The Linear Nested Sequent Framework

Björn Lellmann

TU Wien

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The Problem: Calculi for Modal Logics

Sequent calculi for modal logics are well-established and well-understood – but not entirely satisfactory!

Some desiderata for "good" calculi [Wansing:'02]:

- subformula property: all the material in the premisses is contained in the conclusion
- separation: distinct left and right introduction rules
- locality: no restrictions on the context
- modularity: obtain other logics by changing single rules

The Problem: Calculi for Modal Logics . . . Don't Work

> *It can be easily verified that each of the standard rule systems [for modal logics] fails to satisfy some of the philosophical requirements [...].*
>
> [Wansing:'94]

E.g.:

	$\dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}$ k	$\dfrac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}$ 4 $\dfrac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}$ t
Subformula property:	✓	✓
Separation:	✗	✓
Locality:	✗	✗
Modularity:	✗	

The Solution: Extend the Sequent Framework

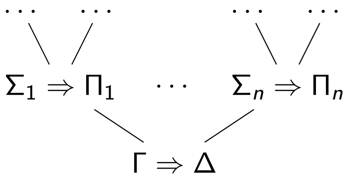
If the sequent structure is not rich enough for modal logics,
extend the structure!

Two of the main contenders:

For the internal approaches:

Nested Sequents

- extend sequent structure



For the external approaches:

Labelled Sequents

- extend formula structure

$$xRy, x : \Box A \Rightarrow y : A$$

An alternative contender for the (mostly) internal approach:

Linear Nested Sequents

Reminder: Modal logics

The formulae of modal logic are given by

$$\varphi ::= \mathsf{Var} \mid \neg\varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \to \varphi \mid \Box\varphi$$

The Hilbert-style presentation of normal modal logic K is given by the axioms and rules for classical propositional logic and

$$\mathsf{k}\ \ \Box(A \to B) \land \Box A \to \Box B \qquad\qquad \frac{\vdash A}{\vdash \Box A}\ \mathsf{nec}$$

A sequent is a tuple of multisets of formulae, written $\Gamma \Rightarrow \Delta$ and interpreted as $\bigwedge\Gamma \to \bigvee\Delta$.

The sequent system for K contains the standard propositional rules together with

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ \mathsf{k}$$

Linear nested sequents

Definition

A linear nested sequent (LNS) is a finite list of sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n$$

and interpreted as $\bigwedge\Gamma_1 \to \bigvee\Delta_1 \vee \Box(\ldots\Box(\bigwedge\Gamma_n \to \bigvee\Delta_n)\ldots)$.

The nested sequent system for K yields the modal rules of $\mathsf{LNS_K}$:

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}}\ \Box_L \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \ \Rightarrow A}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \Box A}\ \Box_R$$

The propositional rules are standard, e.g.:

$$\frac{\mathcal{G} /\!\!/ \Gamma, B \Rightarrow \Delta /\!\!/ \mathcal{H} \quad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, A \to B \Rightarrow \Delta /\!\!/ \mathcal{H}}\ \to_L \qquad \frac{\mathcal{G} /\!\!/ \Gamma, A \Rightarrow \Delta, B /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B /\!\!/ \mathcal{H}}\ \to_R$$

Remark: These are essentially the 2-sequents of [Masini:92]

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Subformula property: ✓

Separation: ✓

Locality: ✓

Completeness for linear nested sequents

We could show completeness via cut elimination ... but it's easier!

Main Observation: The data structure of LNS is the same as that of a history in backwards proof search for a sequent calculus.

So we simply simulate a sequent derivation in the last components: ($\mathcal{G}$ is the history)

$$\dfrac{\dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ \mathsf{k}}{\vdots\ \mathcal{G}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathcal{G} /\!\!/ \Box\Gamma \Rightarrow \Box A /\!\!/ \Gamma \Rightarrow A}{\mathcal{G} /\!\!/ \Box\Gamma \Rightarrow \Box A /\!\!/ \Rightarrow A}\ \Box_L}{\mathcal{G} /\!\!/ \Box\Gamma \Rightarrow \Box A}\ \Box_R$$

Theorem: $\mathsf{LNS_K}$ is sound and cut-free complete for K.

Corollary: Cut-free completeness of the nested sequent calculus.

Extension No.1: Multimodal Logics

The formulae of multimodal logics include modalities $\Box_i$ for $i \in I$.

