Comparing Gentzen Systems via Hilbert Axioms

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Motivation

Fact:
There are a many different extensions of Gentzen's sequent framework.

While it is a lot of fun to play around in the different formalisms we have the following

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Which is the appropriate Gentzen framework for a given logic?

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Which is the appropriate Gentzen framework for a given logic?

- allows sound and complete analytic calculus
- as simple as possible

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There are a many different extensions of Gentzen's sequent framework.

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Problem:
Which is the appropriate Gentzen framework for a given logic?

- allows sound and complete analytic calculus
  (Need to restrict the rule format to avoid triviality!)
- as simple as possible
  (Thus we need to compare the different frameworks.)

How to compare different Gentzen frameworks?

One way of comparing different frameworks is to give translations between them. While very interesting this does not necessarily help for the construction of calculi.

Suggestion:

Let's try to give characterisations of the frameworks in a single simple expressive framework!

A good candidate is that of Hilbert systems aka. "Gentzen systems without structure": given by set $\mathcal{A}$ of axioms and the rules

$$\frac{\vdash A}{\vdash A\sigma}\ \text{Sub} \qquad \frac{\vdash A \quad \vdash A \to B}{\vdash B}\ \text{MP} \qquad \frac{\vdash A \leftrightarrow B}{\vdash \heartsuit A \leftrightarrow \heartsuit B}\ \text{Cong}$$

It's very versatile and lots of logics are given as Hilbert systems.

Let's have a look at the beginnings of such a classification theory!

Sequent calculi

Sequents and Rules

For simplicity we consider classical propositional modal logics with unary monotone connectives $\Box, \heartsuit, \dots \in \Lambda$.

Sequents as usual are tuples $\Gamma \Rightarrow \Delta$ of multisets of formulae with the standard interpretation $\bigwedge \Gamma \to \bigvee \Delta$.

We start our investigations with the following rule formats:

One-step rules:
"Forget the whole context!"

$$\frac{A_1, \dots, A_n \Rightarrow B}{\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta}\ \mathsf{K}_n$$

Shallow rules:
"Copy all or nothing"

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\ \mathsf{T}\Box$$

Rules with context restrictions:
"Copy part of the context"

$$\frac{\Box\Gamma \Rightarrow A}{\Sigma, \Box\Gamma \Rightarrow \Box A, \Pi}\ 4\Box$$

Rules with Context Restrictions Formally

A context restriction is a tuple $\langle F_\ell; F_r \rangle$ of sets of formulae. It restricts a sequent $\Gamma \Rightarrow \Delta$ by allowing only substitution instances of formulae from F_ℓ (resp. F_r) in Γ (resp. Δ).

A rule with context restrictions is of the form

$$\frac{(\Gamma_1 \Rightarrow \Delta_1; \mathcal{C}_1) \quad \ldots \quad (\Gamma_n \Rightarrow \Delta_n; \mathcal{C}_n)}{\Sigma \Rightarrow \Pi}$$

with principal formulae $\Sigma, \Pi \subseteq \heartsuit Var$ and premisses $\Gamma_i, \Delta_i \subseteq Var$ with associated context restrictions $\mathcal{C}_i$.

In an application of such a rule a premiss with associated restriction $\mathcal{C}_i$ carries over only the context restricted according to $\mathcal{C}_i$ from the conclusion.

One-step rules use only the restriction $\langle \emptyset, \emptyset \rangle$ and shallow rules use only $\langle \emptyset, \emptyset \rangle$ and $\langle \{p\}, \{p\} \rangle$.

Properties of the Rule Formats

These rule formats are reasonably natural and capture a number of standard calculi for modal logic such as $\mathsf{K}, \mathsf{KT}, \mathsf{S4}$ (or constructive versions).

Moreover, we have some general results [L.-Pattinson13a]:

Theorem

Under certain (syntactical) conditions we have cut elimination.

Theorem

Under certain (syntactical) conditions we have decidability in PSPACE (one-step / shallow rules) resp. EXP (restrictions).

Sequent Rules and Axioms

Axioms corresponding to rules with context restrictions:

A formula given by the following grammar is translatable:

$$
\begin{aligned}
S &::= L \to R \\
L &::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot \\
P_r &::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top \\
P_\ell &::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p_i \mid \bot \mid \top
\end{aligned}
$$

with $\heartsuit \in \Lambda \cup \{\epsilon\}$ and $\psi_\ell \in C_\ell,\ \psi_r \in C_r$ not containing the p_i such that every ψ_ℓ, ψ_r occurs once on the top level and at least once under a modality and ψ_ℓ (resp. ψ_r) distributes over $\wedge$ (resp. $\vee$).

