The Linear Nested Sequent Framework

Björn Lellmann

TU Wien

Marseille, June 14, 2016

Sequent systems and modal logics

Sequent calculi for modal logics are well-established and well-understood – but not entirely satisfactory!

Some desiderata for “good” calculi [Wansing:’02]:

- subformula property: all the material in the premisses is contained in the conclusion
- separation: distinct left and right introduction rules
- locality: no restrictions on the context
- modularity: obtain other logics by changing single rules

Additionally:

- efficiency: “standard backwards proof search” is complexity-optimal

Sequent systems and modal logics

> *It can be easily verified that each of the standard rule systems [for modal logics] fails to satisfy some of the philosophical requirements [...].*

[Wansing:'94]

E.g.:

	$\dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ \mathsf{k}$	$\dfrac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ 4 \quad \dfrac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\ \mathsf{t}$
Subformula property:	✓	✓
Separation:	✗	✓
Locality:	✗	✗
Modularity:	✗ (for both systems together)	
Efficiency:	✓	✓

Solutions: structures with sequents in them

The solution according to internal approaches:

Extend the sequent structure!

By now, there are many ways to do so:

- Higher-level sequents : Sequents of sequents of sequents of...
  [Došen:'85]
- 2-sequents: Streams of sequents
  [Masini:'92]
- Display calculi: structural connectives for all operators
  [Belnap:'82, Wansing:'94, Kracht:'96]
- Nested sequents: Trees of sequents
  [Kashima:'94, Brünnler:'06, Poggiolesi:'09]
- ...

The Question

What is the simplest extension of the sequent structure satisfying these desiderata for modal logics?

Reminder: Modal logics

The formulae of modal logic are given by

$$\varphi ::= \mathsf{Var} \mid \neg\varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \to \varphi \mid \Box\varphi$$

The Hilbert-style presentation of normal modal logic K is given by the axioms and rules for classical propositional logic and

$$\mathsf{k}\ \ \Box(A \to B) \land \Box A \to \Box B \qquad\qquad \frac{\vdash A}{\vdash \Box A}\ \mathsf{nec}$$

A sequent is a tuple of multisets of formulae, written $\Gamma \Rightarrow \Delta$ and interpreted as $\bigwedge\Gamma \to \bigvee\Delta$.

The standard sequent system contains the standard propositional rules together with

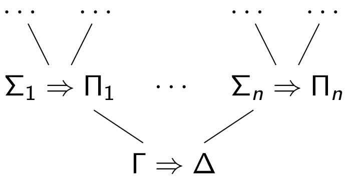
$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ \mathsf{k}$$

Case study: Nested sequents

Definition ([Brünnler:'09,Poggiolesi:'09])

A nested sequent is a finite tree whose nodes are labelled with sequents. The interpretation ι of this nested sequent is

$$\bigwedge\Gamma \to \bigvee\Delta \vee \bigvee_{i=1}^{n} \Box\iota(\Sigma_i \Rightarrow \Pi_i^*) \ .$$



Fact

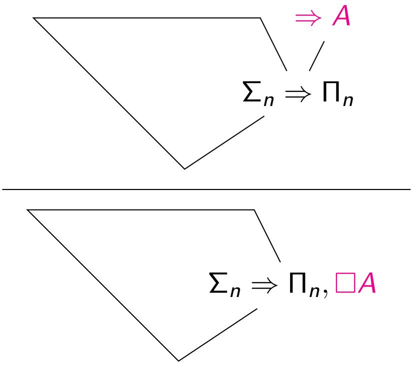
The nested sequent calculus with modal rules $\Box_R$ and $\Box_L$ is sound and cut-free complete for modal logic K.

Case study: Nested sequents

Definition ([Brünnler:'09,Poggiolesi:'09])

A nested sequent is a finite tree whose nodes are labelled with sequents. The interpretation ι of this nested sequent is

$$\bigwedge\Gamma \to \bigvee\Delta \vee \bigvee_{i=1}^{n} \Box\iota(\Sigma_i \Rightarrow \Pi_i^*) \ .$$



Fact

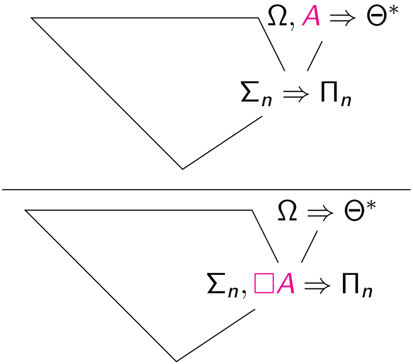
The nested sequent calculus with modal rules $\Box_R$ and $\Box_L$ is sound and cut-free complete for modal logic K.

