Sequent Systems for Lewis' Conditional Logics

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26 September 2012
JELIA 2012 - Toulouse

Motivation: Wobbly Kangaroos

"If kangaroos didn't have tails, they would topple over."

How to analyse this counterfactual implication?

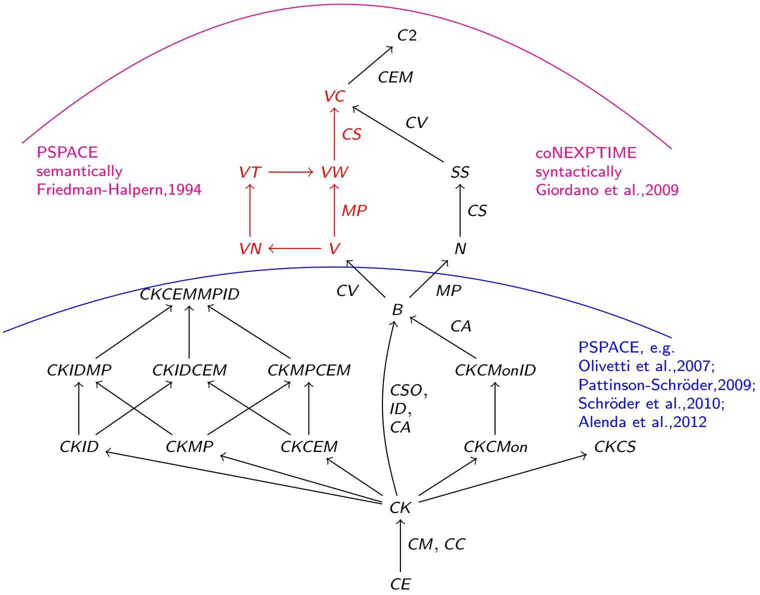
A number of different proposals and logics, e.g. [Lewis,1973]

We adopt a pluralist point of view. Slogan:

"There is a time for every logic"

Here we are interested in deciding validity for the logics.

The Conditional Landscape



Goal:

Systematically construct sequent systems for conditional logics which

- are conceptually simple, i.e. unlabelled
- are cut-free
- give rise to purely syntactical decision procedures of optimal complexity

Preliminaries: Sequent Systems

We consider conditional logics as (non-normal) modal logics over classical propositional logic with the additional binary modalities $\preccurlyeq, \Box\!\!\Rightarrow, \Box\!\!\rightarrow$. Formulae are defined as usual:

$$A, B \ni \mathcal{F} ::= \begin{array}{l} \bot \mid p \mid A \wedge B \mid A \vee B \mid A \rightarrow B \\ \mid A \preccurlyeq B \mid A \Box\!\!\Rightarrow B \mid A \Box\!\!\rightarrow B \end{array}$$

We use sequents $\Gamma \Rightarrow \Delta$, where Γ, Δ are multisets of formulae.

Our sequent systems are based on the system G with axioms $\overline{\Gamma, A \Rightarrow A, \Delta}$ and the standard propositional rules, e.g.

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \wedge\text{R}, \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \wedge\text{L}, \quad \frac{}{\Gamma, \bot \Rightarrow \Delta} \bot\text{L} .$$

Write $\mathrm{G}\mathcal{R}$ for G extended with the rules $\mathcal{R}$. The structural rules are

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \text{ConL}, \quad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} \text{ConR}, \quad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{Cut} .$$

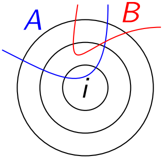
Sphere Semantics, Comparative Possibility And $\mathbb{V}_{\preccurlyeq}$

We make use of the sphere semantics from [Lewis,1973] for $\preccurlyeq$: Intuitively, every world comes with a nested system of spheres, and $A \preccurlyeq B$ holds at a world if for every B-world there is an A-world in the same sphere. E.g. on the right below we have

$$i \vDash (A \preccurlyeq B)$$

but

$$i \nvDash (B \preccurlyeq A)$$



The resulting logic $\mathbb{V}_{\preccurlyeq}$ is given Hilbert-style by the rules and axioms

$$\text{(CP)} \quad \frac{\vdash B \to (A_1 \vee \cdots \vee A_n)}{\vdash (A_1 \preccurlyeq B) \vee \cdots \vee (A_n \preccurlyeq B)} \; (n \geq 1)$$

