Cut Elimination for
Modal Logics with Shallow Axioms

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Bern, July 2011

Motivating Example: Conditional Logics

Conditional logic:
binary operator "$>$" ("conditional implication")

many readings of $(A > B)$, e.g.

- Default logic: "if A, then normally B"
- Counterfactual logic: "if A were the case, then so would B"
- Non-monotonic reasoning: "B is a plausible conclusion of A"

⋮

How To Axiomatise This?

E.g. in Default Logic ("if ..., then normally ..."):

- Should $(monday > work) \to ((monday \wedge sick) > work)$ hold?
- Should $(monday > work) \wedge (monday > sick) \to ((monday \wedge sick) > work)$ hold?
- Should $(monday > monday)$ hold?

Question:

How to check derivability from a set of axioms?

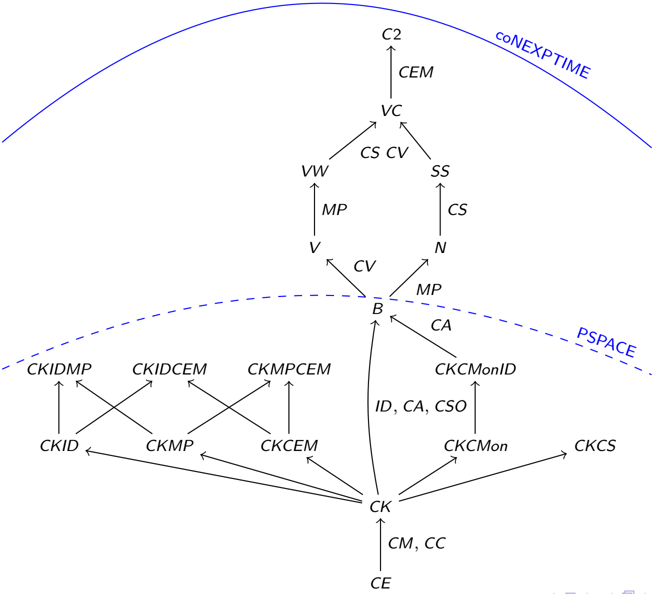
So Many Axioms

Logics given in a Hilbert system $H\mathcal{A}$ with a set $\mathcal{A}$ of axioms, e.g.,

$$
\begin{array}{ll}
\text{(CA)} & (A > B) \wedge (C > B) \to ((A \vee C) > B) \\
\text{(CC)} & (A > B) \wedge (A > C) \to (A > (B \wedge C)) \\
\text{(CEM)} & (A > B) \vee (A > \neg B) \\
\text{(CM)} & (A > (B \wedge C)) \to ((A \wedge B) > C) \\
\text{(CMon)} & (A > B) \wedge (A > C) \to ((A \wedge B) > C) \\
\text{(CS)} & (A \wedge B) \to (A > B) \\
\text{(CSO)} & ((A > B) \wedge (B > A)) \to ((A > C) \leftrightarrow (B > C)) \\
\text{(CV)} & (A > B) \wedge \neg(A > \neg C) \to ((A \wedge C) > B) \\
\text{(ID)} & (A > A) \\
\text{(MP)} & (A > B) \to (A \to B) \\
\vdots & \vdots
\end{array}
$$

Note: All axioms are shallow, i.e. have modal nesting depth 1.

So Many Systems



What's The Problem?

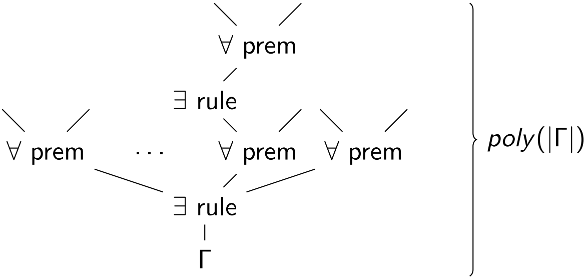
Most algorithms for the derivability problem for these logics use a specifically tailored sequent / tableau system.

How about a generic syntactical treatment?

- Undecidability of some rank 2 logics suggests limitation to shallow axioms
- Aiming for a good complexity (PSPACE) suggests using backwards proof search in a (cut-free) sequent system

Aiming for PSPACE: Tractable Rule Sets

Backwards proof search produces a search tree:



For PSPACE the rule set needs to be tractable: we need to be able to recognise in PSPACE

- the codes of rules matching a given conclusion
- the premisses of a rule given its code

Background on Sequent Systems

Suppose Λ is a set of modalities.

$$\mathcal{F}(\Lambda) \ni A_1, \dots, A_n \ ::= \ p \mid \bot \mid \neg A_1 \mid (A_1 \wedge A_2) \mid \heartsuit(A_1, \dots, A_n)$$

A sequent Γ is a multiset over $\mathcal{F}(\Lambda)$ (read disjunctively)

