Linear Nested Sequents, 2-Sequents and Hypersequents

Björn Lellmann

TU Wien

TABLEAUX 2015
Wrocław, Sep. 22, 2015

Sequent systems and modal logics

Sequent calculi for modal logics are well-established and well-understood – but not entirely satisfactory!

Some desiderata for "good" calculi [Wansing:'02]:

- separation: distinct left and right introduction rules
- locality: no restrictions on the context
- modularity: obtain other logics by adding single rules

> *It can be easily verified that each of the standard rule systems [for modal logics] fails to satisfy some of the philosophical requirements [...].*
>
> [Wansing:'94]

E.g.:

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ \mathsf{k} \qquad\qquad \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ 4$$

Solutions: structures with sequents in them

The solution according to internal approaches:

Extend the sequent structure!

By now, there are many ways to do so:

- Higher-level sequents : Sequents of sequents of sequents of... [Došen:'85]
- 2-sequents: Streams of sequents [Masini:'92]
- Display calculi: structural connectives for all operators [Belnap:'82, Wansing:'94, Kracht:'96]
- Nested sequents: Trees of sequents [Kashima:'94, Brünnler:'06, Poggiolesi:'09]
- ...

The Question

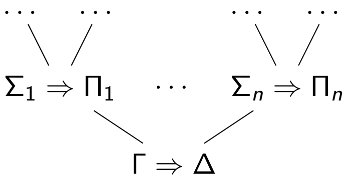
What is the simplest extension of the sequent structure satisfying these desiderata for modal logics?

Case study: Nested sequents

Definition ([Brünnler:'09,Poggiolesi:'09])

A nested sequent is a finite tree whose nodes are labelled with sequents. The interpretation ι of this nested sequent is

$$\bigwedge\Gamma \rightarrow \bigvee\Delta \vee \bigvee_{i=1}^{n} \Box\iota(\Sigma_i \Rightarrow \Pi_i)\ .$$



Fact

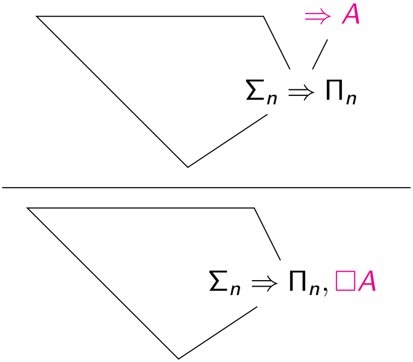
The nested sequent calculus with modal rules $\Box_R$ and $\Box_L$ is sound and cut-free complete for modal logic K.

Case study: Nested sequents

Definition ([Brünnler:'09,Poggiolesi:'09])

A nested sequent is a finite tree whose nodes are labelled with sequents. The interpretation ι of this nested sequent is

$$\bigwedge\Gamma \to \bigvee\Delta \vee \bigvee_{i=1}^{n}\Box\iota(\Sigma_i \Rightarrow \Pi_i)\ .$$



Fact

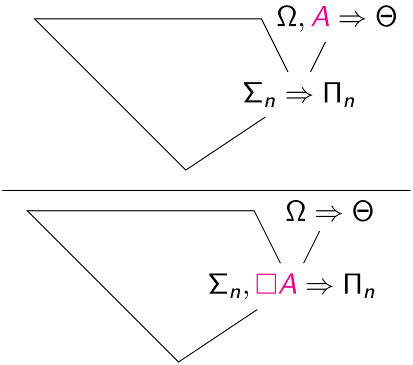
The nested sequent calculus with modal rules $\Box_R$ *and* $\Box_L$ *is sound and cut-free complete for modal logic* K.

Case study: Nested sequents

Definition ([Brünnler:'09,Poggiolesi:'09])

A nested sequent is a finite tree whose nodes are labelled with sequents. The interpretation ι of this nested sequent is

$$\bigwedge \Gamma \to \bigvee \Delta \vee \bigvee_{i=1}^{n} \Box \iota(\Sigma_i \Rightarrow \Pi_i) \ .$$



Fact

The nested sequent calculus with modal rules $\Box_R$ and $\Box_L$ is sound and cut-free complete for modal logic K.

Trees are nice, but can we go simpler?

A different approach: 2-sequents

$$\begin{array}{ccc} \vdots & & \vdots \\ \varepsilon & & \varepsilon \\ \Gamma_n & \Rightarrow & \Delta_n \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}$$

Definition ([Masini:'92])

A 2-sequent is an infinite, eventually empty stream of sequents. It's interpretation is

$$\bigwedge\Gamma_1 \to \bigvee\Delta_1 \vee \Box(\ldots\Box(\bigwedge\Gamma_n \to \bigvee\Delta_n)\ldots)$$

Fact

The 2-sequent calculus with modal rules $\Box_R$ *and* $\Box_L$ *is sound and cut-free complete for modal logic* KD.

