Analytic calculi for intermediate logics: A nested sequent approach

Björn Lellmann

TU Wien

ALCOP VIII, Glasgow
April 12, 2017

Disclaimer: Work in progress - Enter at your own risk!

General methods in proof theory

Recent development: general methods for constructing analytic calculi for non-classical logics in various frameworks. E.g.:

- Modal logics
- Substructural logics
- Intermediate logics
- ...

using

- Sequents
- Hypersequents
- Labelled sequents
- Display calculi

By now these frameworks are (reasonably) well understood ...

$$\begin{array}{c} \textit{low} \longleftarrow \text{expressivity} \longrightarrow \textit{high} \\ \text{Sequents} \leq \text{Hypersequents} \leq \left\{ \begin{array}{l} \text{Labelled sequents} \\ \text{Display calculi} \end{array} \right. \\ \textit{low} \longleftarrow \text{complexity} \longrightarrow \textit{high} \end{array}$$

General methods in proof theory

Recent development: general methods for constructing analytic calculi for non-classical logics in various frameworks. E.g.:

- Modal logics
- Substructural logics
- Intermediate logics
- ...

using

- Sequents
- Hypersequents
- Labelled sequents
- Display calculi

By now these frameworks are (reasonably) well understood ...

$$\begin{array}{c} \textit{low} \longleftarrow \text{expressivity} \longrightarrow \textit{high} \\ \text{Sequents} \leq \text{Hypersequents} \leq ? \leq \begin{cases} \text{Labelled sequents} \\ \text{Display calculi} \end{cases} \\ \textit{low} \longleftarrow \text{complexity} \longrightarrow \textit{high} \end{array}$$

... But is there anything in between?

Reminder: Intermediate logics

The formulae of intermediate logics are given by

$$p \in \mathsf{Var} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi$$

A frame $\mathcal{F} = (W, \preccurlyeq)$ has a reflexive transitive $\preccurlyeq \subseteq W \times W$.
Valuations σ satisfy:

- monotonicity: $\mathcal{F}, \sigma, x \Vdash p$ and $x \preccurlyeq y$ then $\mathcal{F}, \sigma, y \Vdash p$
- $\mathcal{F}, \sigma, x \Vdash A \rightarrow B$ iff
  $$\forall y \, (x \preccurlyeq y \Rightarrow (\mathcal{F}, \sigma, y \nVdash A \text{ or } \mathcal{F}, \sigma, y \Vdash B))$$
- local clauses for $\wedge, \vee, \bot$

Intermediate logics are obtained by restricting the class of frames:

- Bd_k: depth at most k $(x_0 \preccurlyeq \ldots \preccurlyeq x_k \Rightarrow \bigvee_{i=1}^{k} x_{i-1} = x_i)$
- GD: linear frames $(x \preccurlyeq y \vee y \preccurlyeq x)$
- Jan: confluent frames $(\exists z \, (x \preccurlyeq z \wedge y \preccurlyeq z))$
- ...

Reminder: Intermediate logics

The formulae of intermediate logics are given by

$$p \in \mathsf{Var} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi$$

A frame $\mathcal{F} = (W, \preccurlyeq)$ has a reflexive transitive $\preccurlyeq \subseteq W \times W$.
Valuations σ satisfy:

- monotonicity: $\mathcal{F}, \sigma, x \Vdash p$ and $x \preccurlyeq y$ then $\mathcal{F}, \sigma, y \Vdash p$
- $\mathcal{F}, \sigma, x \Vdash A \to B$ iff
  $$\forall y \, (x \preccurlyeq y \Rightarrow (\mathcal{F}, \sigma, y \nVdash A \text{ or } \mathcal{F}, \sigma, y \Vdash B))$$
- local clauses for $\wedge, \vee, \bot$

Intermediate logics are obtained by restricting the class of frames.
Or alternatively as axiomatic extensions of intuitionistic logic:

