Grafting Hypersequents onto Nested Sequents

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(From the point of view of modal logic...)

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- symmetry and transitivity (S5)
- Euclideaness (K5)
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Successful extensions of the framework

In particular two extensions of the sequent framework are useful:

Hypersequents

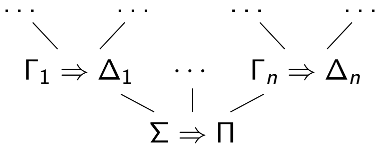
Lists of sequents:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

+ Can be complexity-optimal: coNP for S5
– Do not capture some logics, e.g., K5

Nested sequents

Trees of sequents:



+ Capture all logics in the modal cube, also K5
– Suboptimal complexity: EXP instead of coNP for K5

Can we combine the advantages of hypersequents and nested sequents?

Preliminaries

As usual, the set $\mathcal{F}$ of formulae of modal logic is given by:

$$\mathcal{F} ::= p, q, \ldots \mid \bot \mid \neg\mathcal{F} \mid \Box\mathcal{F} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \rightarrow \mathcal{F}$$

We abbreviate $\neg\Box\neg A$ to $\Diamond A$.

Modal logic K5 is given Hilbert-style by closing the axioms

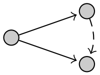
$$(\mathrm{k}) \quad \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \quad \text{and} \quad (5) \quad \Diamond\Box A \rightarrow \Box A$$

and axioms for classical propositional logic under the rules

$$\frac{A \quad A \rightarrow B}{B} \text{ modus ponens, MP} \quad \text{and} \quad \frac{A}{\Box A} \text{ necessitation, nec}$$

Semantically, K5 is the logic of the class of Kripke frames which are euclidean, i.e., satisfy the condition:

$$\forall x, y, z.\, xRy \wedge xRz \rightarrow yRz$$

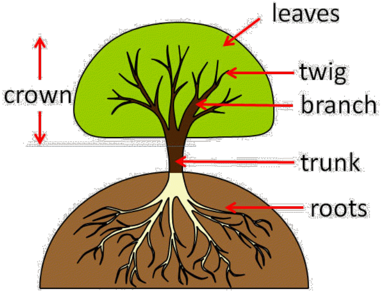


Grafted Hypersequents

Main idea: Graft a hypersequent on top of a nested sequent!

> *Grafting [...] is a horticultural technique whereby tissues from one plant are inserted into those of another so that the two sets of vascular tissues may join together.*

(Wikipedia)



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Definition

A grafted hypersequent is of the form

$$\Gamma \Rightarrow \Delta \mid\mid \Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n$$

with $\Gamma \Rightarrow \Delta$ and the $\Sigma_i \Rightarrow \Pi_i$ sequents (multiset based). The sequent $\Gamma \Rightarrow \Delta$ is its trunk, the rest its crown.

The formula interpretation of the above grafted hypersequent is

$$\bigwedge \Gamma \to \bigvee \Delta \vee \Box(\bigwedge \Sigma_1 \to \bigvee \Pi_1) \vee \cdots \vee \Box(\bigvee \Sigma_n \to \bigvee \Pi_n)\ .$$

(I.e., a "truncated nested sequent" or "rooted hypersequent".)

The grafted hypersequent system $\mathcal{R}_{K5}$ for K5

Trunk rules only work in the trunk, e.g.:

$$\frac{}{\Gamma, \bot \Rightarrow \Delta \mid\mid \mathcal{H}} \bot_L \qquad \frac{}{\Gamma, p \Rightarrow p, \Delta \mid\mid \mathcal{H}} \text{Init}$$

$$\frac{\Gamma, B \Rightarrow \Delta \mid\mid \mathcal{H} \quad \Gamma \Rightarrow A, \Delta \mid\mid \mathcal{H}}{\Gamma, A \to B \Rightarrow \Delta \mid\mid \mathcal{H}} \to_L \qquad \frac{\Gamma, A \Rightarrow B, \Delta \mid\mid \mathcal{H}}{\Gamma \Rightarrow A \to B, \Delta \mid\mid \mathcal{H}} \to_R$$