To capture multimodal logics we include indexed nesting operators:

$$\Gamma_1 \Rightarrow \Delta_1 /\!\!/^{i_1} \dots /\!\!/^{i_n} \Gamma_{n+1} \Rightarrow \Delta_{n+1}$$

interpreted as: $\bigwedge\Gamma_1 \to \bigvee\Delta_1 \vee \Box_{i_1}(\dots\Box_{i_n}(\bigwedge\Gamma_{n+1} \to \bigvee\Delta_{n+1})\dots)$.

E.g.: multimodal logic $\mathsf{K} \oplus \mathsf{K}$

$$\frac{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta /\!\!/^1 \Rightarrow A}{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta, \Box_1 A} \qquad \frac{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta /\!\!/^2 \Rightarrow A}{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta, \Box_2 A}$$

$$\frac{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta /\!\!/^1 \Sigma, A \Rightarrow \Pi /\!\!/^\ell \mathcal{H}}{\mathcal{G} /\!\!/^k \Gamma, \Box_1 A \Rightarrow \Delta /\!\!/^1 \Sigma \Rightarrow \Pi /\!\!/^\ell \mathcal{H}} \qquad \frac{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta /\!\!/^2 \Sigma, A \Rightarrow \Pi /\!\!/^\ell \mathcal{H}}{\mathcal{G} /\!\!/^k \Gamma, \Box_2 A \Rightarrow \Delta /\!\!/^2 \Sigma \Rightarrow \Pi /\!\!/^\ell \mathcal{H}}$$

Extension No.1: Simply Dependent Multimodal Logics

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E.g.: simply dependent multimodal logic $\mathsf{K} \oplus \mathsf{K} \oplus (\Box_2 p \to \Box_1 p)$

$$\frac{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta /\!\!/^1 \ \Rightarrow A}{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta, \Box_1 A} \qquad \frac{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta /\!\!/^2 \ \Rightarrow A}{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta, \Box_2 A}$$

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E.g.: simply dependent multimodal logic $\mathsf{K} \oplus \mathsf{K} \oplus (\Box_2 p \to \Box_1 p)$

Theorem: For $\mathcal{L}$ a suitable simply dependent multimodal logic, the calculus $\mathsf{LNS}_{\mathcal{L}}$ is sound and complete for $\mathcal{L}$.

Proof sketch: Simulate the sequent rules, e.g.:

$$\dfrac{\Gamma, \Sigma \Rightarrow A}{\Box_1\Gamma, \Box_2\Sigma \Rightarrow \Box_1 A}\ \mathsf{k} \qquad \vdots\ \mathcal{G} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\mathcal{G} /\!\!/^{k} \Box_1\Gamma, \Box_2\Sigma \Rightarrow \Box_1 A /\!\!/^{1} \Gamma, \Sigma \Rightarrow A}{\mathcal{G} /\!\!/^{k} \Box_1\Gamma, \Box_2\Sigma \Rightarrow \Box_1 A /\!\!/^{1} \Gamma \Rightarrow A}}{\mathcal{G} /\!\!/^{k} \Box_1\Gamma, \Box_2\Sigma \Rightarrow \Box_1 A /\!\!/^{1} \Rightarrow A}}{\mathcal{G} /\!\!/^{k} \Box_1\Gamma, \Box_2\Sigma \Rightarrow \Box_1 A}$$

Extension No.2: Non-normal Modal Logics

The language of monotone modal logic M is that of modal logic.

The sequent system for M contains the standard propositional rules and the rule

$$\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\ \text{Mon}$$

To capture this rule we use a marker $\backslash\!\!\backslash$ for "unfinished rules": A monotone LNS has the form ($n \geq 1$)