- Bd_k: $\mathsf{Int} \oplus p_k \vee (p_k \to p_{k-1} \vee (\cdots \to (p_1 \vee (p_1 \to \bot))))$
- GD: $\mathsf{Int} \oplus (p \to q) \vee (q \to p)$
- Jan: $\mathsf{Int} \oplus \neg p \vee \neg\neg p$

Proof theory for intermediate logics: A benchmark

A very powerful tool is provided by algebraic proof theory, e.g.:

A formula is a $\mathcal{P}_3$-axiom if it is of the form

$$(\bigwedge_i(\bigwedge \bar{p}^i_1 \to \bigvee \bar{q}^i_1) \to \bigvee \bar{r}_1) \vee \cdots \vee (\bigwedge_i(\bigwedge \bar{p}^i_n \to \bigvee \bar{q}^i_n) \to \bigvee \bar{r}_n)$$

Theorem (Ciabattoni, Galatos, Terui:'08)

An intermediate logic admits a structural hypersequent calculus iff it is axiomatised by $\mathcal{P}_3$ axioms.

Corollary. *GD and Jan admit a structural hypersequent calculus.*

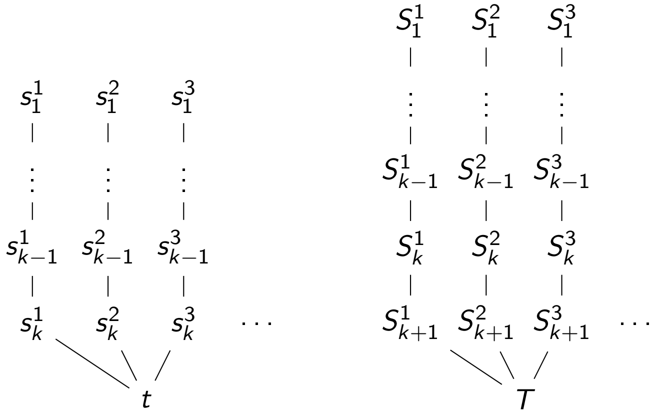
But some interesting logics are not axiomatised in $\mathcal{P}_3$.
In particular:

- Bd_k : $\quad \mathrm{Int} \oplus p_k \vee (p_k \to p_{k-1} \vee (\cdots \to (p_1 \vee (p_1 \to \bot))))$

Example: Limitative result for Bd_k

Theorem. *Bd_k is not axiomatised by $\mathcal{P}_3$ axioms for $k \geq 2$.*

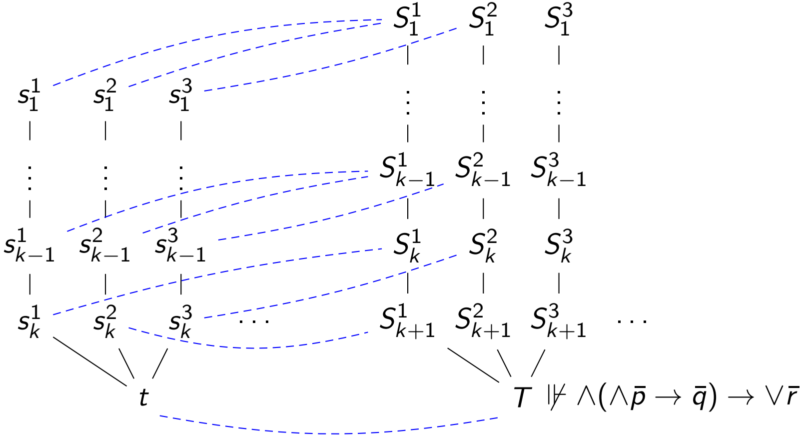
Idea: The frames below validate the same $\mathcal{P}_3$ formulae. One is a Bd_{k+1} frame, the other one is not.