Transfer rules govern the interaction between crown and trunk:

$$\frac{\Gamma \Rightarrow \Delta \mid\mid \mathcal{H} \mid \Sigma, A \Rightarrow \Pi}{\Gamma, \Box A \Rightarrow \Delta \mid\mid \mathcal{H} \mid \Sigma \Rightarrow \Pi} \Box_L \qquad \frac{\Gamma \Rightarrow \Delta \mid\mid \mathcal{H} \mid \; \Rightarrow A}{\Gamma \Rightarrow \Box A, \Delta \mid\mid \mathcal{H}} \Box_R$$

Crown rules only work in the crown (with empty trunk!):

$$\frac{\Rightarrow \mid\mid \mathcal{H} \mid \Sigma, A \Rightarrow \Pi}{\Rightarrow \mid\mid \mathcal{H} \mid \Box A \Rightarrow \; \mid \Sigma \Rightarrow \Pi} 5 \qquad \frac{\Rightarrow \mid\mid \mathcal{H} \mid \Rightarrow A}{\Rightarrow \mid\mid \mathcal{H} \mid \Rightarrow \Box A} \text{K}$$

and similarly for the propositional rules.
We also include (trunk and crown versions of) the structural rules.

The grafted hypersequent system for K5

Example

The axiom (5) $\Diamond\Box p \to \Box p$ is derived via

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \;||\; p \Rightarrow p}{\Rightarrow \;||\; \Box p \Rightarrow \;|\; \Rightarrow p}\; 5}{\Rightarrow \;||\; \Rightarrow \neg\Box p \;|\; \Rightarrow p}\; \neg_R}{\Rightarrow \Box\neg\Box p, \Box p}\; \Box_R, \Box_R}{\Rightarrow \neg\Box\neg\Box p \to \Box p}\; prop$$

Theorem

$\mathcal{R}_{\mathsf{K5}}$ is sound and complete for K5 *in presence of the trunk and crown cut rules:*

$$\dfrac{\Gamma \Rightarrow \Delta, A \;||\; \mathcal{H} \qquad A, \Sigma \Rightarrow \Pi \;||\; \mathcal{G}}{\Gamma, \Sigma \Rightarrow \Delta, \Pi \;||\; \mathcal{H} \;|\; \mathcal{G}}\; \mathrm{Cut_t}$$

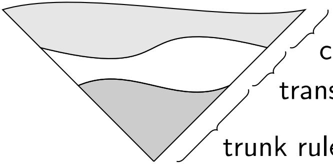
$$\dfrac{\Rightarrow \;||\; \mathcal{H} \;|\; \Gamma \Rightarrow \Delta, A \qquad \Rightarrow \;||\; \mathcal{G} \;|\; A, \Sigma \Rightarrow \Pi}{\Rightarrow \;||\; \mathcal{H} \;|\; \mathcal{G} \;|\; \Gamma, \Sigma \Rightarrow \Delta, \Pi}\; \mathrm{Cut_c}$$

Cut elimination

As expected, cut elimination for $\mathcal{R}_{\mathsf{K5}}$ is a bit complicated...

Main ingredients:

- a layering lemma stating that derivations are layered:



- a standard proof to push up multi-cuts in the trunk layer until they hit the transfer layer
- a step to permute multi-cuts over the transfer layer
- a hypersequent cut elimination proof based on [Ciabattoni, Metcalfe, Montagna: 2010]

Decidability and complexity

For the decision procedure we make the structural rules (except for trunk weakening) admissible by Kleene'ing the rules, e.g.:

$$\frac{\Gamma \Rightarrow \Box A, \Delta \,||\, \mathcal{H} \mid\ \Rightarrow A}{\Gamma \Rightarrow \Box A, \Delta \,||\, \mathcal{H}}\ \Box_R^* \qquad \frac{\Gamma, \Box A \Rightarrow \Delta \,||\, \mathcal{H} \mid \Sigma, A \Rightarrow \Pi}{\Gamma, \Box A \Rightarrow \Delta \,||\, \mathcal{H} \mid \Sigma \Rightarrow \Pi}\ \Box_L^*$$

$$\frac{\Rightarrow \,||\, \mathcal{H} \mid \Gamma, \Box A \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi}{\Rightarrow \,||\, \mathcal{H} \mid \Gamma, \Box A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}\ 5^* \qquad \frac{\Rightarrow \,||\, \mathcal{H} \mid \Gamma, \Box A, A \Rightarrow \Delta}{\Rightarrow \,||\, \mathcal{H} \mid \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{T}^*$$

$$\frac{\Rightarrow \,||\, \mathcal{H} \mid \Gamma \Rightarrow \Box A, \Delta \mid\ \Rightarrow A}{\Rightarrow \,||\, \mathcal{H} \mid \Gamma \Rightarrow \Box A, \Delta}\ \mathsf{K}^*$$

