Hypersequent Calculi for Lewis' Conditional Logics with Uniformity and Reflexivity

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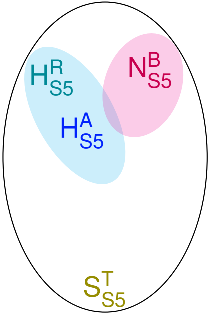
Motivation

Since we're at TABLEAUX: . . . Let's talk about sequent-style systems!

E.g.:

- Avron's H^{A}_{S5} is a hypersequent calculus, as is Restall's H^{R}_{S5}.
- Brünnler's N^{B}_{S5} is a nested sequent calculus
- Takano's S^{T}_{S5} is neither a hypersequent nor a nested sequent calculus



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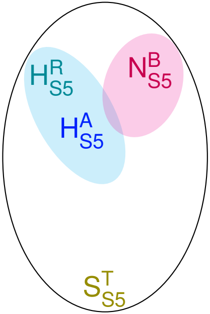
E.g.:

- Avron's $\mathsf{H}^{\mathsf{A}}_{\mathsf{S5}}$ is a hypersequent calculus, as is Restall's $\mathsf{H}^{\mathsf{R}}_{\mathsf{S5}}$.
  $\mathsf{H}^{\mathsf{A}}_{\mathsf{S5}} \vee \mathsf{H}^{\mathsf{R}}_{\mathsf{S5}} \rightarrow \texttt{hypersequent}$
- Brünnler's $\mathsf{N}^{\mathsf{B}}_{\mathsf{S5}}$ is a nested sequent calculus
  $\mathsf{N}^{\mathsf{B}}_{\mathsf{S5}} \rightarrow \texttt{nested}$
- Takano's $\mathsf{S}^{\mathsf{T}}_{\mathsf{S5}}$ is neither a hypersequent nor a nested sequent calculus
  $\mathsf{S}^{\mathsf{T}}_{\mathsf{S5}} \rightarrow \neg(\texttt{hypersequent} \vee \texttt{nested})$

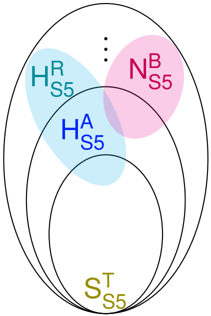


Motivation: Comparing sequent-style systems

However, it's a bit boring if we can't compare them – so let's add similarity into the mix, modelled by a system of nested spheres:
(Things in smaller spheres are more similar than things in larger spheres)

E.g.:

- S^T_{S5} is more similar to H^A_{S5} than to H^R_{S5}
- S^T_{S5} is as similar to H^R_{S5} as to N^B_{S5}
- S^T_{S5} is more similar to hypersequent calculi than to nested sequent calculi



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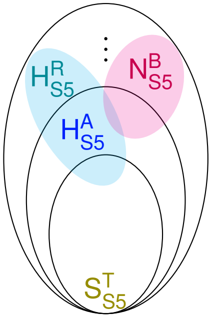
- S^{T}_{S5} is more similar to H^{A}_{S5} than to H^{R}_{S5}
  $S^{T}_{S5} \rightarrow (H^{A}_{S5} \prec H^{R}_{S5})$
- S^{T}_{S5} is as similar to H^{R}_{S5} as to N^{B}_{S5}
  $S^{T}_{S5} \rightarrow (H^{R}_{S5} \preccurlyeq N^{B}_{S5})$
- S^{T}_{S5} is more similar to hypersequent calculi than to nested sequent calculi
  $S^{T}_{S5} \rightarrow (\mathtt{hypersequent} \prec \mathtt{nested})$



The language

The Formulae of conditional logic are given by:

$$A, B \quad ::= \quad p \mid \bot \mid A \to B \mid A \preccurlyeq B$$

A comparative plausibility formula $A \preccurlyeq B$ can be read, e.g., as:

- "A is at least as plausible as B"
- "A is at least as preferable as B"
- "the current state is at least as similar to As as to Bs"

We define $A \prec B$ as $\neg(B \preccurlyeq A)$,
read as "A is more plausible/similar/preferable than B"