$$\text{(TR)} \quad (A \preccurlyeq B) \wedge (B \preccurlyeq C) \to (A \preccurlyeq C)$$

$$\text{(CN)} \quad (A \preccurlyeq B) \vee (B \preccurlyeq A)$$

The Sequent System for $\mathbb{V}_{\preccurlyeq}$

$$\frac{\{\ B_k \Rightarrow A_1, \dots, A_n, D_1, \dots, D_m \mid k \leq n\ \} \cup \{\ C_k \Rightarrow A_1, \dots, A_n, D_1, \dots, D_{k-1} \mid k \leq m\ \}}{\Gamma, (C_1 \preccurlyeq D_1), \dots, (C_m \preccurlyeq D_m) \Rightarrow \Delta, (A_1 \preccurlyeq B_1), \dots, (A_n \preccurlyeq B_n)}\ R_{n,m}$$

We set $\mathcal{R}_{\mathbb{V}_{\preccurlyeq}} := \{R_{n,m} \mid n \geq 1,\ m \geq 0\}$.

Theorem

The sequent system $\mathsf{G}\mathcal{R}_{\mathbb{V}_{\preccurlyeq}}$ is sound for $\mathbb{V}_{\preccurlyeq}$.

Since the axioms and rules of the Hilbert system can be derived in the system with Cut and Contraction we have

Theorem

The system $\mathsf{G}\mathcal{R}_{\mathbb{V}_{\preccurlyeq}}\mathsf{CutCon}$ is complete for $\mathbb{V}_{\preccurlyeq}$.

The Sequent System for $\mathbb{V}_{\preccurlyeq}$

$$\frac{\begin{array}{c}\{\ B_k \Rightarrow A_1,\ldots,A_n,D_1,\ldots,D_m \mid k \leq n\ \} \\ \cup\ \{\ C_k \Rightarrow A_1,\ldots,A_n,D_1,\ldots,D_{k-1} \mid k \leq m\ \}\end{array}}{\Gamma,(C_1 \preccurlyeq D_1),\ldots,(C_m \preccurlyeq D_m) \Rightarrow \Delta,(A_1 \preccurlyeq B_1),\ldots,(A_n \preccurlyeq B_n)}\ R_{n,m}$$

We set $\mathcal{R}_{\mathbb{V}_{\preccurlyeq}} := \{R_{n,m} \mid n \geq 1,\ m \geq 0\}$.

Intuitively the rules capture the axioms and are closed under cuts:

$$\dfrac{\dfrac{B \Rightarrow A, D \quad C \Rightarrow A}{(C \preccurlyeq D) \Rightarrow (A \preccurlyeq B)}\ R_{1,1} \qquad \dfrac{F \Rightarrow E, B \quad A \Rightarrow E}{(A \preccurlyeq B) \Rightarrow (E \preccurlyeq F)}\ R_{1,1}}{(C \preccurlyeq D) \Rightarrow (E \preccurlyeq F)}\ \text{Cut}$$

is replaced by cuts on the premisses and a rule:

$$\dfrac{\dfrac{F \Rightarrow E, B \quad B \Rightarrow A, D \quad A \Rightarrow E}{F \Rightarrow E, D}\ \text{Cut, Con} \qquad \dfrac{C \Rightarrow A \quad A \Rightarrow E}{C \Rightarrow E}\ \text{Cut}}{(C \preccurlyeq D) \Rightarrow (E \preccurlyeq F)}\ R_{1,1}$$

Cut Elimination And Decidability

Cut Elimination and Decidability follow from generic theorems:

Theorem (Generic Cut Elimination, L.-Pattinson, 2011)

If $\mathcal{R}$ is closed under cuts and contractions, then a sequent is derivable in $\mathsf{G}\mathcal{R}\mathsf{ConCut}$ *iff it is derivable in* $\mathsf{G}\mathcal{R}\mathsf{Con}$.