Example: Limitative result for Bd_k

Theorem. *Bd_k is not axiomatised by $\mathcal{P}_3$ axioms for $k \geq 2$.*

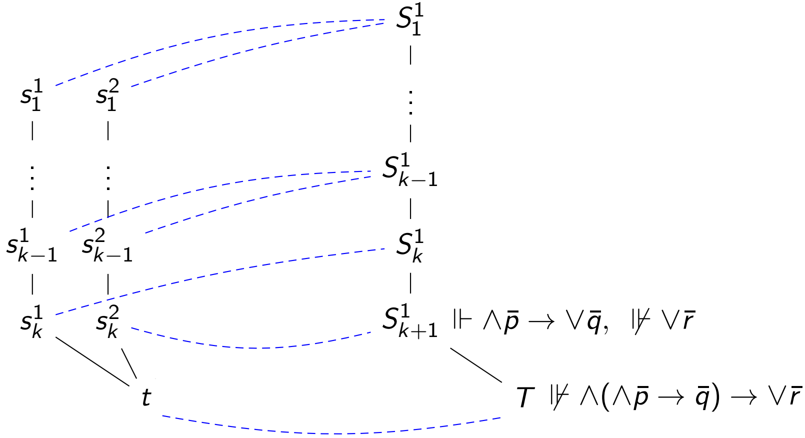
Idea: The frames below validate the same $\mathcal{P}_3$ formulae. One is a Bd_{k+1} frame, the other one is not. E.g, if T falsifies $A = \bigvee(\bigwedge(\bigwedge\bar{p} \to \bigvee\bar{q}) \to \bigvee\bar{r})$: Copy the valuation along --.



Example: Limitative result for Bd_k

Theorem. *Bd_k is not axiomatised by $\mathcal{P}_3$ axioms for $k \geq 2$.*

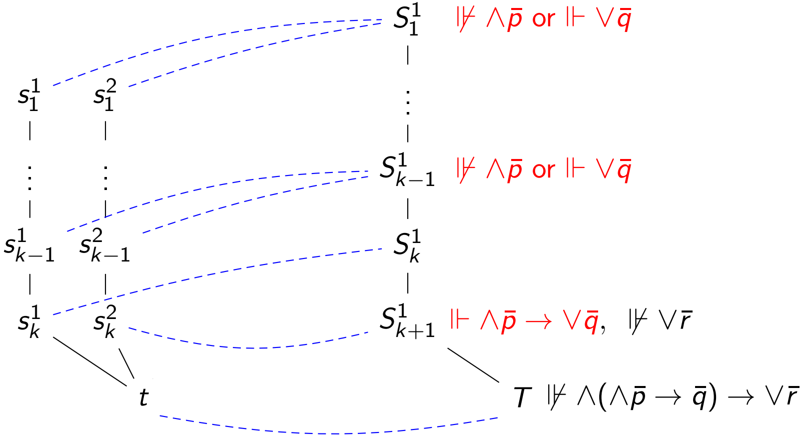
Idea: The frames below validate the same $\mathcal{P}_3$ formulae. One is a Bd_{k+1} frame, the other one is not. E.g, if T falsifies $A = \bigvee(\bigwedge(\bigwedge\bar{p} \to \bigvee\bar{q}) \to \bigvee\bar{r})$: Copy the valuation along --.