$$\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n$$

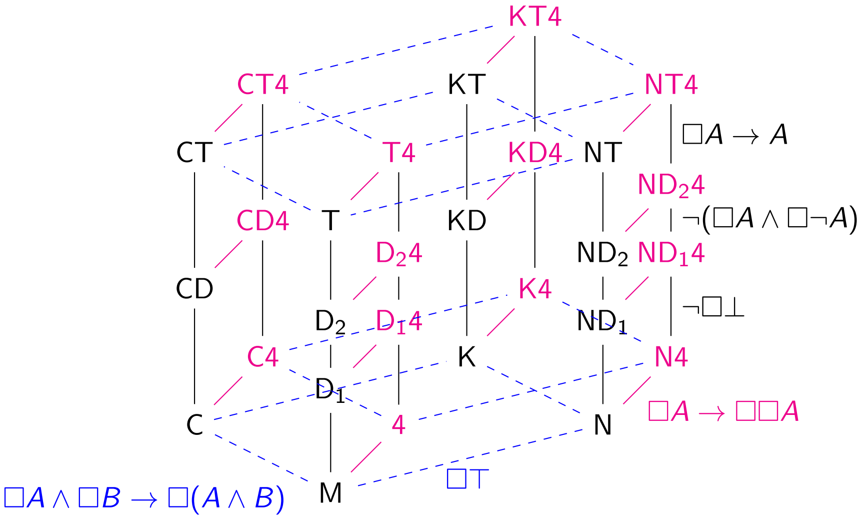
$$\text{or} \quad \Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n \backslash\!\!\backslash \Gamma_{n+1} \Rightarrow \Delta_{n+1}$$

Translating the sequent rule Mon yields the modal rules of $\mathsf{LNS_M}$:

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta \backslash\!\!\backslash \ \Rightarrow B}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \Box B}\ \Box_R \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta \backslash\!\!\backslash \Sigma \Rightarrow \Pi}\ \Box_L$$

The propositional rules cannot be applied inside $\backslash\!\!\backslash$.

Modularity: the Modal Tesseract



Theorem: For $\mathcal{A} \subseteq \{\mathsf{C}, \mathsf{N}, \mathsf{D}_1, \mathsf{D}_2, \mathsf{T}, 4\}$ the calculus $\mathsf{LNS}_{\mathsf{M}\mathcal{A}}$ is sound and complete for $\mathsf{M}\mathcal{A}$.

Adding to the Mix: Substructural Logics

We can change the base logic from classical logic to multiplicative additive linear logic (MALL) with formulae in NNF given by:

$$\varphi ::= \mathsf{Var} \mid \mathsf{Var}^{\perp} \mid 0 \mid 1 \mid \top \mid \perp \mid \varphi \oplus \varphi \mid \varphi \parr \varphi \mid \varphi \,\&\, \varphi \mid \varphi \otimes \varphi$$

(Classical connectives split into additive and multiplicative ones.)

Add to the mix indexed modalities $!^i$ with duals $?^i$ for $i \in I$.
For this we use single-sided LNS of the form ($n \geq 0$):

$$\Gamma_1 \mathbin{/\!\!/}^{i_1} \dots \mathbin{/\!\!/}^{i_n} \Gamma_{n+1}$$
$$\text{or} \quad \Gamma_1 \mathbin{/\!\!/}^{i_1} \dots \mathbin{/\!\!/}^{i_{n-1}} \Gamma_n \mathbin{\backslash\!\!\backslash}^{i_n} \Gamma_{n+1}$$

with interpretation given by: $\parr \Gamma_1 \parr !^{i_1}(\dots !^{i_n}(\parr \Gamma_{n+1}) \dots)$

Remark: See also [Guerrini et al:'98].

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with interpretation given by: $\parr \Gamma_1 \parr !^{i_1}(\ldots !^{i_n}(\parr \Gamma_{n+1}) \ldots)$
and $\parr \Gamma_1 \parr !^{i_1}(\ldots !^{i_{n-1}}(\parr \Gamma_n \parr \parr \Gamma_{n+1}) \ldots)$ respectively

Remark: See also [Guerrini et al:'98].

The propositional rules split into additive and multiplicative ones:

$$\frac{\mathcal{G} \mathbin{/\!/}^{k} \Gamma, F \mathbin{/\!/}^{\ell} \mathcal{H} \quad \mathcal{G} \mathbin{/\!/}^{k} \Gamma, G \mathbin{/\!/}^{\ell} \mathcal{H}}{\mathcal{G} \mathbin{/\!/}^{k} \Gamma, F \mathbin{\&} G \mathbin{/\!/}^{\ell} \mathcal{H}} \qquad \frac{\mathcal{G} \mathbin{/\!/}^{k} \Gamma, F \quad \mathcal{G} \mathbin{/\!/}^{k} \Sigma, G}{\mathcal{G} \mathbin{/\!/}^{k} \Gamma, \Sigma, F \otimes G}$$

$$\frac{}{\mathcal{E} \mathbin{/\!/}^{k} p, p^{\perp}} \qquad \frac{}{\mathcal{E} \mathbin{/\!/}^{k} \Gamma, \top}$$