Case study: Nested sequents

Definition ([Brünnler:'09,Poggiolesi:'09])

A nested sequent is a finite tree whose nodes are labelled with sequents. The interpretation ι of this nested sequent is

$$\bigwedge\Gamma \to \bigvee\Delta \vee \bigvee_{i=1}^{n} \Box\iota(\Sigma_i \Rightarrow \Pi_i^*) \ .$$

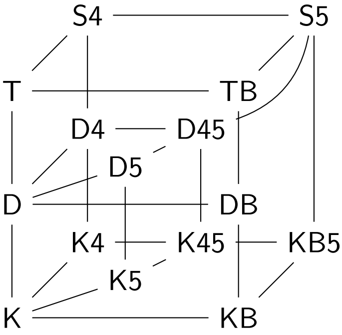


Fact

The nested sequent calculus with modal rules $\Box_R$ and $\Box_L$ is sound and cut-free complete for modal logic K.

Case study: Nested sequents

In fact, there are cut-free modular nested sequent systems for all logics in the (normal) modal cube:



[Marin, Straßburger:'14]

Case study: Nested sequents

In fact, there are cut-free modular nested sequent systems for all logics in the (normal) modal cube.

	seq.	nested seq.
Subformula property:	✓	✓
Separation:	✗	✓
Locality:	✗	✓
Modularity:	✗	✓
Efficiency:	✓	✗

(Nested sequents in proof search may be of exponential size)

Trees are nice, but can we go simpler?

A different approach: 2-sequents

$$\begin{array}{ccc} \vdots & & \vdots \\ \varepsilon & & \varepsilon \\ \Gamma_n & \Rightarrow & \Delta_n \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}$$

Definition ([Masini:'92])

A 2-sequent is an infinite, eventually empty stream of sequents. It's interpretation is

$$\bigwedge\Gamma_1 \to \bigvee\Delta_1 \vee \Box(\dots\Box(\bigwedge\Gamma_n \to \bigvee\Delta_n)\dots)$$

Fact

The 2-sequent calculus with modal rules $\Box_R$ and $\Box_L$ is sound and cut-free complete for modal logic KD.

A different approach: 2-sequents

Definition ([Masini:'92])

A 2-sequent is an infinite, eventually empty stream of sequents. It's interpretation is

$$\bigwedge\Gamma_1 \to \bigvee\Delta_1 \vee \Box(\ldots\Box(\bigwedge\Gamma_n \to \bigvee\Delta_n)\ldots)$$

$$\dfrac{\begin{array}{ccc} \vdots & & \varepsilon \\ \varepsilon & & A \\ \Gamma_n & \Rightarrow & \Delta_n \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}}{\begin{array}{ccc} \vdots & & \vdots \\ \varepsilon & & \varepsilon \\ \Gamma_n & \Rightarrow & \Delta_n, \Box A \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}}$$

Fact

The 2-sequent calculus with modal rules $\Box_R$ and $\Box_L$ is sound and cut-free complete for modal logic KD.

A different approach: 2-sequents

Definition ([Masini:'92])

A 2-sequent is an infinite, eventually empty stream of sequents. It's interpretation is

$$\bigwedge\Gamma_1 \to \bigvee\Delta_1 \vee \Box(\ldots\Box(\bigwedge\Gamma_n \to \bigvee\Delta_n)\ldots)$$

Fact

The 2-sequent calculus with modal rules $\Box_R$ *and* $\Box_L$ *is sound and cut-free complete for modal logic* KD.

$$\frac{\begin{array}{ccc} \vdots & & \vdots \\ \Sigma, A & & \Pi \\ \Gamma_n & \Rightarrow & \Delta_n \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}}{\begin{array}{ccc} \vdots & & \vdots \\ \Sigma & & \Pi \\ \Gamma_n, \Box A & \Rightarrow & \Delta_n \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}}$$

A different approach: 2-sequents

Definition ([Masini:'92])

A 2-sequent is an infinite, eventually empty stream of sequents. It's interpretation is

$$\bigwedge\Gamma_1 \to \bigvee\Delta_1 \vee \Box(\dots\Box(\bigwedge\Gamma_n \to \bigvee\Delta_n)\dots)$$

