Linear Nested Sequents

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Natural and efficient proof systems for modal logics

Some desiderata for "good" or natural (sequent-style) calculi [Wansing:'02]:

- subformula property: all the material in the premisses is contained in the conclusion
- separation: distinct left and right introduction rules
- locality: no restrictions on the context
- modularity: obtain other logics by changing single rules

The desideratum of efficiency:

- "standard backwards proof search" is complexity-optimal (or at least not worse than for sequents)

Reminder: Normal modal logic

The formulae of modal logic are given by

$$A ::= \mathrm{Var} \mid \neg A \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A$$

The Hilbert-style presentation of normal modal logic K is given by the axioms for classical propositional logic, the axiom

$$\mathsf{k} \quad \Box(A \to B) \wedge \Box A \to \Box B$$

and the rules

$$\frac{\vdash A \qquad \vdash A \to B}{\vdash B}\ \mathsf{mp} \qquad\qquad \frac{\vdash A}{\vdash \Box A}\ \mathsf{nec}$$

Sequent calculi

A sequent is a pair of multisets of formulae, written $\Gamma \Rightarrow \Delta$.

The standard sequent system G_K for modal logic K contains the standard propositional rules together with

$$\frac{\Gamma \Rightarrow A}{\Sigma, \Box\Gamma \Rightarrow \Box A, \Delta}\ \mathsf{k}$$

Fact

G_K is sound and cut-free complete for K and is complexity-optimal.

Subformula property: ✓

Separation: ✗

Locality: ✗

(Modularity: ✗)

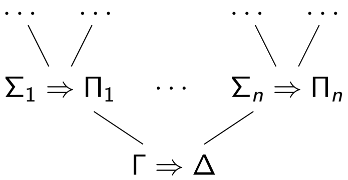
Efficiency: ✓

Nested sequent calculi

Definition ([Brünnler:'09,Poggiolesi:'09])

A nested sequent is a finite tree whose nodes are labelled with sequents. The interpretation ι of this nested sequent is

$$\bigwedge\Gamma \to \bigvee\Delta \vee \bigvee_{i=1}^{n} \Box\iota(\Sigma_i \Rightarrow \Pi_i^*)\ .$$



Fact

The nested sequent calculus with modal rules $\Box_R$ and $\Box_L$ is sound and cut-free complete for modal logic K.

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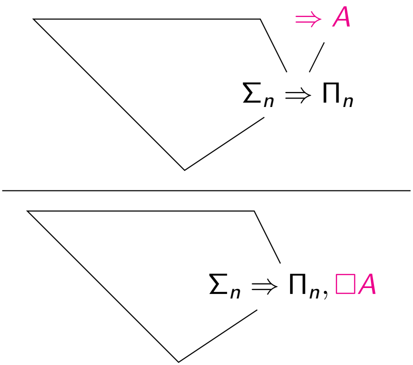

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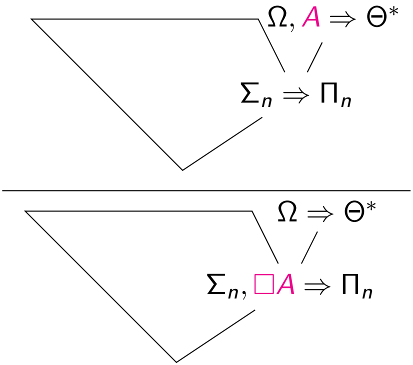

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Subformula property:	✓
Separation:	✓
Locality:	✓
(Modularity:	✓)
Efficiency:	✗

(Nested sequents in proof search may be of exponential size)

Can we combine the efficiency of sequents with the naturalness of nested sequents?

Some observations

- (Folklore) The nested sequent rules are a step-by-step decomposition of the standard sequent rule:

$$
\dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ \mathsf{k}
\qquad \rightsquigarrow \qquad
\dfrac{\dfrac{\begin{array}{c}\Gamma \Rightarrow A \\ | \\ \Box\Gamma \Rightarrow \Box A\end{array}}{\begin{array}{c}\Rightarrow A \\ | \\ \Box\Gamma \Rightarrow \Box A\end{array}}\ \Box_L}{\Box\Gamma \Rightarrow \Box A}\ \Box_R
$$

- Decomposing the sequent rule does not require the full tree structure of nested sequents, but only that of a branch.
- The data structure of a nested sequent branch is that of a history in backwards proof search for sequents.