Theorem (Generic Decidability, L.-Pattinson, 2011)

If $\mathcal{R}$ is closed under contractions and tractable, then backwards proof search in $\mathsf{G}\mathcal{R}\mathsf{Con}$ *can be implemented in polynomial space.*

Theorem

$\mathcal{R}_{\mathbb{V}_{\preccurlyeq}}$ *is closed under cuts and contractions and is tractable.*

Corollary

$\mathsf{G}\mathcal{R}_{\mathbb{V}_{\preccurlyeq}}\mathsf{Con}$ *is complete for* $\mathbb{V}_{\preccurlyeq}$ *and* $\mathbb{V}_{\preccurlyeq}$ *is decidable in pspace.*

Extensions: $\mathbb{VN}_{\preccurlyeq}, \mathbb{VT}_{\preccurlyeq}, \mathbb{VC}_{\preccurlyeq}$

Extensions of $\mathbb{V}_{\preccurlyeq}$ are given by additional axioms / conditions on the sphere systems. Turning the axioms into rules yields

(N) $\neg(\bot \preccurlyeq \top)$

$$\frac{A \Rightarrow \qquad \Rightarrow B}{\Gamma, (A \preccurlyeq B) \Rightarrow \Delta}\ R_N$$

(T) $(\bot \preccurlyeq \neg A) \rightarrow A$

$$\frac{A \Rightarrow \qquad \Gamma \Rightarrow \Delta, B}{\Gamma, (A \preccurlyeq B) \Rightarrow \Delta}\ R_T$$

(C) $((A \preccurlyeq \top) \wedge (\top \preccurlyeq A)) \rightarrow A$

$$\begin{cases} \dfrac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, (A \preccurlyeq B)}\ R_{C1}, \\ \dfrac{\Gamma, A \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, B}{\Gamma, (A \preccurlyeq B) \Rightarrow \Delta}\ R_{C2} \end{cases}$$

Theorem

The systems $\mathsf{G}R_N\mathsf{Con}, \mathsf{G}R_T\mathsf{Con}$ *and* $\mathsf{G}R_{C1}R_{C2}\mathsf{Con}$ *are sound and complete for the logics* $\mathbb{VN}, \mathbb{VT}$ *and* $\mathbb{VC}$ *respectively. Backwards proof search in these systems can be implemented in pspace.*

(For $\mathsf{G}R_{C1}R_{C2}\mathsf{Con}$ see also [Gent,1992])

Extensions: $\mathbb{VW}_{\preccurlyeq}$

For the extension of $\mathbb{V}_{\preccurlyeq}$ with the axiom

$$(\mathrm{W}) \quad ((\bot \preccurlyeq \neg A) \vee \neg(\neg A \preccurlyeq \top)) \to A$$

we need to add all the rules $W_{n,m}$ given by

$$\frac{\{\ \Gamma \Rightarrow \Delta, A_1, \dots, A_n, D_1, \dots, D_m\ \} \cup \{\ C_k \Rightarrow A_1, \dots, A_n, D_1, \dots, D_{k-1} \mid k \leq m\ \}}{\Gamma, (C_1 \preccurlyeq D_1), \dots, (C_m \preccurlyeq D_m) \Rightarrow \Delta, (A_1 \preccurlyeq B_1), \dots, (A_n \preccurlyeq B_n)}\ W_{n,m}$$

$$\mathcal{R}_{\mathbb{VW}_{\preccurlyeq}} := \{R_{n,m} \mid n \geq 1,\ m \geq 0\} \cup \{R_T\} \cup \{W_{n,m} \mid n \geq 1,\ m \geq 0\}$$

Theorem

$\mathcal{R}_{\mathbb{VW}_{\preccurlyeq}}$ *is closed under cut and contraction and is tractable.*

Corollary

G$\mathcal{R}_{\mathbb{VW}_{\preccurlyeq}}$Con *is sound and complete for* $\mathbb{VW}_{\preccurlyeq}$ *and backwards proof search in this system can be implemented in pspace.*

Other Languages: $\Box\!\!\Rightarrow$

Lewis' strong counterfactual $\Box\!\!\Rightarrow$ is expressed in terms of $\preccurlyeq$ by

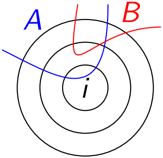
$$(A \Box\!\!\Rightarrow B) \;\longleftrightarrow\; \neg((A \wedge \neg B) \preccurlyeq (A \wedge B))$$