Example: Limitative result for Bd_k

Theorem. *Bd_k is not axiomatised by $\mathcal{P}_3$ axioms for $k \geq 2$.*

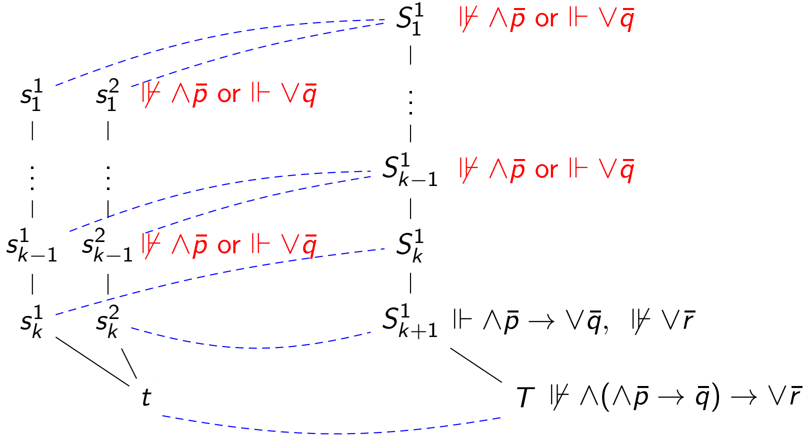
Idea: The frames below validate the same $\mathcal{P}_3$ formulae. One is a Bd_{k+1} frame, the other one is not. E.g, if T falsifies $A = \bigvee(\bigwedge(\bigwedge\bar{p} \to \bigvee\bar{q}) \to \bigvee\bar{r})$: Copy the valuation along --.



Example: Limitative result for Bd_k

Theorem. *Bd_k is not axiomatised by $\mathcal{P}_3$ axioms for $k \geq 2$.*

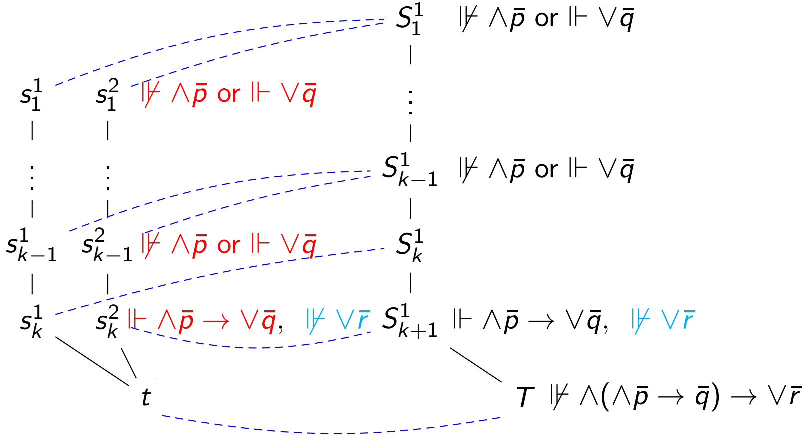
Idea: The frames below validate the same $\mathcal{P}_3$ formulae. One is a Bd_{k+1} frame, the other one is not. E.g, if T falsifies $A = \bigvee(\bigwedge(\bigwedge\bar{p} \to \bigvee\bar{q}) \to \bigvee\bar{r})$: Copy the valuation along --.



Example: Limitative result for Bd_k

Theorem. *Bd_k is not axiomatised by $\mathcal{P}_3$ axioms for $k \geq 2$.*

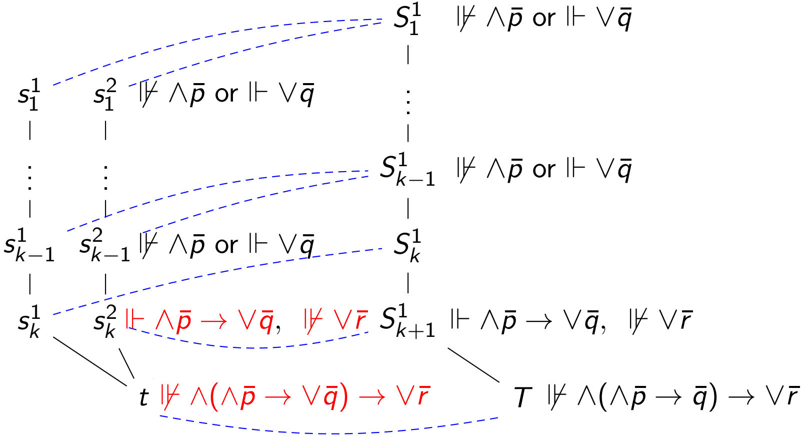
Idea: The frames below validate the same $\mathcal{P}_3$ formulae. One is a Bd_{k+1} frame, the other one is not. E.g, if T falsifies $A = \bigvee(\bigwedge(\bigwedge\bar{p} \to \bigvee\bar{q}) \to \bigvee\bar{r})$: Copy the valuation along --.