Linear nested sequents

Definition

A linear nested sequent is a finite list of sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n$$

and interpreted as $\bigwedge\Gamma_1 \to \bigvee\Delta_1 \vee \Box(\ldots\Box(\bigwedge\Gamma_n \to \bigvee\Delta_n)\ldots)$.

The nested sequent system for K yields the modal rules of LNS_K:

$$\frac{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi}\ \Box_L \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \Box A /\!\!/ \ \Rightarrow A}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \Box A}\ \Box_R$$

In line with the “backwards proof search” perspective we restrict the propositional rules to the last component, e.g.:

$$\frac{\mathcal{G} /\!\!/ \Gamma, A, B \Rightarrow \Delta}{\mathcal{G} /\!\!/ \Gamma, A \wedge B \Rightarrow \Delta}\ \wedge_L \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \qquad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, B}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \wedge B}\ \wedge_R$$

Linear nested sequents for K

Theorem

$\mathsf{LNS_K}$ *is sound and complete for* K *and complexity-optimal.*

Proof.

- Soundness: directly from the full nested sequent calculus.
- Completeness: By simulating a sequent derivation in the last components: ($\mathcal{G}$ is the history)

$$\dfrac{\Gamma \Rightarrow A}{\Sigma, \Box\Gamma \Rightarrow \Box A, \Delta}\ \mathsf{k} \quad \vdots\ \mathcal{G} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\mathcal{G} /\!\!/ \Sigma, \Box\Gamma \Rightarrow \Box A, \Delta /\!\!/ \Gamma \Rightarrow A}{\mathcal{G} /\!\!/ \Sigma, \Box\Gamma \Rightarrow \Box A, \Delta /\!\!/ \Rightarrow A}\ \Box_L}{\mathcal{G} /\!\!/ \Sigma, \Box\Gamma \Rightarrow \Box A, \Delta}\ \Box_R$$

- Complexity: By noticing that $\mathsf{LNS_K}$ derivations are essentially sequent derivations with history. Or directly. □

Corollary

The nested sequent calculus for K *is cut-free complete.*

Taking it further: monotone modal logics

The language of monotone modal logic M is that of modal logic.

The Hilbert-style presentation of M is given by axioms and rules for classical propositional logic and the rule

$$\frac{\vdash A \to B}{\vdash \Box A \to \Box B}\ \text{Mon}$$

The sequent system for M contains the standard propositional rules and the modal rule

$$\frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B, \Delta}\ \text{Mon}$$

Common extensions:

MCN = K

MC $\Box A \wedge \Box B \to \Box(A \wedge B)$ MN $\Box\top$

M

Monotone linear nested sequents

To capture the sequent rule $\dfrac{A \Rightarrow B}{\Box A \Rightarrow \Box B}$ we use a marker $\mathbin{\backslash\!\backslash}$ for "unfinished rules": A monotone LNS has the form ($n \geq 1$)

$$\begin{array}{rl} & \Gamma_1 \Rightarrow \Delta_1 /\!\!/ \dots /\!\!/ \Gamma_n \Rightarrow \Delta_n \\ \text{or} & \Gamma_1 \Rightarrow \Delta_1 /\!\!/ \dots /\!\!/ \Gamma_n \Rightarrow \Delta_n \mathbin{\backslash\!\backslash} \Gamma_{n+1} \Rightarrow \Delta_{n+1} \end{array}$$

Translating the sequent rule Mon yields the modal rules of $\mathsf{LNS_M}$:

$$\dfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \Box B \mathbin{\backslash\!\backslash} \;\Rightarrow B}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \Box B}\ \Box_R \qquad \dfrac{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta \mathbin{\backslash\!\backslash} \Sigma \Rightarrow \Pi}\ \Box_L$$

The propositional rules cannot be applied inside $\mathbin{\backslash\!\backslash}$.