Fact

The 2-sequent calculus with modal rules $\Box_R$ and $\Box_L$ is sound and cut-free complete for modal logic KD.

$$\frac{\begin{array}{ccc} \vdots & & \vdots \\ \varepsilon, A & & \varepsilon \\ \Gamma_n & \Rightarrow & \Delta_n \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}}{\begin{array}{ccc} \vdots & & \vdots \\ \varepsilon & & \varepsilon \\ \Gamma_n, \Box A & \Rightarrow & \Delta_n \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}}$$

A different approach: 2-sequents

This sequent system for KD *admits cut-elimination, [...] and the introduction rules are separate, symmetrical and explicit, but no indication is given of how to present axiomatic extensions [...]. [I]t is not clear how Masini's framework may be modified in order to obtain a 2-sequent calculus for* K*.*

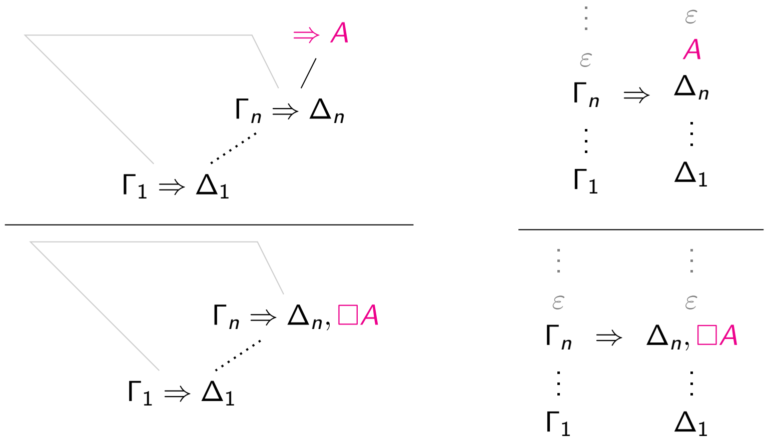
[Wansing:'02]

	seq.	nested seq.	2-seq.
Subformula property:	✓	✓	✓
Separation:	✗	✓	✓
Locality:	✗	✓	✓
Modularity:	✗	✓	(✗)
Efficiency:	✓	✗	?

Infinite linear structures are nice, but can we go simpler?

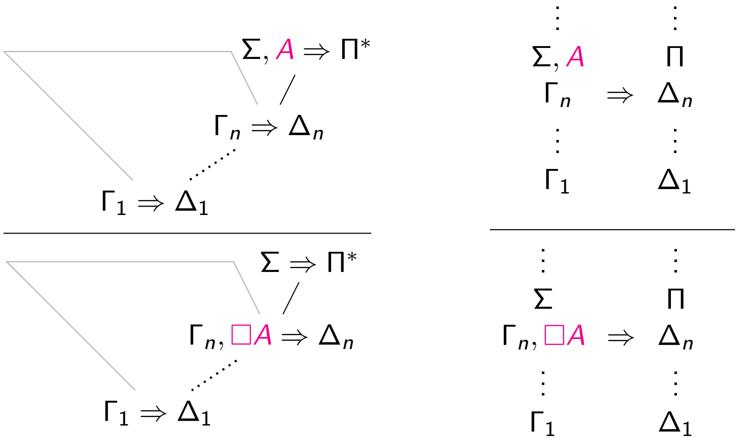
Connections between nested sequents and 2-sequents

Comparing the rules reveals they are essentially the same, e.g.:



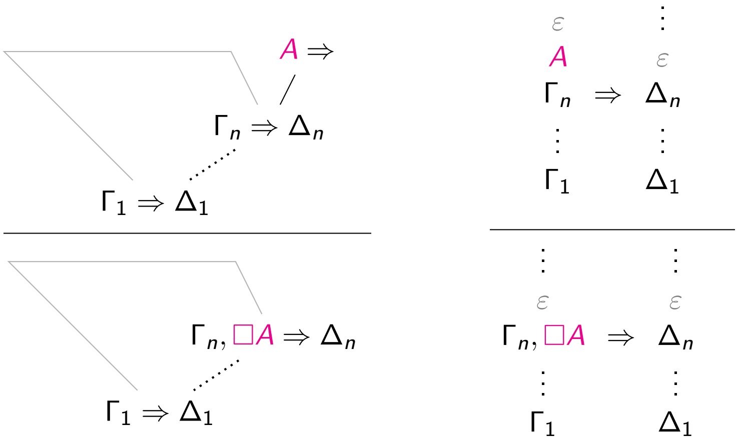
Connections between nested sequents and 2-sequents

Comparing the rules reveals they are essentially the same, e.g.:



Connections between nested sequents and 2-sequents

Comparing the rules reveals they are essentially the same, e.g.:



So the structure of finite lists of sequents is enough for KD!