E.g. on the right we have

$$i \vDash (B \Box\!\!\Rightarrow A)$$

but

$$i \nvDash (A \Box\!\!\Rightarrow B).$$



Using the translation we get sound and cut-free complete sequent systems for all the logics in this language, e.g. for $\mathbb{V}_{\Box\!\!\Rightarrow}$ we have $\mathcal{R}_{\mathbb{V}_{\Box\!\!\Rightarrow}} = \{R'_{n,m} \mid n \geq 1,\ m \geq 0\}$ with $R'_{n,m}$ given by

$$\frac{\left\{ C_k, \{B_i \mid i \in I\} \Rightarrow \begin{matrix} \{A_i \mid i \notin I\}, \{C_j \mid j \in J\}, \\ \{D_j \mid k > j \notin J\} \end{matrix} \;\middle|\; \begin{matrix} k \leq m,\ I \subseteq [n], \\ J \subseteq [k-1] \end{matrix} \right\} \cup \left\{ \begin{matrix} A_k, B_k, \\ \{B_i \mid i \in I\} \end{matrix} \Rightarrow \begin{matrix} \{A_i \mid i \notin I\}, \{C_j \mid j \in J\}, \\ \{D_j \mid j \notin J\} \end{matrix} \;\middle|\; \begin{matrix} k \leq n, I \subseteq [n], \\ J \subseteq [m] \end{matrix} \right\}}{\Gamma, (A_1 \Box\!\!\Rightarrow B_1), \ldots, (A_n \Box\!\!\Rightarrow B_n) \Rightarrow \Delta, (C_1 \Box\!\!\Rightarrow D_1), \ldots, (C_m \Box\!\!\Rightarrow D_m)}$$

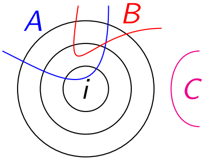
Other Languages: $\Box\!\!\rightarrow$

Lewis' weaker counterfactual $\Box\!\!\rightarrow$ differs from the strong version only if the antecedent is not entertainable:

$$(A \Box\!\!\rightarrow B) \;\longleftrightarrow\; (\,(\bot \preccurlyeq A) \lor \neg((A \land \neg B) \preccurlyeq (A \land B))\,)$$

E.g. on the right again we have $i \models (B \Box\!\!\rightarrow A)$ and $i \not\models (A \Box\!\!\rightarrow B)$, but also for all X

$$i \models (C \Box\!\!\rightarrow X).$$



Since the translation is more complex we don't get sequent systems for the logics in this language. Nevertheless, using formulae in DAG-representation we get

Theorem

There are purely syntactic pspace-decision procedures for all the logics considered in the language with $\Box\!\!\rightarrow$.

Applications: Interpolation

A logic has the Craig Interpolation Property, if whenever we have

$$\vDash A \rightarrow B \ ,$$

then there is an interpolant C with

$$\vDash A \rightarrow C \quad \text{and} \quad \vDash C \rightarrow B \ ,$$

whose variables occur in both A and B.

Using our sequent systems we can establish

Theorem

All the logics considered in all the languages considered have the Craig Interpolation Property.

Applications: Hybrid conditional logic

The strong conditional implication $\Box\!\!\Rightarrow$ can also be interpreted in terms of contextually definite descriptions:
($\texttt{pig} \Box\!\!\Rightarrow \texttt{grunting}$) means "the most salient pig is grunting".

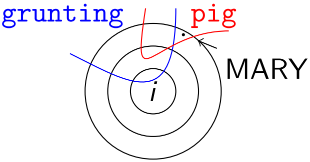
To express that the pig called Mary is not grunting, we need nominals, i.e. names for worlds (see Sano,2009).

Then on the right we have

$$i \models (\texttt{pig} \Box\!\!\Rightarrow \texttt{grunting})$$

and

$$i \models @_{\mathsf{MARY}} \neg \texttt{grunting} \ .$$



Apply the results from (Myers et al., 2009) to $\mathsf{G}\mathcal{R}_{\mathbb{V}_{\Box\Rightarrow}}$ to get

Theorem

The hybrid version $\mathbb{V}^{@}_{\Box\Rightarrow}$ of $\mathbb{V}_{\Box\Rightarrow}$ is decidable in polynomial space.

Summary

- Lewis' conditional logics $\mathbb{V}, \mathbb{VN}, \mathbb{VT}, \mathbb{VW}, \mathbb{VC}$
- Cut free complete unlabelled sequent systems of optimal *pspace*-complexity for the languages with $\preccurlyeq$ and $\Box\!\!\Rightarrow$
- purely syntactic decision procedure of optimal *pspace*-complexity for the language with $\Box\!\!\rightarrow$
- Interpolation for all the logics
- *pspace*-decidability for hybrid conditional logic $\mathbb{V}^{@}_{\Box\!\Rightarrow}$

Thank You!