The completeness proof for $\mathsf{LNS_M}$ then uses the simulation

$$\begin{array}{c} \dfrac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B, \Delta}\ \text{Mon} \\ \vdots\ \mathcal{G} \end{array} \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Box B, \Delta /\!\!/ A \Rightarrow B}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Box B, \Delta \mathbin{\backslash\!\backslash} \;\Rightarrow B}\ \Box_L}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Box B, \Delta}\ \Box_R$$

Modularity for monotone modal logics

C $\Box A \wedge \Box B \to \Box(A \wedge B)$

$$\frac{\mathcal{G} \mathbin{\backslash\!\backslash} \Gamma \Rightarrow \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta}\ \mathsf{c}$$

N $\Box\top$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta}{\mathcal{G} \mathbin{\backslash\!\backslash} \Gamma \Rightarrow \Delta}\ \mathsf{n}$$

D_1 $\neg\Box\bot$

$$\frac{\mathcal{G} \mathbin{\backslash\!\backslash}\ \Rightarrow}{\mathcal{G}}\ \mathsf{d}_1$$

D_2 $\neg(\Box A \wedge \Box\neg A)$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma, \Box A \Rightarrow \Delta \mathbin{\backslash\!\backslash} A \Rightarrow}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{d}_2$$

T $\Box A \to A$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma, \Box A, A \Rightarrow \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{t}$$

4 $\Box A \to \Box\Box A$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma, \Box A \Rightarrow \Delta \mathbin{/\!/} \Sigma, \Box A \Rightarrow \Pi}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box A \Rightarrow \Delta \mathbin{\backslash\!\backslash} \Sigma \Rightarrow \Pi}\ \mathsf{4}$$

5 $\Box\neg A \vee \Box\neg\Box A$

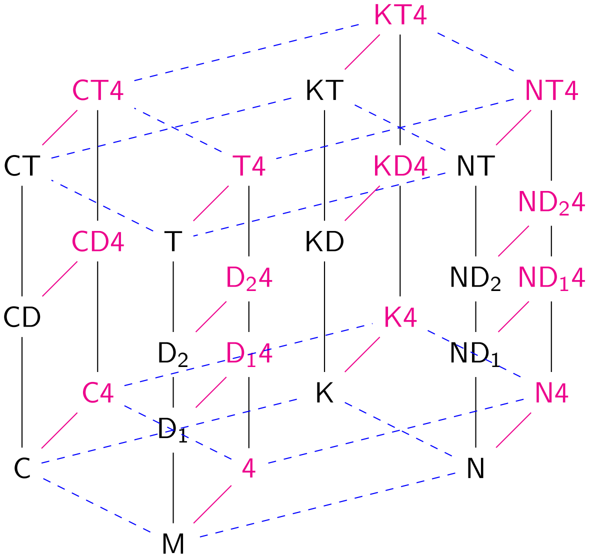
$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta, \Box A \mathbin{/\!/} \Sigma \Rightarrow \Pi, \Box A}{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow \Delta, \Box A \mathbin{\backslash\!\backslash} \Sigma \Rightarrow \Pi}\ \mathsf{5}$$

Theorem

For $\mathcal{A} \subseteq \{\mathsf{C}, \mathsf{N}, \mathsf{D}_1, \mathsf{D}_2, \mathsf{T}, 4\}$ the calculus $\mathsf{LNS}_{\mathsf{M}\mathcal{A}}$ is sound and complete for $\mathsf{M}\mathcal{A}$. Similar for some combinations with 5.

(Use calculi e.g. from [Lavendhomme-Lucas:'00, Indrzejczak:'05].)

Modularity: the modal tesseract



(Restoring the bridge between normal and non-normal logics)

Evaluation

LNS provide efficient and natural proof systems for these logics:

Subformula property:	✓
Separation:	✓
Locality:	✓
Modularity:	✓
Efficiency:	✓

Remark 1: Similarly we get LNS calculi for many other logics, e.g.:

- simply dependent bimodal logics
- (first-order) intuitionistic logic - linking Maehara's LJ' and Fitting's nested calculus

Remark 2: LNS are (almost) the 2-sequents from [Masini:'92]. This induces 2-sequent calculi e.g. for the normal modal logics.

Summing up

Linear nested sequents:

- a good compromise between sequents and nested sequents
- nested sequent systems for non-normal modal logics
- easy cut-free completeness proofs for nested calculi.

Bonus material

Focussed labelled line sequents for K

Sequents over labelled formulae $(v : A)$:

- unfocused: $zRx : \Gamma \,;\, X \Rightarrow Y \,;\, \Delta$
- focused right: $zR[x] : \Gamma \,;\, X \rightarrow \cdot \,;\, \Delta$
- focused left: $[x]Ry : \Gamma \,;\, X \rightarrow Y \,;\, \Delta$

$$\frac{zRx : \Gamma, x : B_b \,;\, X \Rightarrow Y \,;\, \Delta}{zRx : \Gamma \,;\, X, x : B_b \Rightarrow Y \,;\, \Delta}\ \mathtt{str_L} \qquad \frac{zRx : \Gamma \,;\, X \Rightarrow Y \,;\, \Delta, x : A_b}{zRx : \Gamma \,;\, X \Rightarrow Y, x : A_b \,;\, \Delta}\ \mathtt{str_R}$$

$$\frac{}{zR[x] : \Gamma \,;\, X, x : A \rightarrow \cdot \,;\, \Delta, x : A}\ \text{init}$$

$$\frac{zR[x] : \Gamma \,;\, X \rightarrow \cdot \,;\, \Delta}{zRx : \Gamma \,;\, X \Rightarrow \cdot \,;\, \Delta}\ D \qquad \frac{xRy : \cdot \,;\, X \Rightarrow Y \,;\, \Delta}{[x]Ry : \cdot \,;\, X \rightarrow Y \,;\, \Delta}\ R$$

$$\frac{[x]Ry : \Gamma \,;\, X \rightarrow y : A \,;\, \Delta}{zR[x] : \Gamma \,;\, X \rightarrow \cdot \,;\, \Delta, x : \Box A}\ \Box_R \qquad \frac{[x]Ry : \Gamma \,;\, X, y : A \rightarrow Y \,;\, \Delta}{[x]Ry : \Gamma, x : \Box A \,;\, X \rightarrow Y \,;\, \Delta}\ \Box_L$$

(A_b atomic or boxed, B_b boxed)

Simply dependent bimodal logic KT $\oplus_{\subseteq}$ S4

Formulae: $A ::= \text{Var} \mid \neg A \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \heartsuit A$

Axioms: Propositional axioms and rules, KT-axioms for $\Box$, S4-axioms for $\heartsuit$, and:

$$\heartsuit A \to \Box A$$

Linear nested sequent system $\mathsf{LNS}_{\mathsf{KT}\oplus_{\subseteq}\mathsf{S4}}$ (with $/\!/^{\Box}$ and $/\!/^{\heartsuit}$):

$$\frac{\mathcal{G} /\!/^{*} \Gamma \Rightarrow \Delta /\!/^{\Box} \Rightarrow A}{\mathcal{G} /\!/^{*} \Gamma \Rightarrow \Delta, \Box A}\ \Box_R \qquad \frac{\mathcal{G} /\!/^{*} \Gamma \Rightarrow \Delta /\!/^{\Box} \Sigma, A \Rightarrow \Pi}{\mathcal{G} /\!/^{*} \Gamma, \Box A \Rightarrow \Delta /\!/^{\Box} \Sigma \Rightarrow \Pi}\ \Box_L$$

$$\frac{\mathcal{G} /\!/^{*} \Gamma \Rightarrow \Delta /\!/^{\heartsuit} \Sigma, \heartsuit A \Rightarrow \Pi}{\mathcal{G} /\!/^{*} \Gamma, \heartsuit A \Rightarrow \Delta /\!/^{\heartsuit} \Sigma \Rightarrow \Pi}\ \heartsuit_{L\heartsuit} \qquad \frac{\mathcal{G} /\!/^{*} \Gamma \Rightarrow \Delta /\!/^{\Box} \Sigma, \heartsuit A \Rightarrow \Pi}{\mathcal{G} /\!/^{*} \Gamma, \heartsuit A \Rightarrow \Delta /\!/^{\Box} \Sigma \Rightarrow \Pi}\ \heartsuit_{L\Box}$$

$$\frac{\mathcal{G} /\!/^{*} \Gamma \Rightarrow \Delta /\!/^{\heartsuit} \Rightarrow A}{\mathcal{G} /\!/^{*} \Gamma \Rightarrow \Delta, \heartsuit A}\ \heartsuit_R \qquad \frac{\mathcal{G} /\!/^{*} \Gamma, \Box A, A \Rightarrow \Delta}{\mathcal{G} /\!/^{*} \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{t}_{\Box} \qquad \frac{\mathcal{G} /\!/^{*} \Gamma, \heartsuit A, A \Rightarrow \Delta}{\mathcal{G} /\!/^{*} \Gamma, \heartsuit A \Rightarrow \Delta}\ \mathsf{t}_{\heartsuit}$$

Intuitionistic logic

The same idea connects Maehara's multi-succedent calculus and Fitting's nested sequent calculus for first-order intuitionistic logic:

The intuitionistic interpretation of $\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \dots /\!\!/ \Gamma_n \Rightarrow \Delta_n$ is

$$\bigwedge\Gamma_1 \to \bigvee\Delta_1 \vee (\bigwedge\Gamma_2 \to \bigvee\Delta_2 \vee (\dots(\bigwedge\Gamma_n \to \bigvee\Delta_n)\dots))$$

Restricting the nested sequent rules yields the rules of $\mathsf{LNS_{Int}}$, e.g.:

$$\frac{\mathcal{G}/\!\!/\Gamma, B \Rightarrow \Delta/\!\!/\mathcal{H} \quad \mathcal{G}/\!\!/\Gamma \Rightarrow \Delta, A/\!\!/\mathcal{H}}{\mathcal{G}/\!\!/\Gamma, A \to B \Rightarrow \Delta/\!\!/\mathcal{H}} \to_L \qquad \frac{\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta/\!\!/A \Rightarrow B}{\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta, A \to B} \to_R$$

$$\frac{\mathcal{G}/\!\!/\Gamma, A\alpha \Rightarrow \Delta/\!\!/\mathcal{H}}{\mathcal{G}/\!\!/\Gamma, \forall x.Ax \Rightarrow \Delta/\!\!/\mathcal{H}} \forall_L \qquad \frac{\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta/\!\!/ \;\Rightarrow A\alpha}{\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta, \forall x.Ax} \forall_R$$

if α in $\mathcal{H}$ then α in $\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta$ — α not in the conclusion

$$\frac{\mathcal{G}/\!\!/\Gamma \Rightarrow \Delta/\!\!/\Sigma, A \Rightarrow \Pi/\!\!/\mathcal{H}}{\mathcal{G}/\!\!/\Gamma, A \Rightarrow \Delta/\!\!/\Sigma \Rightarrow \Pi/\!\!/\mathcal{H}} \mathsf{Lift}$$

(Rules for $\exists$ analogous and other rules local.)

Intuitionistic logic: Completeness

We simulate Maehara's rules in the last components, e.g.:

$$\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow \Delta, A \to B} \to_R \quad \vdots\ \mathcal{G} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B /\!\!/ \Gamma, A \Rightarrow B}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B /\!\!/ A \Rightarrow B} \text{Lift}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B} \to_R$$

$$\dfrac{\Gamma \Rightarrow A\alpha}{\Gamma \Rightarrow \Delta, \forall x.Ax} \forall_R \quad \vdots\ \mathcal{G} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \forall x.Ax /\!\!/ \Gamma \Rightarrow A\alpha}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \forall x.Ax /\!\!/ \Rightarrow A\alpha} \text{Lift}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \forall x.Ax} \forall_R$$

The other rules are easy.

Theorem

$\mathsf{LNS}_{\mathsf{Int}}$ *is sound and complete for first-order intuitionistic logic.*

Corollary: Cut-free completeness of Fitting's calculus.

Hypersequents

The data structure of LNS is rather familiar from another setting:

Definition ([Avron:'96])

A hypersequent is a finite list of sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

and interpreted as $\Box(\wedge\Gamma_1 \to \vee\Delta_1) \vee \cdots \vee \Box(\wedge\Gamma_n \to \vee\Delta_n)$.

This interpretation suggests the external structural rules:

$$\frac{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi \mid \mathcal{H}} \text{ EEX}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}} \text{ EC} \qquad \frac{\mathcal{G} \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}} \text{ EW}$$

They are part of almost all hypersequent calculi for modal logics.

Hypersequents and linear nested sequents

Observation 1: EC and EW are the structural nested sequent rules for (4) and (t) (modulo internal structural rules):

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}\ \mathsf{EC} \quad \text{vs.} \quad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Sigma \Rightarrow \Delta, \Pi /\!\!/ \mathcal{H}}\ \dot{\mathsf{t}}$$

$$\frac{\mathcal{G} \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}\ \mathsf{EW} \quad \text{vs.} \quad \frac{\mathcal{G} /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Rightarrow /\!\!/ \mathcal{H}}\ \bar{4}$$

Observation 2: $\mathsf{LNS_K} + \dot{\mathsf{t}} + \bar{4} + \mathsf{EEX}$ is (essentially) the hypersequent calculus for S5 from [Restall:'07].

Theorem

$\mathsf{LNS_K} + \dot{\mathsf{t}} + \bar{4} + \mathsf{EEX}$ *is sound and cut-free complete for* S5 *(under the* LNS*-interpretation).*

(Similarly we obtain e.g. Avron's calculus etc.)