Let's try finite lists of sequents!

Linear nested sequents

Definition

A linear nested sequent (LNS) is a finite list of sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \dots /\!\!/ \Gamma_n \Rightarrow \Delta_n$$

and interpreted as $\wedge\Gamma_1 \rightarrow \vee\Delta_1 \vee \Box(\dots\Box(\wedge\Gamma_n \rightarrow \vee\Delta_n)\dots)$.

The nested sequent system for K yields the modal rules of $\mathsf{LNS_K}$:

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}} \;\Box_L \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \;\Rightarrow A}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \Box A} \;\Box_R$$

The propositional rules are standard, e.g.:

$$\frac{\mathcal{G} /\!\!/ \Gamma, B \Rightarrow \Delta /\!\!/ \mathcal{H} \quad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, A \rightarrow B \Rightarrow \Delta /\!\!/ \mathcal{H}} \;\rightarrow_L \qquad \frac{\mathcal{G} /\!\!/ \Gamma, A \Rightarrow \Delta, B /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \rightarrow B /\!\!/ \mathcal{H}} \;\rightarrow_R$$

Completeness for linear nested sequents

We could show completeness via cut elimination . . . but it's easier!

Main Observation: The data structure of LNS is the same as that of a history in backwards proof search for a sequent calculus.

So we simply simulate a sequent derivation in the last components:
($\mathcal{G}$ is the history)

$$\dfrac{\dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ \mathsf{k}}{\vdots\ \mathcal{G}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathcal{G} /\!\!/ \Box\Gamma \Rightarrow \Box A /\!\!/ \Gamma \Rightarrow A}{\mathcal{G} /\!\!/ \Box\Gamma \Rightarrow \Box A /\!\!/ \Rightarrow A}\ \Box_L}{\mathcal{G} /\!\!/ \Box\Gamma \Rightarrow \Box A}\ \Box_R$$

Theorem
LNS_K *is sound and cut-free complete for* K *and complexity-optimal.*

Corollary: Cut-free completeness of the nested sequent calculus.

Extensions

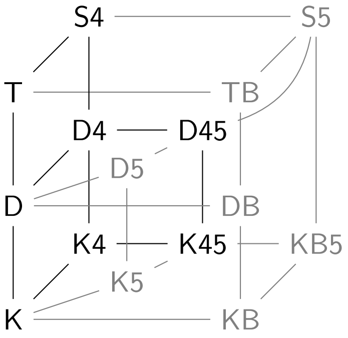
Extensions, e.g. (lifted shamelessly from nested sequent calculi):

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{d} \qquad \frac{\mathcal{G} /\!\!/ \Gamma, A \Rightarrow \Delta /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/ \mathcal{H}}\ \mathsf{t}$$

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma, \Box A \Rightarrow \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}}\ 4 \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi, \Box A /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \Box A /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}}\ 5$$

Theorem

The LNS calculi for extensions of K *with axioms from* $\mathsf{d}, \mathsf{t}, 4$ *or* $\mathsf{d}, 4, (4 \wedge 5)$ *are cut-free complete and modular.*



Application: intuitionistic logic

The same idea connects Maehara's multi-succedent calculus and Fitting's nested sequent calculus for first-order intuitionistic logic:

The intuitionistic interpretation of $\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n$ is

$$\bigwedge\Gamma_1 \to \bigvee\Delta_1 \vee (\bigwedge\Gamma_2 \to \bigvee\Delta_2 \vee (\ldots(\bigwedge\Gamma_n \to \bigvee\Delta_n)\ldots))$$

Restricting the nested sequent rules yields the rules of $\mathsf{LNS}_{\mathsf{Int}}$, e.g.:

$$\frac{\mathcal{G} /\!\!/ \Gamma, B \Rightarrow \Delta /\!\!/ \mathcal{H} \quad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, A \to B \Rightarrow \Delta /\!\!/ \mathcal{H}} \to_L \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow B}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B} \to_R$$

$$\frac{\mathcal{G} /\!\!/ \Gamma, A\alpha \Rightarrow \Delta /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \forall x.Ax \Rightarrow \Delta /\!\!/ \mathcal{H}} \forall_L \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Rightarrow A\alpha}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \forall x.Ax} \forall_R$$

if α in $\mathcal{H}$ then α in $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta$ — α not in the conclusion

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}} \text{Lift}$$

(Rules for $\exists$ analogous and other rules local.)