Example: Limitative result for Bd_k

Theorem. *Bd_k is not axiomatised by $\mathcal{P}_3$ axioms for $k \geq 2$.*

Idea: The frames below validate the same $\mathcal{P}_3$ formulae. One is a Bd_{k+1} frame, the other one is not. E.g, if T falsifies $A = \bigvee(\bigwedge(\bigwedge\bar{p} \to \bigvee\bar{q}) \to \bigvee\bar{r})$: Copy the valuation along --.



Example: Limitative result for Bd_k

Theorem. *Bd_k is not axiomatised by $\mathcal{P}_3$ axioms for $k \geq 2$.*

...or talk to Nick and Frederik for a more algebraic proof!

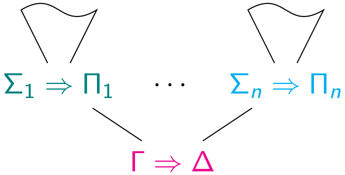
How to capture logics like Bd_k?

Move to a more expressive framework!

(. . . but not too expressive)

Nested Sequents

Nested sequents are trees of (multi-set based) sequents:



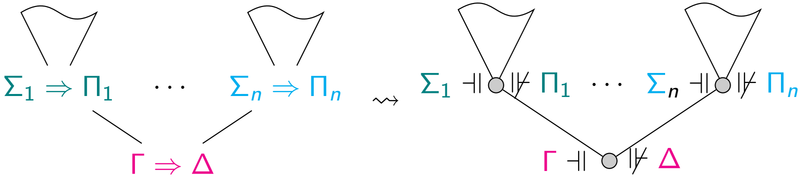
interpreted as $\bigwedge\Gamma \rightarrow \bigvee\Delta \vee (\bigwedge\Sigma_1 \rightarrow \bigvee\Pi_1^*) \vee \cdots \vee (\bigwedge\Sigma_n \rightarrow \bigvee\Pi_n^*)$.

A bit of history:

- Precursors: [Bull:'92], [Kashima:'94], [Masini:'92]
- Current form in modal logics: [Brünnler:'09], [Poggiolesi:'09]
- For intuitionistic modal logics: [Straßburger et al:'12 - now]
- Adapted to intuitionistic logic in [Fitting:'14]

Nested Sequents

Nested sequents are trees of (multi-set based) sequents:

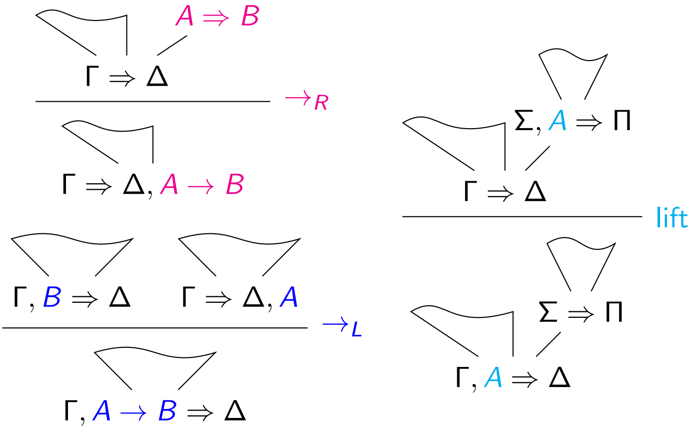


interpreted as $\bigwedge\Gamma \to \bigvee\Delta \vee (\bigwedge\Sigma_1 \to \bigvee\Pi_1^*) \vee \cdots \vee (\bigwedge\Sigma_n \to \bigvee\Pi_n^*)$.

