarrow$$

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \Gamma \Rightarrow \Delta \mid\ \Rightarrow B\end{array} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\ \mathcal{D}'\\ \Gamma \Rightarrow \Delta \mid\ \Rightarrow B\end{array}}{\Gamma \Rightarrow \Delta, \Box B}\ \Box_R \quad \begin{array}{c}\vdots\ \mathcal{E}'\\ \Box B^{k_1-1}, \Sigma_1 \Rightarrow \Pi_1 \mid B, \Box B^{k_2}, \Sigma_2 \Rightarrow \Pi_2\end{array}}{\Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid B, \Gamma, \Sigma_2 \Rightarrow \Delta, \Pi_2}\ IH}{\Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid \Gamma, \Sigma_2 \Rightarrow \Delta, \Pi_2}\ \text{hcut}, W, EC$$

Cut Elimination for HS5 - Shift Left Lemma

Lemma (Shift Left Lemma)

If there are HS5hcut-*derivations*

$$\vdots\ \mathcal{D} \qquad\qquad\qquad\qquad \vdots\ \mathcal{E}$$

$$\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1, A^{k_1} \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n, A^{k_n} \qquad \textit{and} \qquad \mathcal{H} \mid A, \Sigma \Rightarrow \Pi$$

of cut rank $< |A|$, *then there is a derivation of cut rank* $< |A|$ *of*

$$\mathcal{G} \mid \mathcal{H} \mid \Gamma_1, \Sigma \Rightarrow \Delta_1, \Pi \mid \ldots \mid \Gamma_n, \Sigma \Rightarrow \Delta_n, \Pi \,.$$

Proof (Shift Left Lemma)

By induction on the depth of the derivation $\mathcal{D}$, distinguishing cases according to the last rule in $\mathcal{D}$. An interesting case:

- $A = \Box B$ and last applied rule $\Box_R$ with $\Box B$ principal (omitting side hypersequents and assuming only two components):

$$\dfrac{\dfrac{\vdots\ \mathcal{D}'}{\Gamma_1 \Rightarrow \Delta_1, \Box B^{k_1} \mid \Gamma_2 \Rightarrow \Delta_2, \Box B^{k_2-1} \mid\ \Rightarrow B}}{\Gamma_1 \Rightarrow \Delta_1, \Box B^{k_1} \mid \Gamma_2 \Rightarrow \Delta_2, \Box B^{k_2}}\ \Box_R \qquad \dfrac{\vdots\ \mathcal{E}}{\Box B, \Sigma \Rightarrow \Pi}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}'}{\Gamma_1 \Rightarrow \Delta_1, \Box B^{k_1} \mid \Gamma_2 \Rightarrow \Delta_2, \Box B^{k_2-1} \mid\ \Rightarrow B} \quad \dfrac{\vdots\ \mathcal{E}}{\Box B, \Sigma \Rightarrow \Pi}}{\Gamma_1, \Sigma \Rightarrow \Delta_1, \Pi \mid \Gamma_2, \Sigma \Rightarrow \Delta_2, \Pi \mid\ \Rightarrow B}\ IH}{\Gamma_1, \Sigma \Rightarrow \Delta_1, \Pi \mid \Gamma_2, \Sigma \Rightarrow \Delta_2, \Pi, \Box B}\ \Box_R \qquad \dfrac{\vdots\ \mathcal{E}}{\Box B, \Sigma \Rightarrow \Pi}}{\Gamma_1, \Sigma \Rightarrow \Delta_1, \Pi \mid \Gamma_2, \Sigma \Rightarrow \Delta_2, \Pi}\ SRL$$

Cut Elimination for HS5 - Main Theorem

Theorem

Every derivation in HS5hcut *can be converted into a derivation in* HS5 *with the same conclusion.*

Proof.

By double induction on the cut rank r of the derivation and the number of cuts on formulae with complexity r. Topmost cuts of maximal complexity are eliminated using the Shift Left Lemma. □

Corollary (Cut-free Completeness)

If A is S5*-valid, then $\Rightarrow A$ is derivable in* HS5*.*

Applications: Decidability and Complexity

In order to use the calculus HS5 in a decision procedure for S5 we also need to deal with the contraction rules.