Intuitionistic logic: Completeness

We simulate Maehara's rules in the last components, e.g.:

$$\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow \Delta, A \to B} \to_R \quad \vdots\ \mathcal{G} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B /\!\!/ \Gamma, A \Rightarrow B}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B /\!\!/ A \Rightarrow B} \text{Lift}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B} \to_R$$

$$\dfrac{\Gamma \Rightarrow A\alpha}{\Gamma \Rightarrow \Delta, \forall x.Ax} \forall_R \quad \vdots\ \mathcal{G} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \forall x.Ax /\!\!/ \Gamma \Rightarrow A\alpha}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \forall x.Ax /\!\!/ \Rightarrow A\alpha} \text{Lift}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \forall x.Ax} \forall_R$$

The other rules are easy.

Theorem

$\mathsf{LNS}_{\mathsf{Int}}$ *is sound and complete for first-order intuitionistic logic.*

Corollary: Cut-free completeness of Fitting's calculus.

Taking it further: non-normal modal logics

The language of monotone modal logic M is that of modal logic.

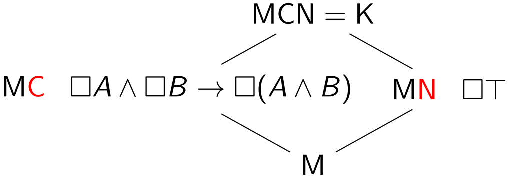
The Hilbert-style presentation of M is given by the axioms and rules for classical propositional logic and the rule

$$\frac{\vdash A \to B}{\vdash \Box A \to \Box B}\ \text{Mon}$$

The sequent system for M contains the standard propositional rules and the modal rule

$$\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\ \text{Mon}$$

Common extensions:



Monotone linear nested sequents

To capture the sequent rule $\dfrac{A \Rightarrow B}{\Box A \Rightarrow \Box B}$ we use a marker $\mathbin{\backslash\!\backslash}$ for "unfinished rules": A monotone LNS has the form ($n \geq 1$)

$$\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n$$
$$\text{or} \quad \Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n \backslash\!\!\backslash \Gamma_{n+1} \Rightarrow \Delta_{n+1}$$

Translating the sequent rule Mon yields the modal rules of $\mathsf{LNS_M}$:

$$\dfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta \backslash\!\!\backslash \Rightarrow B}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \Box B}\ \Box_R \qquad \dfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta \backslash\!\!\backslash \Sigma \Rightarrow \Pi}\ \Box_L$$

The propositional rules cannot be applied inside $\backslash\!\!\backslash$.

The completeness proof for $\mathsf{LNS_M}$ then uses the simulation

$$\begin{array}{c} \dfrac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\ \text{Mon} \\ \vdots\ \mathcal{G} \end{array} \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathcal{G} /\!\!/ \Box A \Rightarrow \Box B /\!\!/ A \Rightarrow B}{\mathcal{G} /\!\!/ \Box A \Rightarrow \Box B \backslash\!\!\backslash \Rightarrow B}\ \Box_L}{\mathcal{G} /\!\!/ \Box A \Rightarrow \Box B}\ \Box_R$$

Modularity for non-normal modal logics

C $\Box A \land \Box B \to \Box(A \land B)$

$$\frac{\mathcal{G} \mathbin{\backslash\!\backslash} \Gamma \Rightarrow \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta}\ \mathsf{c}$$

N $\Box\top$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta}{\mathcal{G} \mathbin{\backslash\!\backslash} \Gamma \Rightarrow \Delta}\ \mathsf{n}$$

D_1 $\neg\Box\bot$

$$\frac{\mathcal{G} \mathbin{\backslash\!\backslash} \ \Rightarrow}{\mathcal{G}}\ \mathsf{d}_1$$

D_2 $\neg(\Box A \land \Box\neg A)$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta \mathbin{\backslash\!\backslash} A \Rightarrow}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{d}_2$$

T $\Box A \to A$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma, A \Rightarrow \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{t}$$