A different approach: 2-sequents

Definition ([Masini:'92])

A 2-sequent is an infinite, eventually empty stream of sequents. It's interpretation is

$$\bigwedge \Gamma_1 \to \bigvee \Delta_1 \vee \Box(\ldots \Box(\bigwedge \Gamma_n \to \bigvee \Delta_n) \ldots)$$

$$\dfrac{\begin{array}{ccc} \vdots & & \varepsilon \\ \varepsilon & & A \\ \Gamma_n & \Rightarrow & \Delta_n \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}}{\begin{array}{ccc} \vdots & & \vdots \\ \varepsilon & & \varepsilon \\ \Gamma_n & \Rightarrow & \Delta_n, \Box A \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}}$$

Fact

The 2-sequent calculus with modal rules $\Box_R$ *and* $\Box_L$ *is sound and cut-free complete for modal logic* KD.

A different approach: 2-sequents

Definition ([Masini:'92])

A 2-sequent is an infinite, eventually empty stream of sequents. It's interpretation is

$$\bigwedge\Gamma_1 \to \bigvee\Delta_1 \vee \Box(\dots\Box(\bigwedge\Gamma_n \to \bigvee\Delta_n)\dots)$$

$$\dfrac{\begin{array}{ccc} \vdots & & \vdots \\ \Sigma, A & & \Pi \\ \Gamma_n & \Rightarrow & \Delta_n \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}}{\begin{array}{ccc} \vdots & & \vdots \\ \Sigma & & \Pi \\ \Gamma_n, \Box A & \Rightarrow & \Delta_n \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}}$$

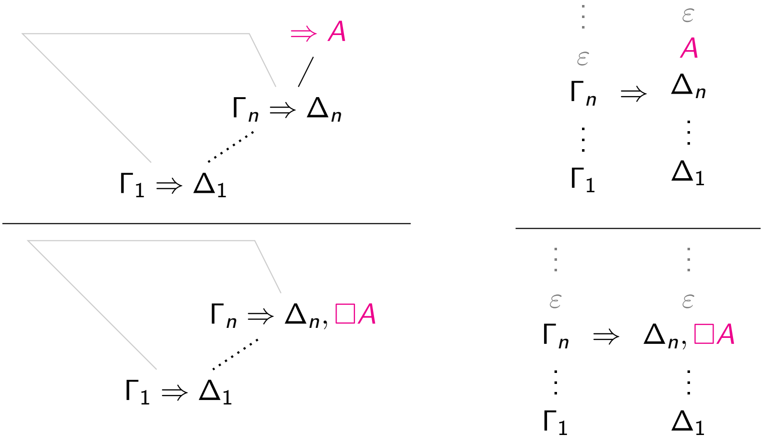
Fact

The 2-sequent calculus with modal rules $\Box_R$ and $\Box_L$ is sound and cut-free complete for modal logic KD.

Infinite linear structures are nice, but can we go simpler?

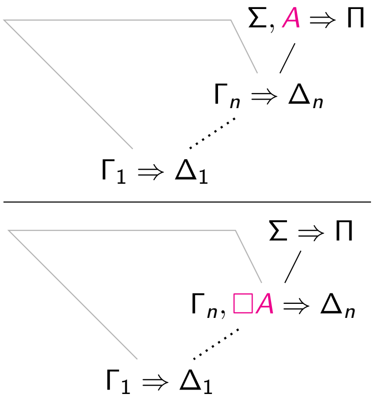
Connections between nested sequents and 2-sequents

Comparing the rules reveals they are essentially the same, e.g.:



Connections between nested sequents and 2-sequents

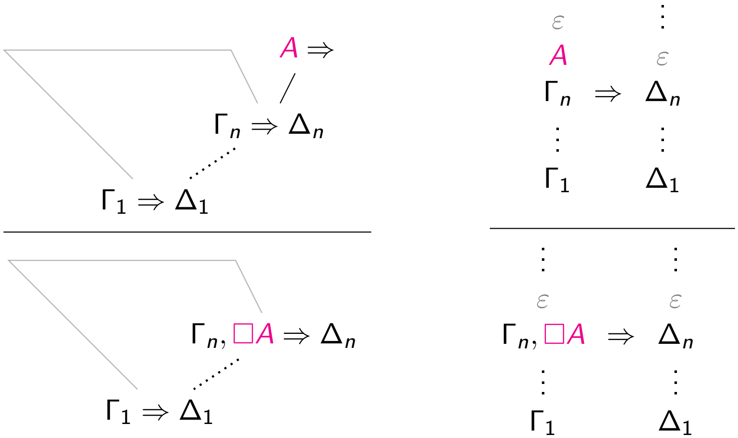
Comparing the rules reveals they are essentially the same, e.g.:



$$\dfrac{\begin{array}{ccc} \vdots & & \vdots \\ \Sigma, A & & \Pi \\ \Gamma_n & \Rightarrow & \Delta_n \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}}{\begin{array}{ccc} \vdots & & \vdots \\ \Sigma & & \Pi \\ \Gamma_n, \Box A & \Rightarrow & \Delta_n \\ \vdots & & \vdots \\ \Gamma_1 & & \Delta_1 \end{array}}$$

Connections between nested sequents and 2-sequents

Comparing the rules reveals they are essentially the same, e.g.:



So the structure of finite lists of sequents is enough for KD!