For this we consider the modified system HS5* with rules:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Box B \mid \ \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Box B}\ \Box^*_R \qquad \frac{\mathcal{G} \mid \Gamma, \Box A \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \Box A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}\ \Box^*_L$$

$$\frac{\mathcal{G} \mid \Gamma, \Box A, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{T}^*$$

and propositional rules with principal formulae copied to premisses.

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{p, \neg p \Rightarrow p, \Box\neg\Box\neg p \mid \Box\neg p \Rightarrow \neg\Box\neg p}\ \text{init}}{p, \neg p \Rightarrow \Box\neg\Box\neg p \mid \Box\neg p \Rightarrow \neg\Box\neg p}\ \neg_L}{p \Rightarrow \Box\neg\Box\neg p \mid \Box\neg p \Rightarrow \neg\Box\neg p}\ \Box_L}{p \Rightarrow \Box\neg\Box\neg p \mid \ \Rightarrow \neg\Box\neg p}\ \neg_R}{p \Rightarrow \Box\neg\Box\neg p}\ \Box_R$$

Soundness and Completeness of HS5*

Lemma (Equivalence)

In presence of the structural rules, a hypersequent is derivable in HS5 *iff it is derivable in* HS5*.*

Proof.

Simulate the rules. E.g.:

$$\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Box B}\ \Box_R \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Box B \mid \Rightarrow B}\ \mathrm{W}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Box B}\ \Box_R^*$$

$$\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Box B \mid \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Box B}\ \Box_R^* \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Box B \mid \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Box B, \Box B}\ \Box_R}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Box B}\ \mathrm{IC}$$

□

Admissibility of the structural rules

Lemma

The internal and external structural rules are admissible in HS5*.

Proof.

By induction on the depth of the derivation. E.g.:

$$\dfrac{\dfrac{\Gamma, \Box A, \Box A \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Delta}{\Gamma, \Box A, \Box A \Rightarrow \Delta \mid \Sigma \Rightarrow \Delta}\ \Box_L^*}{\Gamma, \Box A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}\ \mathsf{IC} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, \Box A, \Box A \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi}{\Gamma, \Box A \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi}\ IH}{\Gamma, \Box A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}\ \Box_L^*$$

□

Thus when trying to construct a derivation for a hypersequent

- we don't need to consider the structural rule, in particular the contraction rules
- we don't need to consider rules which only duplicate formulae.

Applications: Decidability and Complexity

To decide whether a formula is valid in S5 we do a backwards proof search in HS5*, applying rules (backwards) only if they create new formulae:

On input $\mathcal{G}$:

$$\frac{}{\Box\Box p \Rightarrow \Box q}$$

Applications: Decidability and Complexity

To decide whether a formula is valid in S5 we do a backwards proof search in HS5*, applying rules (backwards) only if they create new formulae:

On input $\mathcal{G}$:

- apply all propositional rules, universally choose a premiss

$$\overline{\qquad\qquad\qquad} \\ \Box\, \Box\, p \Rightarrow \Box q$$

Applications: Decidability and Complexity

To decide whether a formula is valid in S5 we do a backwards proof search in HS5*, applying rules (backwards) only if they create new formulae:

On input $\mathcal{G}$:

- apply all propositional rules, universally choose a premiss
- apply $\Box_R^*$ in all ways

$$\cfrac{\overline{\overline{\Box\Box p \Rightarrow \Box q \mid \Rightarrow q}}}{\Box\Box p \Rightarrow \Box q}\ \Box_R^*$$

Applications: Decidability and Complexity

To decide whether a formula is valid in S5 we do a backwards proof search in HS5*, applying rules (backwards) only if they create new formulae:

On input $\mathcal{G}$:

- apply all propositional rules, universally choose a premiss
- apply $\Box_R^*$ in all ways
- apply $\Box_L^*$ and T* in all ways

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Box\Box p, \Box p \Rightarrow \Box q \mid \Box p, p \Rightarrow q}}{\Box\Box p \Rightarrow \Box q \mid \Box p \Rightarrow q}\ \mathrm{T}^*}{\Box\Box p \Rightarrow \Box q \mid \Rightarrow q}\ \Box_L^*}{\Box\Box p \Rightarrow \Box q}\ \Box_R^*
$$