Sequent Rules and Axioms

Axioms corresponding to rules with context restrictions:

$$\Box p_1 \wedge \Box p_2 \to \Box(\Box p_1 \wedge p_2) \quad \rightsquigarrow$$

A formula given by the following grammar is translatable:

$$S ::= L \to R$$

$$L ::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

$$P_\ell ::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p_i \mid \bot \mid \top$$

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$$\Box p_1 \wedge \Box p_2 \to \Box(\Box p_1 \wedge p_2) \quad \rightsquigarrow \quad \frac{}{\Box p_1, \Box p_2 \Rightarrow \Box(\Box p_1 \wedge p_2)}$$

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with $\heartsuit \in \Lambda \cup \{\epsilon\}$ and $\psi_\ell \in C_\ell$, $\psi_r \in C_r$ not containing the p_i such that every ψ_ℓ, ψ_r occurs once on the top level and at least once under a modality and ψ_ℓ (resp. ψ_r) distributes over $\wedge$ (resp. $\vee$).

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A formula given by the following grammar is translatable:

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Sequent Rules and Axioms

Axioms corresponding to rules with context restrictions:

$$\Box p_1 \wedge \Box p_2 \to \Box(\Box p_1 \wedge p_2) \quad \rightsquigarrow \quad \frac{\Box p_1 \wedge p_2 \Rightarrow q}{\Box p_1, \Box p_2 \Rightarrow \Box q}$$

A formula given by the following grammar is translatable:

$$S ::= L \to R$$

$$L ::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

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Sequent Rules and Axioms

Axioms corresponding to rules with context restrictions:

$$\Box p_1 \wedge \Box p_2 \to \Box(\Box p_1 \wedge p_2) \quad \rightsquigarrow \quad \frac{\Box p_1, p_2 \Rightarrow q}{\Box p_1, \Box p_2 \Rightarrow \Box q}$$

A formula given by the following grammar is translatable:

$$S ::= L \to R$$

$$L ::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

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with $\heartsuit \in \Lambda \cup \{\epsilon\}$ and $\psi_\ell \in C_\ell,\ \psi_r \in C_r$ not containing the p_i such that every ψ_ℓ, ψ_r occurs once on the top level and at least once under a modality and ψ_ℓ (resp. ψ_r) distributes over $\wedge$ (resp. $\vee$).

Sequent Rules and Axioms

Axioms corresponding to rules with context restrictions:

$$\Box p_1 \land \Box p_2 \to \Box(\Box p_1 \land p_2) \quad \rightsquigarrow \quad \frac{\Box\Gamma, p_2 \Rightarrow q}{\Box\Gamma, \Box p_2 \Rightarrow \Box q}$$

A formula given by the following grammar is translatable:

$$S ::= L \to R$$

$$L ::= L \land L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \lor R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

$$P_r ::= P_r \lor P_r \mid P_r \land P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

$$P_\ell ::= P_\ell \lor P_\ell \mid P_\ell \land P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p_i \mid \bot \mid \top$$

with $\heartsuit \in \Lambda \cup \{\epsilon\}$ and $\psi_\ell \in C_\ell,\ \psi_r \in C_r$ not containing the p_i such that every ψ_ℓ, ψ_r occurs once on the top level and at least once under a modality and ψ_ℓ (resp. ψ_r) distributes over $\land$ (resp. $\lor$).

Sequent Rules and Axioms

Axioms corresponding to shallow rules:

$$\Box p_1 \wedge \Box p_2 \to \Box(\Box p_1 \wedge p_2) \quad \rightsquigarrow \quad \frac{\Box\Gamma, p_2 \Rightarrow q}{\Box\Gamma, \Box p_2 \Rightarrow \Box q}$$

A formula given by the following grammar is non-nested:

$$S ::= L \to R$$

$$L ::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

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with $\heartsuit \in \Lambda \cup \{\epsilon\}$ and $\psi_\ell \in C_\ell$, $\psi_r \in C_r$ not containing the p_i such that every ψ_ℓ, ψ_r occurs once on the top level and at least once under a modality and ψ_ℓ (resp. ψ_r) distributes over $\wedge$ (resp. $\vee$).

