Constructing Cut-free Sequent Systems with Context Restrictions

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Motivation

Many propositional modal logics are given by axioms in a Hilbert system.

Question:

How can we construct an equivalent cut-free sequent system?

Strategy:

- Translate the axioms
- Isolate general criteria from Gentzen's proof
- Saturate the rule set until it satisfies the general criteria

Here we concentrate on the latter two points.

Basics

Consider (intuitionistic or classical) propositional modal logics.
Formulae are defined as usual:

$$A_1, \ldots, A_n \ni \mathcal{F} ::= \begin{array}{l} p \mid \bot \mid \neg A_1 \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid A_1 \rightarrow A_2 \\ \mid \Box A_1 \mid \Diamond A_1 \mid A_1 \preccurlyeq A_2 \mid \heartsuit(A_1, \ldots, A_n) \mid \ldots \end{array}$$

We use (single- or multi-succedent) sequents $\Gamma \vdash \Delta$.

Our sequent systems have axioms $\overline{\Gamma, A \vdash A, \Delta}$, the structural rules

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \mathrm{con}_L, \quad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ \mathrm{con}_R, \quad \frac{\Gamma \vdash \Delta, A \quad A, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}\ \mathrm{cut},$$

the congruence rules for all operators $\heartsuit$ given by

$$\frac{A_1 \vdash B_1 \quad B_1 \vdash A_1 \quad \ldots \quad A_n \vdash B_n \quad B_n \vdash A_n}{\Gamma, \heartsuit(A_1, \ldots, A_n) \vdash \heartsuit(B_1, \ldots, B_n), \Delta}$$

and additional rules of a specific format.

Rules with Context Restrictions

A rule with context restrictions is of the form

$$\frac{(\Gamma_1 \vdash \Delta_1; \mathcal{C}_1) \quad \dots \quad (\Gamma_n \vdash \Delta_n; \mathcal{C}_n)}{\Sigma \vdash \Pi}$$

with principal formulae $\Sigma \vdash \Pi \in Seq(Mod(Var))$ and premisses $\Gamma_i \vdash \Delta_i \in Seq(Var)$, where the context restrictions $\mathcal{C}_i$ are sets of signed formulae $(X : \ell)$ or $(X : r)$.

In an application of such a rule a premiss with $(X : \ell)$ (resp. $(X : r)$) in its associated context restriction carries over all the substitution instances of X from the left (resp. right) side of the conclusion.

In shallow rules all context restrictions are $\{(p : \ell), (q : r)\}$ or $\emptyset$.

Examples of Context Restrictions

Some well known rules as rules with context restrictions:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge_L^c \qquad \frac{(A, B \vdash\, ; \{(p : \ell), (q : r)\})}{A \wedge B \vdash}$$

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash D}{\Gamma, A \to B \vdash D} \to_L^i \qquad \frac{(\vdash A\, ; \{(p : \ell)\}) \quad (B \vdash\, ; \{(p : \ell), (q : r)\})}{A \to B \vdash}$$

$$\frac{\Box\Gamma \vdash A}{\Sigma, \Box\Gamma \vdash \Box A, \Pi} 4_\Box^c \qquad \frac{(\vdash A\, ; \{(\Box p : \ell)\})}{\vdash \Box A}$$

$$\frac{A, B \vdash C}{\Sigma, \Box A, \Box B \vdash \Box C, \Pi} K_2^c \qquad \frac{(A, B \vdash C\, ; \emptyset)}{\Box A, \Box B \vdash \Box C}$$

We often use the more suggestive notation on the left.
The rules $\wedge_L^c$ and K_2^c are shallow.

Fact

Axioms without nested modalities can be translated into shallow rules using the rules in G3cp.[1]

[1] See [L., Pattinson 2011]

Cut and Contraction as Operations on Rules

For two rules $\dfrac{\mathcal{P}_R}{\Gamma \vdash \Delta, \heartsuit p}\ R$, $\dfrac{\mathcal{P}_Q}{\heartsuit p, \Sigma \vdash \Pi}\ Q$ the cut between R and Q on $\heartsuit p$ is the rule

$$\frac{(\mathcal{P}_R \cup \mathcal{P}_Q)_p}{\Gamma, \Sigma \vdash \Delta, \Pi}\ \mathrm{cut}(R, Q, \heartsuit p) .$$

where the p-elimination of a set $\mathcal{P}$ of premisses with restrictions is

$$\mathcal{P}_p := \begin{array}{l} \{(\Gamma, \Sigma \vdash \Delta, \Pi\ ; \mathcal{C} \cup \mathcal{D}) \mid (\Gamma \vdash \Delta, p; \mathcal{C}) \in \mathcal{P},\ (p, \Sigma \vdash \Pi; \mathcal{D}) \in \mathcal{P}\} \\ \cup \{(\Gamma \vdash \Delta\ ; \mathcal{C}) \in \mathcal{P} \mid p \notin \Gamma \cup \Delta\} \end{array}$$

Slogan: Cut the conclusion, cut the premisses!