Rules of the basic system G:

$$\frac{}{\Gamma, p, \neg p} \quad (p \in V) \qquad \frac{}{\Gamma, \neg\bot} \qquad \frac{\Gamma, A}{\Gamma, \neg\neg A}$$

$$\frac{\Gamma, A \quad \Gamma, B}{\Gamma, (A \wedge B)} \qquad \frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)} \qquad \frac{\Gamma, A, A}{\Gamma, A}\ Con \qquad \frac{\Gamma, A \quad \neg A, \Delta}{\Gamma, \Delta}\ Cut$$

$$\frac{A_1 = B_1 \quad \dots \quad A_n = B_n}{\Gamma, \neg\heartsuit(A_1, \dots, A_n), \heartsuit(B_1, \dots, B_n)}\ Cg$$

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From Axioms to Rules

Example: (CC)

$$(A > B) \wedge (A > C) \rightarrow (A > (B \wedge C))$$

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$$\rightsquigarrow \quad \frac{(A > B) \land (A > C) \to (A > (B \land C))}{\Gamma, \neg(A > B), \neg(A > C), (A > (B \land C))}$$

From Axioms to Rules

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$$(A > B) \wedge (A > C) \to (A > (B \wedge C))$$

$$\rightsquigarrow \quad \frac{}{\Gamma, \neg(A > B), \neg(A > C), (A > (B \wedge C))}$$

$$\rightsquigarrow \quad \frac{B \wedge C = D}{\Gamma, \neg(A \;\; > B), \neg(A \;\; > C), (A \;\; > D)}$$

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$$\rightsquigarrow \quad \frac{B \wedge C = D \quad A_1 = A_2 = A_3}{\Gamma, \neg(A_1 > B), \neg(A_2 > C), (A_3 > D)} = R_{(CC)}$$

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Example: (CS)

$$\rightsquigarrow \quad \frac{A \wedge B \to (A > B)}{\Gamma, \neg A, \neg B, (A > B)}$$

From Axioms to Rules

Example: (CC)

$$\rightsquigarrow \quad \frac{(A > B) \land (A > C) \to (A > (B \land C))}{\Gamma, \neg(A > B), \neg(A > C), (A > (B \land C))}$$

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Example: (CS)

$$\rightsquigarrow \quad \frac{A \land B \to (A > B)}{\Gamma, \neg A, \neg B, (A > B)}$$

$$\rightsquigarrow \quad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, (A > B)} \; = \; R_{(CS)}$$

Axioms And Rules

Definition

A shallow rule is a rule of the form

$$\frac{\Gamma, \Gamma_1 \quad \dots \quad \Gamma, \Gamma_\ell \quad \Delta_1 \quad \dots \quad \Delta_m}{\Gamma, M_1, \dots, M_n},$$

where Γ is a sequent, $\Gamma_1, \dots, \Gamma_\ell, \Delta_1, \dots, \Delta_m$ are sequents of literals over variables, and $M_1, \dots, M_n$ are literals over modalised variables.

Theorem

Let $\mathcal{A}$ be a set of shallow axioms, and $\mathcal{R}_\mathcal{A}$ the corresponding rule set. Then for all sequents Γ

$$G\mathcal{R}_\mathcal{A} + Cut + Con \vdash \Gamma \iff H\mathcal{A} \vdash \bigvee \Gamma .$$

Side Remark: Decidability

Definition

A (context-sensitive) pseudo-analytic cut is a cut

$$\frac{\Gamma, \heartsuit(A_1, \ldots, A_n) \quad \neg\heartsuit(A_1, \ldots, A_n), \Gamma}{\Gamma} \; PAC \; ,$$

where the A_i are propositional combinations of formulae B with $(\neg)\spadesuit(\ldots, (\neg)B, \ldots) \in \Gamma$.

Theorem

Pseudo-analytic cuts suffice.

Theorem

If $\mathcal{R}$ is a tractable and contraction closed set of shallow rules, then derivability in $G\mathcal{R} + Cut + Con$ is in 3EXPTIME.

Admissibility of Contraction

- For contraction between principal formulae: close the rule set,

$$\text{e.g. } R_{(CC)} \quad \frac{B_1 \wedge B_1 = B \quad A_1 = A_1 = A}{\Gamma, \neg(A_1 > B_1), \neg(A_1 > B_1), (A > B)} \rightsquigarrow (Cg)\ .$$

- For contraction between principal formulae and context: Add principal formulae to premisses involving context,

$$\text{e.g. } R_{(CS)} \quad \frac{\Gamma, A, (A > B) \quad \Gamma, B, (A > B)}{\Gamma, (A > B)}\ .$$

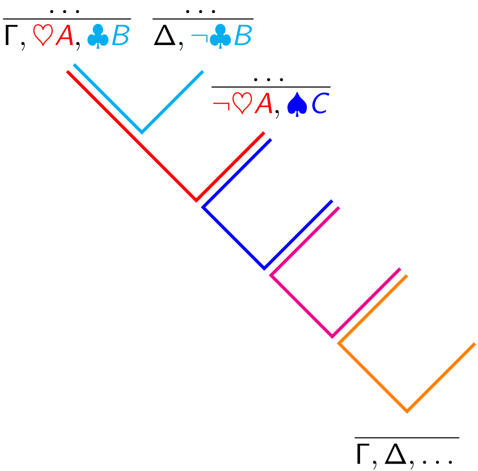
This guarantees admissibility of Contraction.
Then a blocking technique in backwards proof search gives

Theorem

If $\mathcal{R}$ is a contraction closed and tractable set of shallow rules, then the derivability problem for $G\mathcal{R} + Con$ is in PSPACE.