Sequent Rules and Axioms

Axioms corresponding to one-step rules:

$$\Box p_1 \wedge \Box p_2 \to \Box(\Box p_1 \wedge p_2) \quad \rightsquigarrow \quad \frac{\Box\Gamma, p_2 \Rightarrow q}{\Box\Gamma, \Box p_2 \Rightarrow \Box q}$$

A formula given by the following grammar is rank-one:

$$S ::= L \to R$$

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Overview over Results

This gives a classification [L.-Pattinson13b]:

Theorem

We have the following precise correspondences between classes of Hilbert axioms and rules:

translatable rank-1	$\longleftrightarrow$	one-step rule
translatable non-nested	$\longleftrightarrow$	shallow rule
translatable	$\longleftrightarrow$	rule with (normal) restrictions
translatable scheme	$\longleftrightarrow$	rule with general restrictions

Applications: Limitative Results (GL)

Theorem

Gödel-Löb logic cannot be captured by rules with simple context restrictions, i.e. restrictions $\langle G, F\rangle$ *with* $G, F \in \{\emptyset, \{p\}, \{\Box p\}\}$.

(Note that $\dfrac{\Box\Gamma, \Box A \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}$ is not a rule with context restrictions!)

Proof sketch:

Translations of such rules have the form

$$p \wedge \Box q \wedge P \wedge \bigwedge_{i \in I} \Box C_i \to \bigvee_{j \in J} \Box D_j \vee \Box r \vee s$$

with $p, \Box q$ (resp. $s, \Box r$) occurring only negatively (resp. positively) in C_i and vice versa for D_j.

But such formulae are not expressive enough to characterise GL-frames and hence cannot axiomatise GL.

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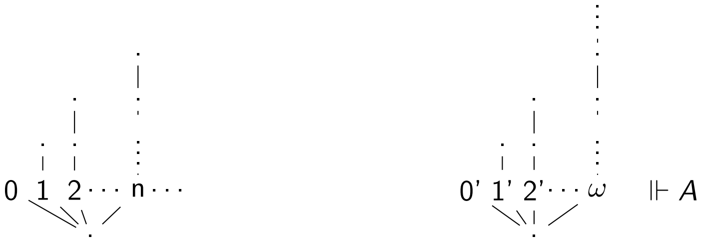
Proof sketch:

E.g. negating $p \wedge \Box q \rightarrow \Box(p \wedge \Box q \rightarrow \Box r) \vee \Box r$ we get $A := p \wedge \Box q \wedge \Diamond(p \wedge \Box q \wedge \Diamond\neg r) \wedge \Diamond\neg r$ which is satisfiable in the non GL-frame (right) iff satisfiable in the GL-frame (left):



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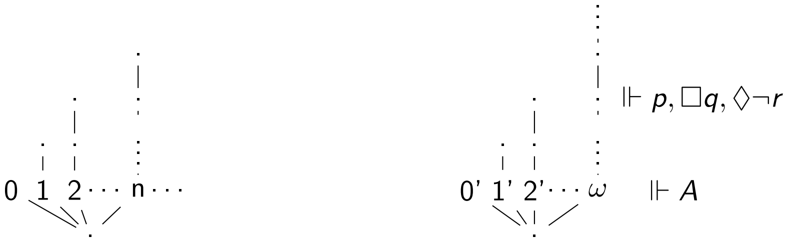

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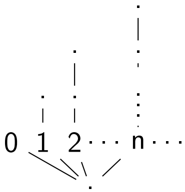

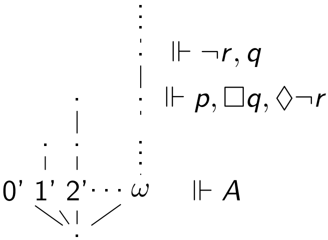

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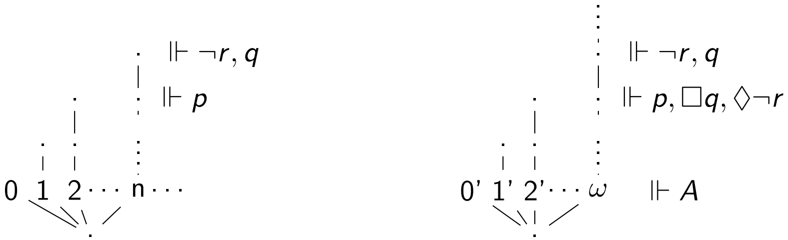

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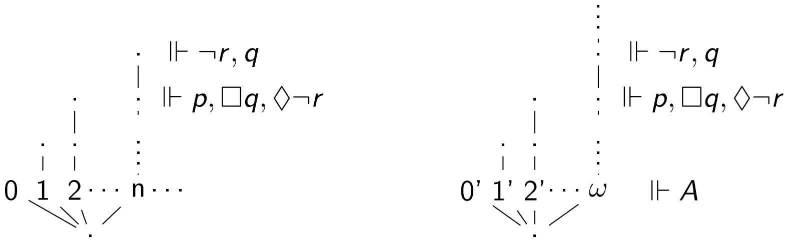