Theorem

Proof search in the Kleene'd system $\mathcal{R}^*_{\mathsf{K5}}$ can be implemented in (optimal) complexity coNP.

Cut-free completeness semantically

... via equivalence to a grafted tableaux system:

Labelled formulae F A or T A are prefixed with either the trunk prefix $\bullet$, a limb prefix $\mathbb{1}, \mathbb{2}, \ldots$ or a twig prefix $1, 2, \ldots$.

The interesting rules (the propositional rules are standard):

$$\frac{\bullet : \mathrm{F}\,\Box A}{\mathbb{n} : \mathrm{F}\,A}\ \mathbb{n}\text{ new} \qquad \frac{\bullet : \mathrm{T}\,\Box A}{\mathbb{n} : \mathrm{T}\,A}\ \mathbb{n}\text{ occurs} \qquad \frac{\mathrm{c} : \mathrm{F}\,\Box A}{n : \mathrm{F}\,A}\ n\text{ new} \qquad \frac{\mathrm{c} : \mathrm{T}\,\Box A}{\mathrm{c}' : \mathrm{T}\,A}\ \mathrm{c}'\text{ occurs}$$

where c and c′ are limb or twig prefixes.

- A branch is closed if it contains ℓ : T A and ℓ : F A for some label ℓ and formula A.
- A tableau is closed if every branch in it is closed.
- A formula is A derivable if there is a closed tableau starting with $\bullet$: F A.

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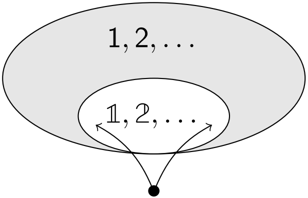
Intuition: Models for K5 have the shape



- twigs are accessible from twigs and limbs but not from the root
- limbs are accessible from the root, from twigs and from limbs.

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Example

The following closed tableau shows derivability of shift transitivity:

$\bullet : \mathsf{F}\, \Box(\Box p \to \Box\Box p)$
$\mathbb{1} : \mathsf{F}\, \Box p \to \Box\Box p$
$\mathbb{1} : \mathsf{T}\, \Box p$
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$2 : \mathsf{F}\, \Box p$
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The grafted tableaux system for K5 *is sound and complete and equivalent to the grafted hypersequent system* $\mathcal{R}^*_{\mathsf{K5}}$*. Hence the latter is cut-free complete.*

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Summary

- A framework combining nested sequents and hypersequents
- Complexity optimal cut-free calculi for K5, KD5, SDL^+
- A corresponding simplified prefixed tableaux system.

R. Kuznets and B. Lellmann.
Grafting hypersequents onto nested sequents.
Arxiv preprint arXiv:1502.00814 [cs.LO], 2015.

Extensions and Modifications

The same ideas yield complexity-optimal grafted hypersequent calculi for the logics

- KD5, axiomatised by the K5-axioms and

$$\text{seriality} \qquad \Box A \to \Diamond A\,.$$

$$\left(\text{Add the rule} \quad \frac{\Gamma \Rightarrow \Delta \mid\mid \mathcal{H} \mid A \Rightarrow}{\Gamma, \Box A \Rightarrow \Delta \mid\mid \mathcal{H}}\ \Box_L^D\,.\right)$$

- SDL^+ or $\text{KT}_\Box$, axiomatised by the K-axioms and

$$\text{shift reflexivity} \qquad \Box(\Box A \to A)\,.$$

  (Use a hypersequent calculus for KT as graft.)

- $\text{KDT}_\Box$, axiomatised by the $\text{KT}_\Box$-axioms and seriality. (Add the rule $\Box_L^D$.)