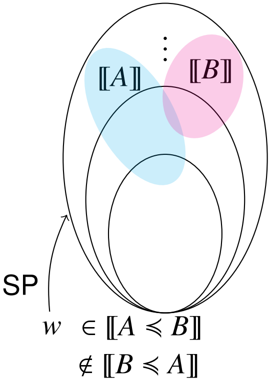
The logic of universal sphere models $\mathbb{VTU}$

A universal sphere model consists of:

- a non-empty universe W
- A valuation $[\![.]\!] : \mathsf{Var} \to \mathcal{P}(W)$
- a system of spheres $\mathsf{SP} : W \to \mathcal{PP}(W)$

with for all $w, v \in W$:

- $\forall \alpha \in \mathsf{SP}(w) . \alpha \neq \emptyset$
- $\forall \alpha, \beta \in \mathsf{SP}(w) . \alpha \subseteq \beta \vee \beta \subseteq \alpha$
- $w \in \bigcup \mathsf{SP}(w)$ (reflexivity)
- $\bigcup \mathsf{SP}(w) = \bigcup \mathsf{SP}(v)$ (uniformity)



The valuation is extended to comparative plausibility formulae by:

$$[\![A \preccurlyeq B]\!] \quad := \quad \{ w \in W : \forall \alpha \in \mathsf{SP}(w) . [\![B]\!] \cap \alpha \neq \emptyset \Rightarrow [\![A]\!] \cap \alpha \neq \emptyset \}$$

Lewis' conditional logic $\mathbb{VTU}$ is the logic of all universal sphere models.

How to construct a sequent-style proof system for $\mathbb{VTU}$?

Hypersequents for $\mathbb{VTU}$

The proximity to modal logic S5 suggests we use an extension of sequents:

A hypersequent is a non-empty multiset of (multiset-based) sequents

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

The conditional formula interpretation of a hypersequent is

$$\Box(\bigwedge \Gamma_1 \to \bigvee \Delta_1) \vee \ldots \vee \Box(\bigwedge \Gamma_n \to \bigvee \Delta_n)$$

where $\Box$ is the outer modality defined by $\Box A \equiv (\bot \preccurlyeq \neg A)$.

Hypersequents for $\mathbb{VTU}$

The hypersequent calculus $\mathsf{H}_{\mathbb{VTU}}$ contains the propositional rules, internal contraction, and:

$$\dfrac{\{\, \mathcal{G} \mid \Sigma \Rightarrow \Pi \mid C_k \Rightarrow D_1, \ldots, D_{k-1}, A_1, \ldots, A_n : k \leq m \,\} \quad \cup \quad \{\, \mathcal{G} \mid \Sigma \Rightarrow \Pi \mid B_k \Rightarrow D_1, \ldots, D_m, A_1, \ldots, A_n : k \leq n \,\}}{\mathcal{G} \mid \Sigma, C_1 \preccurlyeq D_1, \ldots, C_m \preccurlyeq D_m \Rightarrow A_1 \preccurlyeq B_1, \ldots, A_n \preccurlyeq B_n, \Pi} \; R_{m,n}$$

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Soundness and completeness

Theorem

The calculus $\mathsf{H}_{\mathbb{VTU}}$ *is sound for* $\mathbb{VTU}$.

Idea of Proof: From a model falsifying the conclusion construct one falsifying a premiss. E.g., for the transfer rule:

$$\dfrac{\begin{array}{l}\{\, \mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow \Theta \mid C_k \Rightarrow D_1, \ldots, D_{k-1} : k \leq m \,\} \\ \cup \quad \{\, \mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow D_1, \ldots, D_m, \Theta \,\}\end{array}}{\mathcal{G} \mid \Sigma, C_1 \preccurlyeq D_1, \ldots, C_m \preccurlyeq D_m \Rightarrow \Pi \mid \Omega \Rightarrow \Theta} \ \mathsf{trf}_m$$

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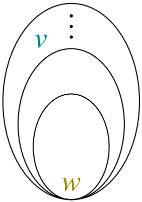

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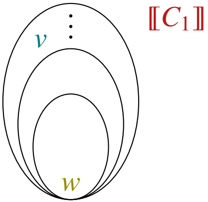