Let's try finite lists of sequents!

Linear nested sequents

Definition

A linear nested sequent is a finite list of sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n$$

and interpreted as $\bigwedge\Gamma_1 \to \bigvee\Delta_1 \vee \Box(\ldots\Box(\bigwedge\Gamma_n \to \bigvee\Delta_n)\ldots)$.

The nested sequent system for K yields the modal rules of LNS_K:

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}}\ \Box_L \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Rightarrow A}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \Box A}\ \Box_R$$

Extensions, e.g. (lifted shamelessly from nested sequent calculi):

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{d} \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma, \Box A \Rightarrow \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}}\ 4$$

Completeness for linear nested sequents

We could show completeness via cut elimination . . . but it's easier!

Observation: The data structure of LNS is the same as that of a history in backwards proof search for a sequent calculus.

So we simply simulate a sequent derivation in the last components: ($\mathcal{G}$ is the history)

$$\dfrac{\dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ \mathsf{k}}{\vdots\ \mathcal{G}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathcal{G} /\!\!/ \Box\Gamma \Rightarrow \Box A /\!\!/ \Gamma \Rightarrow A}{\mathcal{G} /\!\!/ \Box\Gamma \Rightarrow \Box A /\!\!/ \Rightarrow A}\ \Box_L}{\mathcal{G} /\!\!/ \Box\Gamma \Rightarrow \Box A}\ \Box_R$$

Theorem

The LNS calculi for K *and extensions with axioms from* $\mathsf{d}, \mathsf{t}, 4$ *or* $\mathsf{d}, 4, (4 \wedge 5)$ *are cut-free complete and modular.*

Corollary: Cut-free completeness of the nested sequent calculi.

Application: intuitionistic logic

The same idea connects Maehara's multi-succedent calculus and Fitting's nested sequent calculus for intuitionistic logic, e.g.:

Maehara:

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \rightarrow_R$$

Fitting (restricted to LNS):

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}} \text{ Lift}$$

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow B}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \rightarrow B} \rightarrow_R$$

Maehara's rule is simulated by Fitting's $\rightarrow_R$ and Lift.
The quantifier rules are similar.

Theorem

The LNS calculus for (full) first-order intuitionistic logic (and hence also Fitting's nested sequent calculus) is cut-free complete.

Hypersequents

The data structure of LNS is rather familiar from another setting:

Definition ([Avron:'96])

A hypersequent is a finite list of sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

and interpreted as $\Box(\bigwedge\Gamma_1 \rightarrow \bigvee\Delta_1) \vee \cdots \vee \Box(\bigwedge\Gamma_n \rightarrow \bigvee\Delta_n)$.

This interpretation suggests the external structural rules:

$$\frac{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi \mid \mathcal{H}} \text{ EEX}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}} \text{ EC} \qquad \frac{\mathcal{G} \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}} \text{ EW}$$

They are part of almost all hypersequent calculi for modal logics.

Hypersequents and linear nested sequents

Observation 1: EC and EW are the structural nested sequent rules for (4) and (t) (modulo internal structural rules):

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}} \; \mathsf{EC} \quad \text{vs.} \quad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Sigma \Rightarrow \Delta, \Pi /\!\!/ \mathcal{H}} \; \dot{\mathsf{t}}$$

$$\frac{\mathcal{G} \mid \mathcal{H}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \mathcal{H}} \; \mathsf{EW} \quad \text{vs.} \quad \frac{\mathcal{G} /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Rightarrow /\!\!/ \mathcal{H}} \; \bar{4}$$

Observation 2: $\mathsf{LNS_K} + \dot{\mathsf{t}} + \bar{4} + \mathsf{EEX}$ is (essentially) the hypersequent calculus for S5 from [Restall:'07].

Theorem

$\mathsf{LNS_K} + \dot{\mathsf{t}} + \bar{4} + \mathsf{EEX}$ *is sound and cut-free complete for* S5 *(under the* LNS*-interpretation).*

(Similarly we obtain e.g. Avron's calculus etc.)

Conclusion

Summing up:

- Finite lists of sequents give good systems for modal logics and intuitionistic logic
- An easy method to show cut-free completeness
- A connection to hypersequents via external exchange.

Future work:

- Complexity of proof search (partly done)
- Non-normal modal logics (partly done)
- Syntactic cut elimination for LNS
- "Proper" LNS systems for logics without cut-free sequent calculi (e.g., modal logic B).