Substructural Logics

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$$\frac{}{\mathcal{E} \mathbin{/\!\!/}^k p, p^\perp} \qquad \frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma \mathbin{\backslash\!\!\backslash}^t \top}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, \top} \qquad \frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma \mathbin{\backslash\!\!\backslash}^t \Delta}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, F \mathbin{\backslash\!\!\backslash}^t \Delta} \qquad \frac{}{\mathcal{E} \mathbin{\backslash\!\!\backslash}^t \top}$$

Modal Substructural Logics

The propositional rules split into additive and multiplicative ones:

$$\frac{\mathcal{G} /\!\!/^{k} \Gamma, F /\!\!/^{\ell} \mathcal{H} \quad \mathcal{G} /\!\!/^{k} \Gamma, G /\!\!/^{\ell} \mathcal{H}}{\mathcal{G} /\!\!/^{k} \Gamma, F \,\&\, G /\!\!/^{\ell} \mathcal{H}} \qquad \frac{\mathcal{G} /\!\!/^{k} \Gamma, F \quad \mathcal{G} /\!\!/^{k} \Sigma, G}{\mathcal{G} /\!\!/^{k} \Gamma, \Sigma, F \otimes G}$$

$$\frac{}{\mathcal{E} /\!\!/^{k} p, p^{\perp}} \qquad \frac{\mathcal{G} /\!\!/^{k} \Gamma \backslash\!\!\backslash^{t} \top}{\mathcal{G} /\!\!/^{k} \Gamma, \top} \qquad \frac{\mathcal{G} /\!\!/^{k} \Gamma \backslash\!\!\backslash^{t} \Delta}{\mathcal{G} /\!\!/^{k} \Gamma, F \backslash\!\!\backslash^{t} \Delta} \qquad \frac{}{\mathcal{E} \backslash\!\!\backslash^{t} \top}$$

The modal rules ...

$$\frac{\mathcal{G} /\!\!/^{k} \Gamma \backslash\!\!\backslash^{i} \Delta, F}{\mathcal{G} /\!\!/^{k} \Gamma, ?^{i} F \backslash\!\!\backslash^{i} \Delta}\ ?_{i} \qquad \frac{\mathcal{G} /\!\!/^{k} \Gamma \backslash\!\!\backslash^{i} F}{\mathcal{G} /\!\!/^{k} \Gamma, !^{i} F}\ !_{i} \qquad \frac{\mathcal{G} /\!\!/^{i} \Gamma}{\mathcal{G} \backslash\!\!\backslash^{i} \Gamma}\ \mathsf{r}_{i}$$

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The modal rules with Con / W . . .

$$\frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma \mathbin{\backslash\!\!\backslash}^i \Delta, F}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, ?^i F \mathbin{\backslash\!\!\backslash}^i \Delta}\ ?_i \qquad \frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma \mathbin{\backslash\!\!\backslash}^i F}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, !^i F}\ !_i \qquad \frac{\mathcal{G} \mathbin{/\!\!/}^i \Gamma}{\mathcal{G} \mathbin{\backslash\!\!\backslash}^i \Gamma}\ \mathsf{r}_i$$

$$\frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, ?^i F, ?^i F \mathbin{/\!\!/}^\ell \mathcal{H}}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, ?^i F \mathbin{/\!\!/}^\ell \mathcal{H}}\ \mathsf{Con}_i \qquad \frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma \mathbin{/\!\!/}^\ell \mathcal{H}}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, ?^i F \mathbin{/\!\!/}^\ell \mathcal{H}}\ \mathsf{W}_i$$

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The modal rules with Con / W, properties ...