Example:

$$\frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\ 4_\Box \quad \frac{A, B \vdash C}{\Box A, \Box B \vdash \Box C}\ K_2 \quad \frac{\Box\Gamma, B \vdash C}{\Box\Gamma, \Box B \vdash \Box C}\ \mathrm{cut}(4_\Box, K_2, \Box A)$$

Contractions of rules are defined similarly.

The General Conditions: Saturation

A set $\mathcal{R}$ of rules is

- principal-cut closed if all rules $\text{cut}(R, Q, \heartsuit\bar{A})$ are $\mathcal{R}$ConW-derivable;
- context-cut closed if whenever context restrictions of R and Q overlap (i.e. there is X with $(X : r) \in \mathcal{C}_R$ and $(X : \ell) \in \mathcal{C}_Q$), then principal formulae and all restrictions of one rule satisfy all restrictions of the other rule overlapping on X;
- mixed-cut closed if whenever a principal formula A of R satisfies a context restriction of Q then all restrictions and principal formulae of R (except for A) satisfy all restrictions of Q overlapping on A;
- contraction closed if for $R \in \mathcal{R}$ the contractions of R are derivable using at most one application of a rule in $\mathcal{R}$;[2]
- saturated if it is all of the above.

[2] Compare the closure condition in [Negri, von Plato 2001]

Generic Cut Elimination

Theorem (Generic Cut Elimination)

In saturated rule sets the cut rule can be eliminated.

Copying all formulae obeying the associated context restriction into the premisses, yields admissibility of Contraction. For shallow and tractable rule sets (codes of applicable rules / their premisses can be computed in pspace from the conclusion / the rule code) we also have

Theorem (Complexity)

For saturated and tractable sets of shallow rules the derivability problem is decidable in PSpace.[3]

[3] See [L., Pattinson 2011]

Example: Constructive $S4$

Idea: Intuitionistic $S4$ without $\Diamond(A \vee B) \equiv (\Diamond A \vee \Diamond B)$ and $\neg\Diamond\bot$.[4]

Rules: single succedent rules for intuitionistic logic with

$$\frac{\Box\Gamma \vdash B}{\Delta, \Box\Gamma \vdash \Box B}\; 4_\Box \quad \frac{\Box\Gamma, A \vdash \Diamond B}{\Delta, \Box\Gamma, \Diamond A \vdash \Diamond B}\; 4_\Diamond \quad \frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B}\; T_\Box \quad \frac{\Gamma \vdash B}{\Gamma \vdash \Diamond B}\; T_\Diamond$$

Now easy to check that this is saturated:

- principal-cut closed: e.g. cut$(4_\Box, T_\Box)$
- context-cut closed: e.g. for $4_\Diamond$ and $T_\Box$: push up in premisses of $T_\Box$
- mixed-cut closed: e.g. for $4_\Box$ and $4_\Diamond$: push up in $4_\Diamond$
- contraction closed: trivial

Corollary

$\mathcal{R}_{CS4}$ *has cut elimination.*

[4]See e.g. [Pfenning, Davies 2001] or [Mendler, Scheele 2011]

Constructing Cut-free Calculi

Question: How to construct saturated rule sets from axioms?

Lemma (Cuts preserve soundness)

For multi- (resp. single-)succedent sequent systems: if G3*cp (resp.* G3*ip)* $\in \mathcal{R}$*, then cuts between rules in* $\mathcal{R}$ *are* $\mathcal{R}$ConCut*-derivable.*

This suggests the following algorithm:

1. Translate the axioms into sequent rules
2. Add cuts and contractions until no more new rules are found (guarantees principal-cut closure and contraction closure)
3. check that the rule set is context- and mixed-cut closed

Example: Conditional Logics $\mathbb{V}$ and $\mathbb{VA}$[5]

Consider formulation in the entrenchment connective $\preccurlyeq$.
Semantics intuitively: $A \preccurlyeq B$ iff A is not more far fetched than B.