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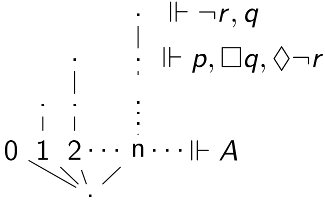

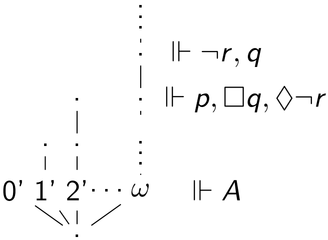

Hypersequent calculi

Hypersequents and Rules

As usual, hypersequents are multisets $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ of sequents – but now the intended interpretation for $\mid$ is not clear!

An interpretation for a logic $\mathcal{L}$ is a set $\{\varphi_n(p_1, \ldots, p_n) : n \in \mathbb{N}\}$ of formulae which respects the structural rules, (e.g. $\models_{\mathcal{L}} \varphi_n(\xi_1, \xi_2, \vec{\chi})$ iff $\models_{\mathcal{L}} \varphi_n(\xi_2, \xi_1, \vec{\chi})$ etc) such that $\models_{\mathcal{L}} \psi$ iff $\models_{\mathcal{L}} \varphi_1(\psi)$ (regularity).

Examples: $\iota_\Box = \{\bigvee_{i \leq n} \Box p_i : n \in \mathbb{N}\}$ for reflexive modal logics or $\iota_{\mathsf{i}} = \{\bigvee_{i \leq n} p_i : n \in \mathbb{N}\}$ for intuitionistic logics.

Simple hypersequent rules with context restrictions are of the form

$$\frac{(\Gamma_1 \Rightarrow \Delta_1; \mathcal{C}_1^1 \ldots \mathcal{C}_n^1) \quad \ldots \quad (\Gamma_m \Rightarrow \Delta_m; \mathcal{C}_1^m \ldots \mathcal{C}_n^m)}{\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n}$$

with $\mathcal{C}_j^i \in \{\langle \emptyset, \emptyset \rangle, \langle \{p\}, \{p\} \rangle, \langle \{\Box p\}, \emptyset \rangle$ and $\Gamma_i, \Delta_i \subseteq Var$ and $\Sigma_i, \Pi_i \subseteq \Box(Var)$. In an application the premiss with restriction $\mathcal{C}_1^i \ldots \mathcal{C}_n^i$ copies the context of the jth component restricted by $\mathcal{C}_j^i$.

Hypersequent Rules and Axioms

The axioms corresponding to simple sequent rules are the formulae given by the following grammar:

$$S ::= L \to R$$

$$L ::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

$$P_\ell ::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell p_i \mid \bot \mid \top$$

with $\heartsuit \in \Lambda \cup \{\epsilon\}$ and $\psi_\ell \in \{q_i, \Box q_i : i \in \mathbb{N}\}$, $\psi_r \in \{r_i : i \in \mathbb{N}\}$ such that every ψ_ℓ, ψ_r occurs once on the top level and at least once under a modality.

Hypersequent Rules and Axioms

The axioms corresponding to simple hypersequent rules for $\iota_\Box = \{\bigvee_{i \leq n} \Box p_i : n \in \mathbb{N}\}$ are the $\iota_\Box$-simple formulae given by the following grammar:

$$S ::= \varphi_n(L \to R, \ldots, L \to R)$$

$$L ::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

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with $\heartsuit \in \Lambda \cup \{\epsilon\}$ and $\psi_\ell \in \{q_i, \Box q_i : i \in \mathbb{N}\}$, $\psi_r \in \{r_i : i \in \mathbb{N}\}$ such that every ψ_ℓ, ψ_r occurs under φ_n once on the top level and at least once under a modality.

Examples: S4.2, S4.3, S5, …

Summing Up

Hilbert-axioms

- help comparing and classifying Gentzen-style systems
- provide limitative results.

translatable rank-1	$\longleftrightarrow$	one-step rule
translatable non-nested	$\longleftrightarrow$	shallow rule
translatable	$\longleftrightarrow$	rule with (normal) restrictions
translatable scheme	$\longleftrightarrow$	rule with general restrictions
$\iota_\Box$-simple	$\longleftrightarrow$	simple hypersequent rule

Thank you very much.