4 $\Box A \to \Box\Box A$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta \mathbin{/\!/} \Sigma, \Box A \Rightarrow \Pi}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box A \Rightarrow \Delta \mathbin{\backslash\!\backslash} \Sigma \Rightarrow \Pi}\ 4$$

5 $\Box\neg A \lor \Box\neg\Box A$

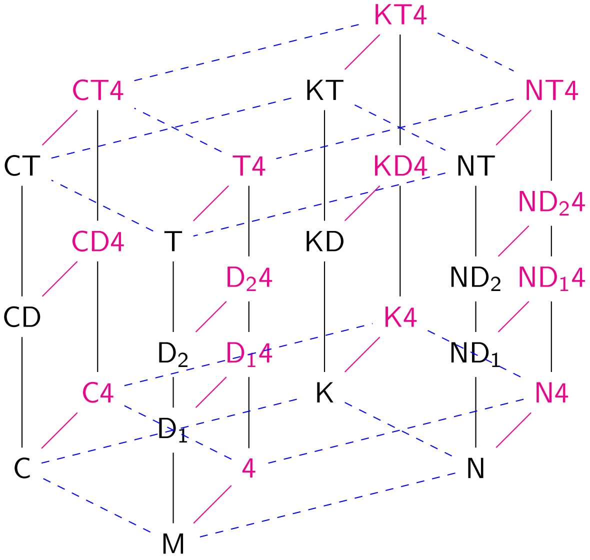
$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta \mathbin{/\!/} \Sigma \Rightarrow \Pi, \Box A}{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta, \Box A \mathbin{\backslash\!\backslash} \Sigma \Rightarrow \Pi}\ 5$$

Theorem

For $\mathcal{A} \subseteq \{\mathsf{C}, \mathsf{N}, \mathsf{D}_1, \mathsf{D}_2, \mathsf{T}, 4\}$ the calculus $\mathsf{LNS}_{\mathsf{M}\mathcal{A}}$ is sound and complete for $\mathsf{M}\mathcal{A}$. Similar for some combinations with 5.

(Use calculi e.g. from [Lavendhomme-Lucas:'00, Indrzejczak:'05].)

Modularity: the modal tesseract



(Restoring the bridge between normal and non-normal logics)

The desiderata

	seq.	nested seq.	2-seq	LNS
Subformula property:	✓	✓	✓	✓
Separation:	✗	✓	✓	✓
Locality:	✗	✓	✓	✓
Modularity:	✗	✓	(✓)	✓
Efficiency:	✓	✗	(✓)	✓

What about connections to other frameworks?

Hypersequents

The data structure of LNS is rather familiar from another setting:

Definition ([Avron:'96])

A hypersequent is a finite list of sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

and interpreted as $\Box(\bigwedge\Gamma_1 \to \bigvee\Delta_1) \vee \cdots \vee \Box(\bigwedge\Gamma_n \to \bigvee\Delta_n)$.

This interpretation suggests the external structural rules:

$$\frac{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi \mid \mathcal{H}} \text{ EEX}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}} \text{ EC} \qquad \frac{\mathcal{G} \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}} \text{ EW}$$

They are part of almost all hypersequent calculi for modal logics.

Hypersequents and linear nested sequents

Observation 1: EC and EW are the structural nested sequent rules for (4) and (t) (modulo internal structural rules):

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}\ \mathsf{EC} \quad \text{vs.} \quad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Sigma \Rightarrow \Delta, \Pi /\!\!/ \mathcal{H}}\ \dot{\mathsf{t}}$$

$$\frac{\mathcal{G} \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}\ \mathsf{EW} \quad \text{vs.} \quad \frac{\mathcal{G} /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Rightarrow /\!\!/ \mathcal{H}}\ \overline{4}$$

Observation 2: $\mathsf{LNS_K} + \dot{\mathsf{t}} + \overline{4} + \mathsf{EEX}$ is (essentially) the hypersequent calculus for S5 from [Restall:'07].

Theorem

$\mathsf{LNS_K} + \dot{\mathsf{t}} + \overline{4} + \mathsf{EEX}$ *is sound and cut-free complete for* S5 *(under the* LNS-*interpretation).*

(Similarly we obtain e.g. Avron's calculus etc.)

Summing up

Linear nested sequents:

- A good compromise between sequents and nested sequents
- Nested sequent systems for non-normal modal logics
- Easy cut-free completeness proof for nested sequents
- A connection to hypersequents via external exchange.