Nested sequents give rise to models for intuitionistic logic.

Nested sequents: The standard rules

Fitting's rules (applied anywhere inside the nested sequent):



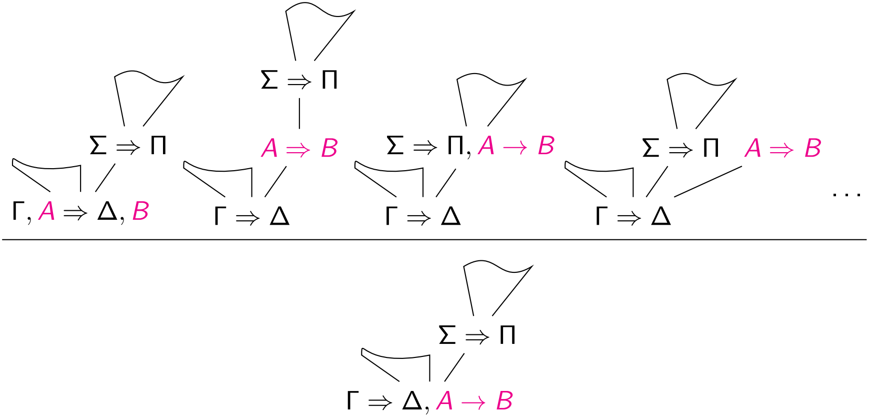
Together with local rules for $\wedge, \vee, \bot$, init, and contraction.

Problem: Rule $\rightarrow_R$ loses control over the structure of the models.

Suggestion here: be more explicit

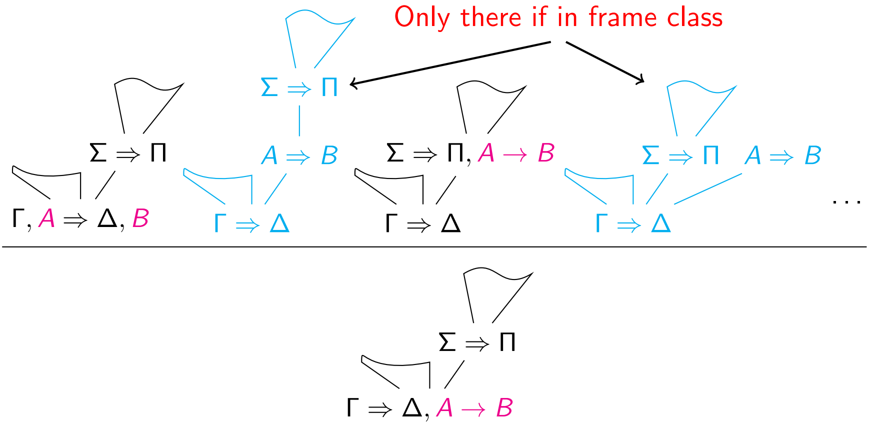
To regain control over the structure of the models we incorporate all different possibilities in the implication right rule

...



Suggestion here: be more explicit

To regain control over the structure of the models we incorporate all different possibilities in the implication right rule and restrict according to the class of frames!

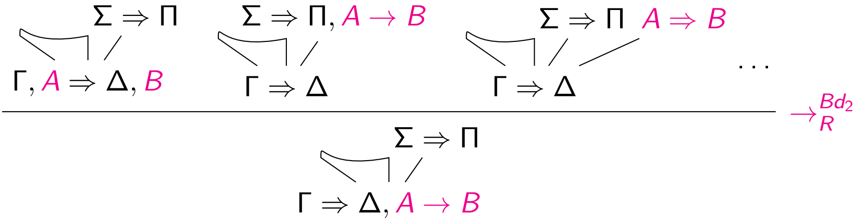


Example: Bounded depth Bd_2

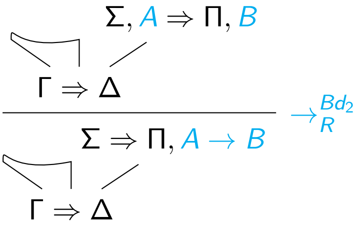
Reminder: Bd_2 frames have depth at most 2.