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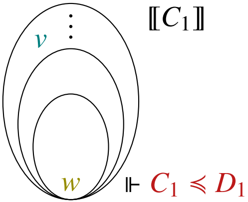

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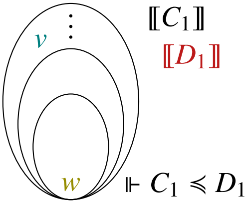

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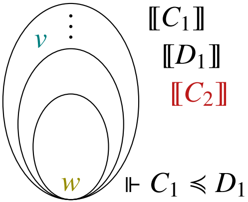

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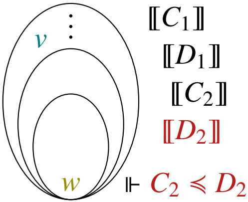

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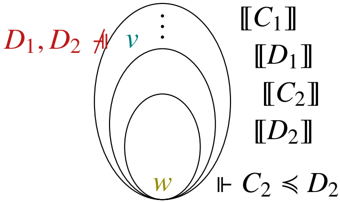

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The calculus $\mathsf{H}_{\mathbb{VTU}}$ *has cut elimination and is cut-free complete for* $\mathbb{VTU}$.

Proof: Non-trivial and technical (...as usual).

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Unfortunately, our calculi are non-standard in the sense that

- they include an infinite number of rules
- the rules introduce more than one principal formula at a time.

So . . .

How could we massage our calculi to become standard?

How to construct a standard calculus?

Main idea:

Decompose the rules so they are simulated one formula at a time!

E.g., for the transfer rule:

$$\dfrac{\{\, \mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow \Theta \mid C_1 \Rightarrow \;\} \;\cup\; \{\, \mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow \Theta \mid C_2 \Rightarrow D_1 \,\} \;\cup\; \{\, \mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow D_1, D_2, \Theta \,\}}{\mathcal{G} \mid \Sigma, C_1 \preccurlyeq D_1, C_2 \preccurlyeq D_2 \Rightarrow \Pi \mid \Omega \Rightarrow \Theta} \; \mathsf{trf}_2$$

We need to transfer the whole block D_1, D_2 to another component, so introduce a block for temporary storage, written $\langle . \rangle$.
Then simulate the rule by:

- initialising
- storing
- transferring and closing

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$$\dfrac{\dfrac{\ldots \mid C_1 \Rightarrow \bot \qquad \dfrac{\ldots \mid C_2 \Rightarrow D_1, \bot \qquad \dfrac{\Sigma \Rightarrow \Pi \mid \Omega \Rightarrow D_1, D_2, \bot, \Theta}{\Sigma \Rightarrow \Pi, \langle D_1, D_2, \bot \rangle \mid \Omega \Rightarrow \Theta}}{\Sigma, C_2 \preccurlyeq D_2 \Rightarrow \Pi, \langle D_1, \bot \rangle \mid \Omega \Rightarrow \Theta}}{\Sigma, C_1 \preccurlyeq D_1, C_2 \preccurlyeq D_2 \Rightarrow \Pi, \langle \bot \rangle \mid \Omega \Rightarrow \Theta}}{\Sigma, C_1 \preccurlyeq D_1, C_2 \preccurlyeq D_2 \Rightarrow \Pi \mid \Omega \Rightarrow \Theta}$$

How to construct a standard calculus?

Main idea:

Decompose the rules so they are simulated one formula at a time!

E.g., for the transfer rule:

$$\dfrac{\{\, \mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow \Theta \mid C_1 \Rightarrow \;\} \;\cup\; \{\, \mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow \Theta \mid C_2 \Rightarrow D_1 \,\} \;\cup\; \{\, \mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow D_1, D_2, \Theta \,\}}{\mathcal{G} \mid \Sigma, C_1 \preccurlyeq D_1, C_2 \preccurlyeq D_2 \Rightarrow \Pi \mid \Omega \Rightarrow \Theta}\ \mathsf{trf}_2$$

We need to transfer the whole block D_1, D_2 to another component, so introduce a block for temporary storage, written $\langle . \rangle$.
Then simulate the rule by:

- initialising
- storing
- transferring and closing

$$\dfrac{\dfrac{\ldots \mid C_1 \Rightarrow \bot \qquad \dfrac{\ldots \mid C_2 \Rightarrow D_1, \bot \qquad \dfrac{\Sigma \Rightarrow \Pi \mid \Omega \Rightarrow D_1, D_2, \bot, \Theta}{\Sigma \Rightarrow \Pi, \langle D_1, D_2, \bot \rangle \mid \Omega \Rightarrow \Theta}}{\Sigma, C_2 \preccurlyeq D_2 \Rightarrow \Pi, \langle D_1, \bot \rangle \mid \Omega \Rightarrow \Theta}}{\Sigma, C_1 \preccurlyeq D_1, C_2 \preccurlyeq D_2 \Rightarrow \Pi, \langle \bot \rangle \mid \Omega \Rightarrow \Theta}}{\Sigma, C_1 \preccurlyeq D_1, C_2 \preccurlyeq D_2 \Rightarrow \Pi \mid \Omega \Rightarrow \Theta}$$

The standard calculus for $\mathbb{VTU}$

An extended sequent is a sequent whose right hand side also contains conditional blocks and transfer blocks.

$$\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd C_1], \ldots, [\Sigma_n \lhd C_n], \langle\Theta_1\rangle, \ldots, \langle\Theta_m\rangle$$

An extended hypersequent is a hypersequent of extended sequents.

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

The standard calculus for $\mathbb{VTU}$

An extended sequent is a sequent whose right hand side also contains conditional blocks and transfer blocks.
Its formula interpretation is given by:

$$\iota_e(\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd C_1], \dots, [\Sigma_n \lhd C_n], \langle\Theta_1\rangle, \dots, \langle\Theta_m\rangle)$$
$$:= \bigwedge \Gamma \rightarrow \bigvee \Delta \vee \bigvee_{i=1}^{n} \bigvee_{B\in\Sigma_i}(B \preccurlyeq C_i) \vee \bigvee_{j=1}^{m} \Diamond(\bigvee \Theta_j)$$

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The standard calculus for $\mathbb{VTU}$

The calculus $\text{SH}_{\mathbb{VTU}}$ contains propositional rules, contraction, and:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B} \preccurlyeq_R \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A] \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]} \text{com}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd A]}{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C]} \preccurlyeq_L \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid A \Rightarrow \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd A]} \text{jump}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \bot \rangle}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \text{in}_{\text{trf}} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid A \Rightarrow \Theta \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta, B \rangle}{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, \langle \Theta \rangle} \text{T}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Theta, \Pi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle \mid \Sigma \Rightarrow \Pi} \text{jump}_U \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Theta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle} \text{jump}_T$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd A]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \lhd A]} \text{Con}_S \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, A, A \lhd B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, A \lhd B]} \text{Con}_B$$

The standard calculus for $\mathbb{VTU}$

Theorem

The calculus $\mathsf{SH}_{\mathbb{VTU}}$ *is sound for* $\mathbb{VTU}$.

Proof:

By showing that all the rules preserve validity (as usual).

Theorem

The calculus $\mathsf{SH}_{\mathbb{VTU}}$ *is cut-free complete for* $\mathbb{VTU}$.

Proof:

By simulating derivations in the hypersequent system.

Alternative Proof:

By constructing a countermodel from failed proof search (non-trivial...).

Wrapping up

So what have we achieved?

- Hypersequent calculi for Lewis' conditional logics $\mathbb{VTU}, \mathbb{VWU}, \mathbb{VCU}, \mathbb{VTA}, \mathbb{VWA}, \mathbb{VCA}$.
- Syntactic cut elimination for these calculi
- Applications of the calculi in proving connections to modal logic
- Standard calculi for all the logics
- Completeness proofs via simulation
- For $\mathbb{VTU}, \mathbb{VWU}, \mathbb{VCU}$: An alternative completeness proof via countermodel construction.

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$$(\texttt{questions} \land \texttt{happy}) \prec (\neg\texttt{questions} \land \texttt{happy})$$