Translating the axioms for $\mathbb{V}$ into sequent rules yields the rules

$$\frac{B \vdash A}{\vdash A \preccurlyeq B}\ CP_1\ , \quad \frac{A_1 = B_2 \quad A_2 = B_1}{\Gamma \vdash \Delta, (A_1 \preccurlyeq B_1), (A_2 \preccurlyeq B_2)}\ CO\ ,$$

$$\frac{B_1 = A_2 \quad B_2 = B_3 \quad A_1 = A_3}{\Gamma, (A_1 \preccurlyeq B_1), (A_2 \preccurlyeq B_2) \vdash \Delta, (A_3 \preccurlyeq B_3)}\ TR\ ,$$

$$\frac{\{B_i \vdash A_1, A_2 \mid i \in \{1,2\}\} \ \cup\ \{A_i \vdash B_j \mid i,j \in \{1,2\}\}}{\Gamma \vdash \Delta, (A_1 \preccurlyeq B_1), (A_2 \preccurlyeq B_2)}\ CP_2$$

For $\mathbb{VA}$ add the rules

$$\frac{C \vdash \qquad (A \preccurlyeq B), D \vdash}{(A \preccurlyeq B) \vdash (C \preccurlyeq D)}\ A_1\ , \quad \frac{A \vdash \qquad B \vdash (C \preccurlyeq D)}{\vdash (A \preccurlyeq B), (C \preccurlyeq D)}\ A_2$$

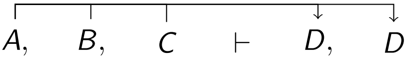
Unfortunately, computing cuts quickly becomes rather messy...

[5] See [Lewis 1973]

A Graphical Representation

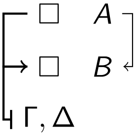
Can we represent sequents and sequent rules more intuitively?

- Represent sequents by doodles: Take $\vdash$ as an arrow with multiple heads and tails:

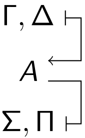


- Represent sequent rules as rule doodles: formulae in polish notation, premisses on the right and conclusion on the left:

$$\frac{A \vdash B}{\Gamma, \Box A \vdash \Box B, \Delta}\ K_1$$



Thus cut $\dfrac{\Gamma \vdash \Delta, A \quad A, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}$ becomes



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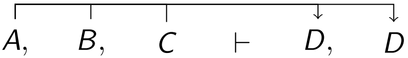

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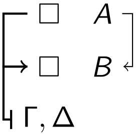

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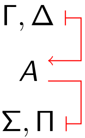

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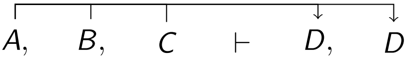

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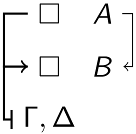

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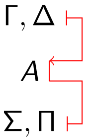

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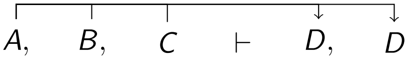

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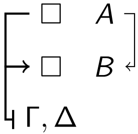

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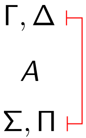

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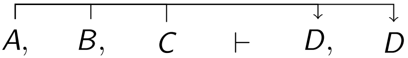

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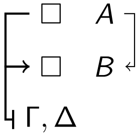

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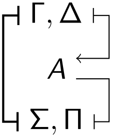

Slogan: "Connect heads and tails and yank the wire!"

Constructing Rules for $\mathbb{V}$ and $\mathbb{VA}$

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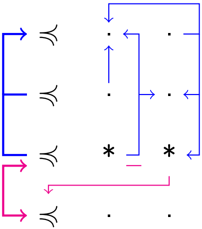
Thus we can compute the cut between two rules by connecting heads and tails and yanking the wire.

A typical step in the construction of the rule set for $\mathbb{VA}$:



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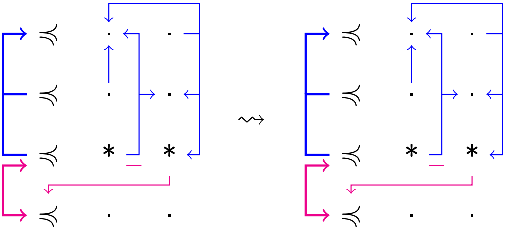

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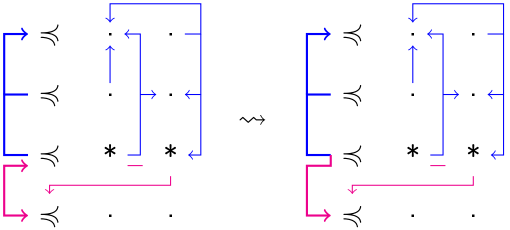

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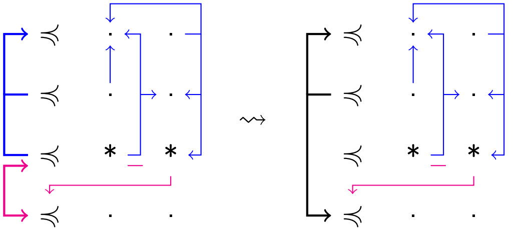