Thus the rules work only on nested sequents of depth ≤ 2.

The rule with principal formula in the root:

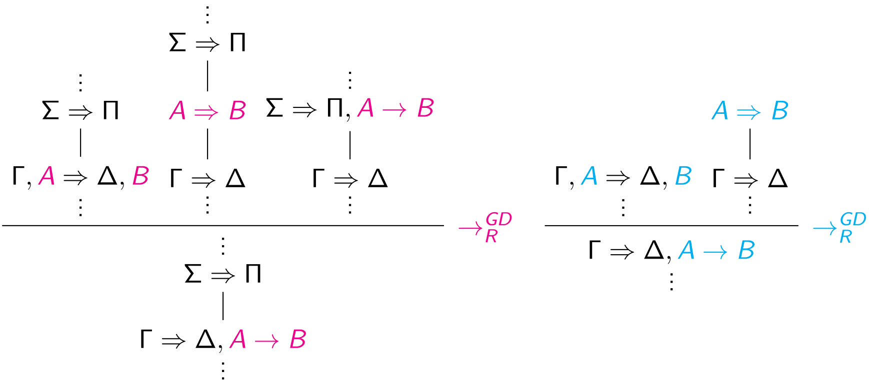


And the rule with principal formula in a leaf:



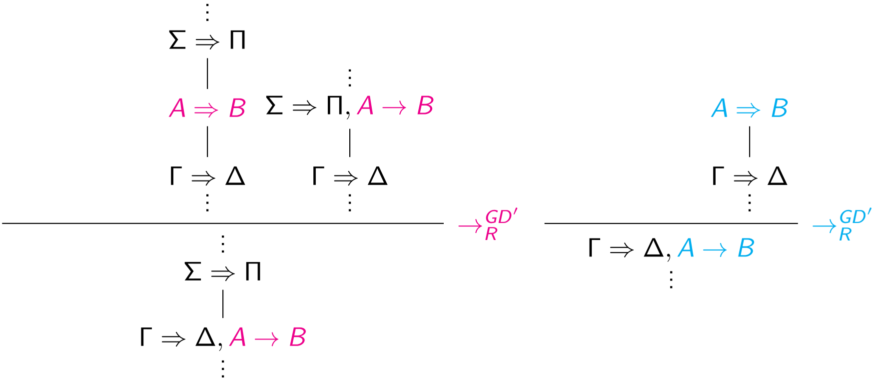
Example: Gödel-Dummett logic *GD*

Reminder: *GD* frames are linear: every node has ≤ 1 successor.



Example: Gödel-Dummett logic *GD*

Reminder: *GD* frames are linear: every node has ≤ 1 successor.



In this case we can even omit more premisses.

Theorem. *This calculus for GD has (syntactic) cut elimination.*

What do derivations look like?

. . . Let the implementation work that out!

Logic: Bd_2

Input sequent: $\Rightarrow (p \vee (p \rightarrow (q \vee (q \rightarrow \bot))))$

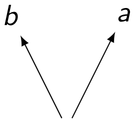
Derivation found!

$$\dfrac{\dfrac{\dfrac{}{p \Rightarrow (q \vee (q \rightarrow \bot)), p,}\ \text{init} \qquad \dfrac{\dfrac{\dfrac{}{\Rightarrow p, [q, p \Rightarrow \bot, q,]}\ \text{init}}{\Rightarrow p, [p \Rightarrow q, (q \rightarrow \bot),]}\ \rightarrow_R}{\Rightarrow p, [p \Rightarrow (q \vee (q \rightarrow \bot)),]}\ \vee_R}{\Rightarrow p, (p \rightarrow (q \vee (q \rightarrow \bot))),}\ \rightarrow_R}{\Rightarrow (p \vee (p \rightarrow (q \vee (q \rightarrow \bot))))}\ \vee_R$$

Logic: Bd_2

Input sequent: $\Rightarrow ((a \to b) \lor (b \to a))$

Countermodel found!