Admissibility of Cut via Cut-trees

Idea: implicitly represent the missing cuts between principal formulas of shallow rules as trees ...



too big...

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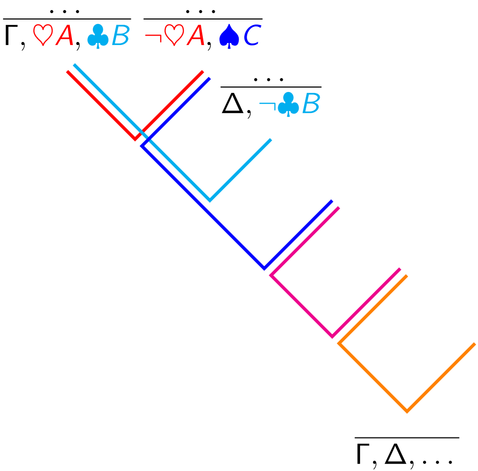
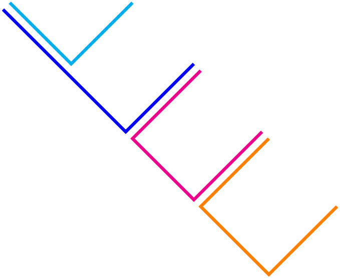


too big...

Admissibility of Cut via Cut-trees

Idea: implicitly represent the missing cuts between principal formulas of shallow rules as trees and close under small cuts.

$$\dfrac{\dots}{\Gamma, \clubsuit B, \spadesuit C} \qquad \dfrac{\dots}{\Delta, \neg\clubsuit B}$$



$$\dfrac{}{\Gamma, \Delta, \dots}$$

still too deep...

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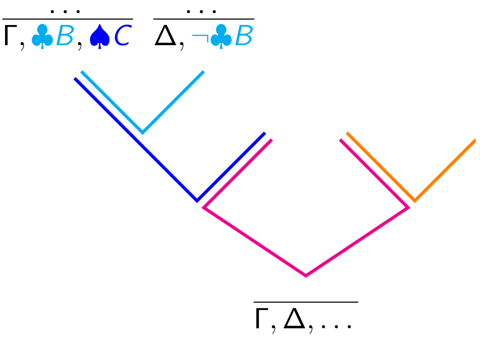

that's it!

Admissibility of Cut via Cut-trees

Observation

For finite $\mathcal{R}$ we may assume closure under cuts with "small" rules.

Definition

If $\mathcal{R}$ is a set of shallow rules, then $\mathcal{R}^*$ is the set of rules represented by cut trees with linear size and logarithmic depth.

Theorem

If $\mathcal{R}$ is finite, $G\mathcal{R}^* + Con + Cut$ is equivalent to $G\mathcal{R} + Con + Cut$. Furthermore, $\mathcal{R}^*$ is tractable, and $G\mathcal{R}^* + Con$ has cut elimination.

Corollary

If $\mathcal{A}$ is finite, and $\mathcal{R}^*_{\mathcal{A}}$ is contraction closed, then the derivability problem for $H\mathcal{A}$ is in PSPACE.

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If $\mathcal{A}$ is finite, and $\mathcal{R}^*_{\mathcal{A}}$ is contraction closed, then the derivability problem for $H\mathcal{A}$ is in PSPACE.

Applications: Conditional Logics

Theorem

If $\mathcal{R}^*$ is contraction closed, and Q is a shallow rule with one principal formula, then $(\mathcal{R} \cup \{Q\})^*$ is contraction closed.

CEM	$(A > B) \vee (A > \neg B)$	ID	$(A > A)$
MP	$(A > B) \rightarrow (A \rightarrow B)$	CS	$(A \wedge B) \rightarrow (A > B)$
CC	$(A > B) \wedge (A > C) \rightarrow (A > (B \wedge C))$		
CM	$(A > (B \wedge C)) \rightarrow (A > B) \wedge (A > C)$		

Theorem (Olivetti, Schwind, 2001; Olivetti, Pozzato, Schwind, 2007; Pattinson, Schröder, 2009)

If $\mathcal{S} \subseteq \{CC, CEM, ID, MP, CS\}$, then the conditional logic axiomatised by $CM + \mathcal{S}$ is decidable in polynomial space.

Conclusion

Results:

- An algorithm to turn shallow axioms into sequent rules.
- A generic 3EXPTIME decidability result for finitely axiomatised shallow logics.
- A generic PSPACE decidability result for "good" logics.

Future Work:

Make the PSPACE decidability result unconditional!

Thank you for your attention!