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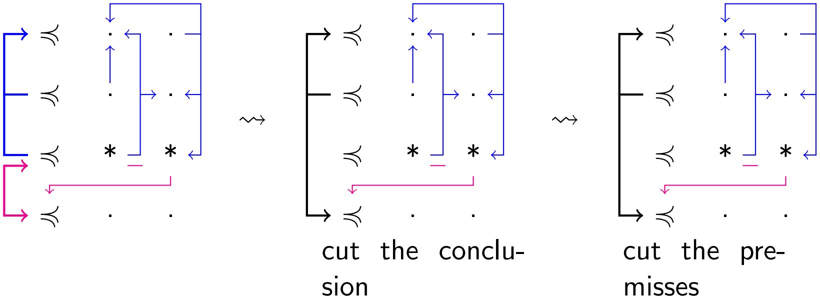

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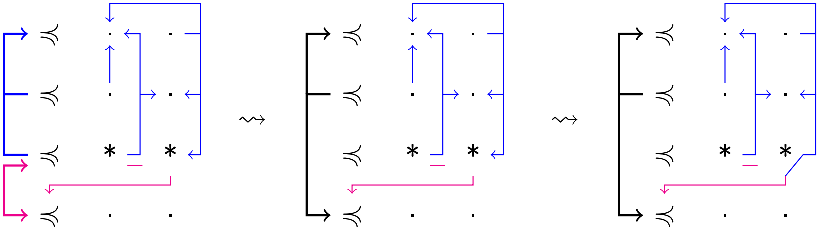

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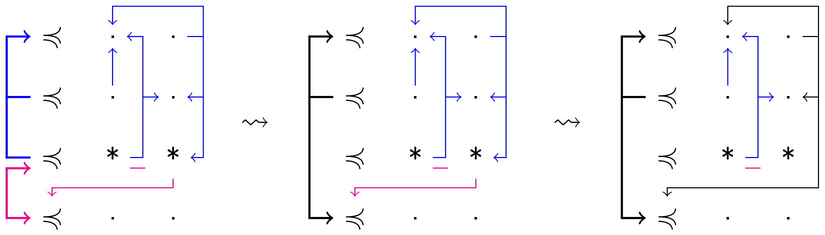

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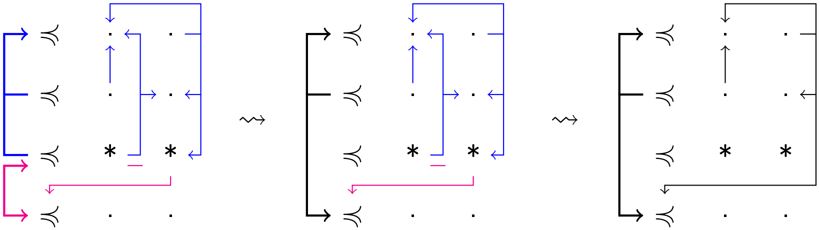

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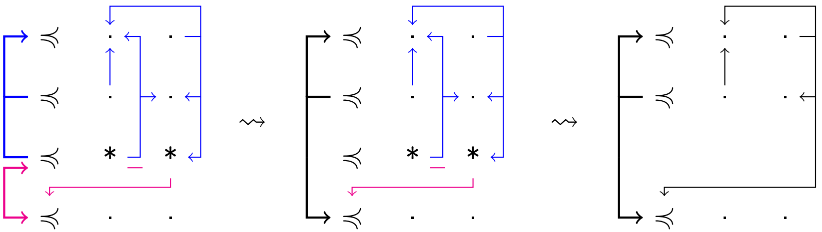

cut the conclusion

cut the premisses

The Rules for $\mathbb{V}$ and $\mathbb{VA}$

This yields for $\mathbb{V}$ the rules of G3*cp* and for $n \geq 1, m \geq 0$ the rules

$$\dfrac{\begin{array}{c}\{\ (B_k \vdash A_1, \ldots, A_n, D_1, \ldots, D_m\ ; \emptyset) \mid k \leq n\ \} \\ \cup\ \{\ (C_k \vdash A_1, \ldots, A_n, D_1, \ldots, D_{k-1}\ ; \emptyset) \mid k \leq m\ \}\end{array}}{(C_1 \preccurlyeq D_1), \ldots, (C_m \preccurlyeq D_m) \vdash (A_1 \preccurlyeq B_1), \ldots, (A_n \preccurlyeq B_n)}\ R_{n,m}$$

For $\mathbb{VA}$ add the context restriction $\{(p \preccurlyeq q : \ell), (r \preccurlyeq s : r)\}$ to every premiss of rule $R_{n,m}$.

Theorem

The systems $\mathcal{R}_{\mathbb{V}}$ and $\mathcal{R}_{\mathbb{VA}}$ have cut elimination. Furthermore, $\mathbb{V}$ is decidable in PSpace and has the Craig interpolation property.

Summing Up

- General criteria guaranteeing cut elimination
- Inspecting these criteria yields heuristic to construct cut-free sequent systems
- The graphical representation greatly simplifies this process
- Applications: new sequent systems for conditional logics.

Thank you very much.