Suming up

The main idea:

- A semantic view of nested sequents for intermediate logics.
- Control the structure of the frames via the premisses of $\rightarrow_R$.
- Restrict the general rule scheme according to the frame class.

Questions:

- Is this known already?
- Could there be connections to algebraic semantics?
- Is this cheating?

Thank You!

Logic: Bd_3

Input sequent: $\Rightarrow (p \vee (p \rightarrow (q \vee (q \rightarrow (r \vee (r \rightarrow \bot))))))$

Derivation found!

$$\dfrac{}{\Rightarrow p, [p \Rightarrow q, [r, p, q \Rightarrow \bot, r,]]}\ \text{init}$$

$$\dfrac{\Rightarrow p, [p \Rightarrow q, [r, p, q \Rightarrow \bot, r,]]}{\Rightarrow p, [p \Rightarrow q, [p, q \Rightarrow r, (r \rightarrow \bot),]]}\ \rightarrow_R$$

$$\dfrac{\Rightarrow p, [p \Rightarrow q, [p, q \Rightarrow r, (r \rightarrow \bot),]]}{\Rightarrow p, [p \Rightarrow q, [q \Rightarrow r, (r \rightarrow \bot),]]}\ \text{lift}$$

$$\dfrac{\Rightarrow p, [p \Rightarrow q, [q \Rightarrow r, (r \rightarrow \bot),]]}{\Rightarrow p, [p \Rightarrow q, [q \Rightarrow (r \vee (r \rightarrow \bot)),]]}\ \vee_R$$

$$\dfrac{}{\Rightarrow p, [q, p \Rightarrow (r \vee (r \rightarrow \bot)), q,]}\ \text{init}$$

$$\dfrac{\Rightarrow p, [q, p \Rightarrow (r \vee (r \rightarrow \bot)), q,] \qquad \Rightarrow p, [p \Rightarrow q, [q \Rightarrow (r \vee (r \rightarrow \bot)),]]}{\Rightarrow p, [p \Rightarrow q, (q \rightarrow (r \vee (r \rightarrow \bot))),]}\ \rightarrow_R$$

$$\dfrac{\Rightarrow p, [p \Rightarrow q, (q \rightarrow (r \vee (r \rightarrow \bot))),]}{\Rightarrow p, [p \Rightarrow (q \vee (q \rightarrow (r \vee (r \rightarrow \bot)))),]}\ \vee_R$$

$$\dfrac{}{p \Rightarrow (q \vee (q \rightarrow (r \vee (r \rightarrow \bot)))), p,}\ \text{init}$$

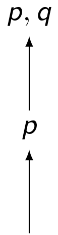
$$\dfrac{p \Rightarrow (q \vee (q \rightarrow (r \vee (r \rightarrow \bot)))), p, \qquad \Rightarrow p, [p \Rightarrow (q \vee (q \rightarrow (r \vee (r \rightarrow \bot)))),]}{\Rightarrow p, (p \rightarrow (q \vee (q \rightarrow (r \vee (r \rightarrow \bot))))),}\ \rightarrow_R$$

$$\dfrac{\Rightarrow p, (p \rightarrow (q \vee (q \rightarrow (r \vee (r \rightarrow \bot))))),}{\Rightarrow (p \vee (p \rightarrow (q \vee (q \rightarrow (r \vee (r \rightarrow \bot))))))}\ \vee_R$$

Logic: Bd_3

Input sequent: $\Rightarrow (p \vee (p \rightarrow (q \vee (q \rightarrow \bot))))$

Countermodel found!



Logic: Sm

Input sequent: $\Rightarrow (((q \to \bot) \to p) \to (((p \to q) \to p) \to p))$

Derivation found!