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$$\frac{\mathcal{G} \mathbin{/\!/}^{k} \Gamma \mathbin{/\!/}^{i} F}{\mathcal{G} \mathbin{/\!/}^{k} \Gamma, ?^{i} F}\ \mathsf{d}_i \qquad \frac{\mathcal{G} \mathbin{/\!/}^{k} \Gamma, F}{\mathcal{G} \mathbin{/\!/}^{k} \Gamma, ?^{i} F}\ \mathsf{t}_i \qquad \frac{\mathcal{G} \mathbin{/\!/}^{k} \Gamma \mathbin{/\!/}^{i} \Delta, ?^{i} F}{\mathcal{G} \mathbin{/\!/}^{k} \Gamma, ?^{i} F \mathbin{/\!/}^{i} \Delta}\ 4_i$$

Simply Dependent Multimodal Substructural Logics

The propositional rules split into additive and multiplicative ones:

$$\frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, F \mathbin{/\!\!/}^\ell \mathcal{H} \quad \mathcal{G} \mathbin{/\!\!/}^k \Gamma, G \mathbin{/\!\!/}^\ell \mathcal{H}}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, F \mathbin{\&} G \mathbin{/\!\!/}^\ell \mathcal{H}} \qquad \frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, F \quad \mathcal{G} \mathbin{/\!\!/}^k \Sigma, G}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, \Sigma, F \otimes G}$$

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The modal rules with Con / W, properties, and $!^j F \multimap\, !^i F$

$$\frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma \mathbin{\backslash\!\!\backslash}^i \Delta, F}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, ?^j F \mathbin{\backslash\!\!\backslash}^i \Delta}\ ?^j_i \qquad \frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma \mathbin{\backslash\!\!\backslash}^i F}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, !^i F}\ !_i \qquad \frac{\mathcal{G} \mathbin{/\!\!/}^i \Gamma}{\mathcal{G} \mathbin{\backslash\!\!\backslash}^i \Gamma}\ \mathsf{r}_i$$

$$\frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, ?^i F, ?^i F \mathbin{/\!\!/}^\ell \mathcal{H}}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, ?^i F \mathbin{/\!\!/}^\ell \mathcal{H}}\ \mathsf{Con}_i \qquad \frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma \mathbin{/\!\!/}^\ell \mathcal{H}}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, ?^i F \mathbin{/\!\!/}^\ell \mathcal{H}}\ \mathsf{W}_i$$

$$\frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma \mathbin{/\!\!/}^i F}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, ?^j F}\ \mathsf{d}^j_i \qquad \frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, F}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, ?^i F}\ \mathsf{t}_i \qquad \frac{\mathcal{G} \mathbin{/\!\!/}^k \Gamma \mathbin{/\!\!/}^i \Delta, ?^j F}{\mathcal{G} \mathbin{/\!\!/}^k \Gamma, ?^j F \mathbin{/\!\!/}^i \Delta}\ 4^j_i$$

Summing Up

The Linear Nested Sequent framework provides standard analytic calculi for large classes of

- simply dependent normal multimodal logics,
- non-normal modal logics,
- simply dependent normal multimodal extensions of MALL.

What we haven't seen:

- calculi for conditional logics,
- calculi for intuitionistic and intermediate logics,
- connections to labelled sequent calculi,
- connections to hypersequents.

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A Glimpse at Conditional Logics

The formulae of Lewis' Conditional Logic have a binary operator $\preccurlyeq$.
Conditional linear nested sequents have the form:

$$\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n$$
$$\text{or } \mathcal{G} /\!\!/ \Sigma \Rightarrow \Pi, [\Omega_1 \lhd A_1], \ldots, [\Omega_k \lhd A_k]$$

with interpretation $\wedge\Gamma_1 \to \vee\Delta_1 \vee \Box(\ldots\Box(\wedge\Gamma_n \to \vee\Delta_n)\ldots)$ and
$\ldots\Box(\wedge\Sigma \to \vee\Pi \vee \bigvee_{B\in\Omega_1}(B \preccurlyeq A) \vee \cdots \vee \bigvee_{B\in\Omega_k}(B \preccurlyeq A_k))\ldots$
where $\Box A \equiv (\bot \preccurlyeq \neg A)$

The modal rules:

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, [A \lhd B]}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \preccurlyeq B} \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, [\Sigma, D \lhd A] \quad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta\,[\Sigma \lhd C]}{\mathcal{G} /\!\!/ \Gamma, C \preccurlyeq D \Rightarrow \Delta, [\Sigma \lhd A]}$$

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow \Sigma}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, [\Sigma \lhd A]} \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta\,[\Sigma, \Omega \lhd A] \quad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta\,[\Sigma, \Omega \lhd B]}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Omega \lhd B]}$$

Modularity for non-normal modal logics

C $\Box A \land \Box B \to \Box(A \land B)$

$$\frac{\mathcal{G} \mathbin{\backslash\!\backslash} \Gamma \Rightarrow \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta}\ \mathsf{c}$$