Applications: Decidability and Complexity

To decide whether a formula is valid in S5 we do a backwards proof search in HS5*, applying rules (backwards) only if they create new formulae:

On input $\mathcal{G}$:

- apply all propositional rules, universally choose a premiss
- apply $\Box^*_R$ in all ways
- apply $\Box^*_L$ and T* in all ways
- reject if no rule applied
- accept if you see an initial sequent

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Box\Box p, \Box p \Rightarrow \Box q \mid \Box p, p \Rightarrow q}}{\Box\Box p \Rightarrow \Box q \mid \Box p \Rightarrow q}\ \mathrm{T}^*}{\Box\Box p \Rightarrow \Box q \mid\ \Rightarrow q}\ \Box^*_L}{\Box\Box p \Rightarrow \Box q}\ \Box^*_R
$$

Applications: Decidability and Complexity

To decide whether a formula is valid in S5 we do a backwards proof search in HS5*, applying rules (backwards) only if they create new formulae:

On input $\mathcal{G}$:

- apply all propositional rules, universally choose a premiss
- apply $\Box^*_R$ in all ways
- apply $\Box^*_L$ and T* in all ways
- reject if no rule applied
- accept if you see an initial sequent
- repeat

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{no rule applies}}{\Box\Box p, \Box p, p \Rightarrow \Box q \mid \Box p, p \Rightarrow q}}{\Box\Box p, \Box p \Rightarrow \Box q \mid \Box p, p \Rightarrow q}\ \Box^*_L}{\Box\Box p \Rightarrow \Box q \mid \Box p \Rightarrow q}\ T^*}{\Box\Box p \Rightarrow \Box q \mid \ \Rightarrow q}\ \Box^*_L}{\Box\Box p \Rightarrow \Box q}\ \Box^*_R$$

Applications: Decidability and Complexity

To decide whether a formula is valid in S5 we do a backwards proof search in HS5*, applying rules (backwards) only if they create new formulae:

On input $\mathcal{G}$:

- apply all propositional rules, universally choose a premiss
- apply $\Box^*_R$ in all ways
- apply $\Box^*_L$ and T* in all ways
- reject if no rule applied
- accept if you see an initial sequent
- repeat

Complexity (input size $= n$):

- $\rightsquigarrow$ $\leq n$ new forrmulae, universal choices
- $\rightsquigarrow$ $\leq n$ formulae, components
- $\rightsquigarrow$ $\leq n^2$ steps
- $\rightsquigarrow$ $\leq$ steps
- $\rightsquigarrow$ $\leq n$ times

In total: $p(n)$ steps $\rightsquigarrow$ coNP.

Hypersequents for Other Logics

Hypersequent calculi also capture other extensions of S4:
E.g., take the rules

$$\frac{\mathcal{G} \mid \Box\Gamma \Rightarrow A}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box A} \qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box A \Rightarrow \Delta}$$

and for the following logics and frame conditions extend them with:

S4.2	$\forall x, y \exists z : xRz \mathbin{\&} yRz$	$\dfrac{\mathcal{G} \mid \Box\Gamma, \Box\Delta \Rightarrow}{\mathcal{G} \mid \Box\Gamma \Rightarrow \mid \Box\Delta \Rightarrow}$
S4.3	$\forall x, y : xRy$ or yRx	$\dfrac{\mathcal{G} \mid \Sigma, \Box\Gamma \Rightarrow \Pi \quad \mathcal{G} \mid \Theta, \Box\Delta \Rightarrow \Lambda}{\mathcal{G} \mid \Sigma, \Box\Delta \Rightarrow \Pi \mid \Theta, \Box\Gamma \Rightarrow \Lambda}$
S5	$\forall x, y : xRy$	$\dfrac{\mathcal{G} \mid \Box\Gamma, \Delta \Rightarrow \Pi}{\mathcal{G} \mid \Box\Gamma \Rightarrow \mid \Delta \Rightarrow \Pi}$

(from [Kurokawa:'14])

Cut elimination is shown as we did for S5.

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