N $\Box\top$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta}{\mathcal{G} \mathbin{\backslash\!\backslash} \Gamma \Rightarrow \Delta}\ \mathsf{n}$$

D_1 $\neg\Box\bot$

$$\frac{\mathcal{G} \mathbin{\backslash\!\backslash} \ \Rightarrow}{\mathcal{G}}\ \mathsf{d}_1$$

D_2 $\neg(\Box A \land \Box\neg A)$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta \mathbin{\backslash\!\backslash} A \Rightarrow}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{d}_2$$

T $\Box A \to A$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma, A \Rightarrow \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{t}$$

4 $\Box A \to \Box\Box A$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta \mathbin{/\!/} \Sigma, \Box A \Rightarrow \Pi}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box A \Rightarrow \Delta \mathbin{\backslash\!\backslash} \Sigma \Rightarrow \Pi}\ 4$$

5 $\Box\neg A \lor \Box\neg\Box A$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta \mathbin{/\!/} \Sigma \Rightarrow \Pi, \Box A}{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta, \Box A \mathbin{\backslash\!\backslash} \Sigma \Rightarrow \Pi}\ 5$$

Theorem

For $\mathcal{A} \subseteq \{\mathsf{C}, \mathsf{N}, \mathsf{D}_1, \mathsf{D}_2, \mathsf{T}, 4\}$ the calculus $\mathsf{LNS}_{\mathsf{M}\mathcal{A}}$ is sound and complete for $\mathsf{M}\mathcal{A}$. Similar for some combinations with 5.

(Use calculi e.g. from [Lavendhomme-Lucas:'00, Indrzejczak:'05].)

The Rules for Linear Logic

$$\frac{}{\mathcal{E} /\!\!/ p, p^{\perp}}\ \text{init} \qquad \frac{}{\mathcal{E} /\!\!/ 1}\ 1 \qquad \frac{}{\mathcal{E} /\!\!/ \Gamma, \top}\ \top \qquad \frac{\mathcal{G} /\!\!/ \Gamma_1, F \quad \mathcal{G} /\!\!/ \Gamma_1, F^{\perp}}{\mathcal{G} /\!\!/ \Gamma_1, \Gamma_2}\ \text{cut}$$

$$\frac{\mathcal{S}\{\Gamma\}}{\mathcal{S}\{\Gamma, \perp\}}\ \perp \quad \frac{\mathcal{S}\{\Gamma, F, G\}}{\mathcal{S}\{\Gamma, F \parr G\}}\ \parr \quad \frac{\mathcal{S}\{\Gamma, F\} \quad \mathcal{S}\{\Gamma, G\}}{\mathcal{S}\{\Gamma, F \mathbin{\&} G\}}\ \& \quad \frac{\mathcal{G} /\!\!/ \Gamma, F[y/x]}{\mathcal{G} /\!\!/ \Gamma, \forall x.F}\ \forall$$

$$\frac{\mathcal{G} /\!\!/ \Gamma_1, F \quad \mathcal{G} /\!\!/ \Gamma_2, G}{\mathcal{G} /\!\!/ \Gamma_1, \Gamma_2, F \otimes G}\ \otimes \qquad \frac{\mathcal{G} /\!\!/ \Gamma, F_i}{\mathcal{G} /\!\!/ \Gamma, F_1 \oplus F_2}\ \oplus_i \qquad \frac{\mathcal{G} /\!\!/ \Gamma, F[t/x]}{\mathcal{G} /\!\!/ \Gamma, \exists F}\ \exists$$

$$\frac{\mathcal{S}\{\Gamma, ?F, ?F\}}{\mathcal{S}\{\Gamma, ?F\}}\ \text{Con} \qquad \frac{\mathcal{S}\{\Gamma\}}{\mathcal{S}\{\Gamma, ?F\}}\ \text{W} \qquad \frac{\mathcal{S}\{\Gamma, F\}}{\mathcal{S}\{\Gamma, ?F\}}\ \text{der}$$

$$\frac{\mathcal{S}\{\Gamma /\!\!/ \Delta, ?F\}}{\mathcal{S}\{\Gamma, ?F /\!\!/ \Delta\}}\ ? \qquad \frac{\mathcal{G} /\!\!/ \Gamma /\!\!/ F}{\mathcal{G} /\!\!/ \Gamma, !F}\ !$$