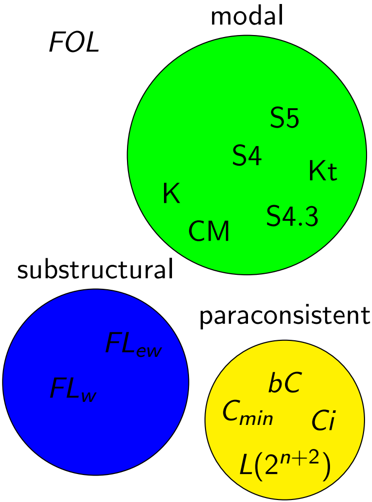
Towards a Theory of Hypersequent Calculi for Modal Logics

Björn Lellmann

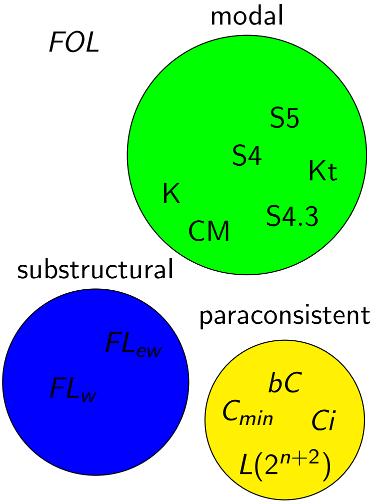
TU Wien

Vienna, 5 November 2014

Motivation 1: The Zoo of Logics



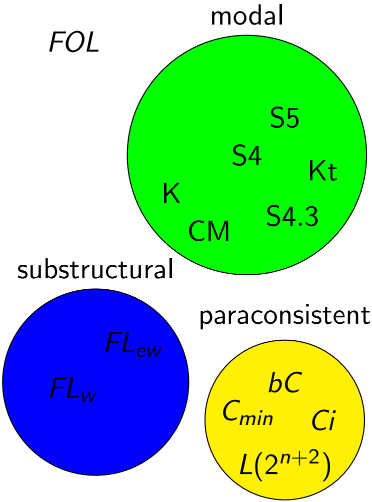
Motivation 1: The Zoo of Logics



Fact 1

There are many logics.

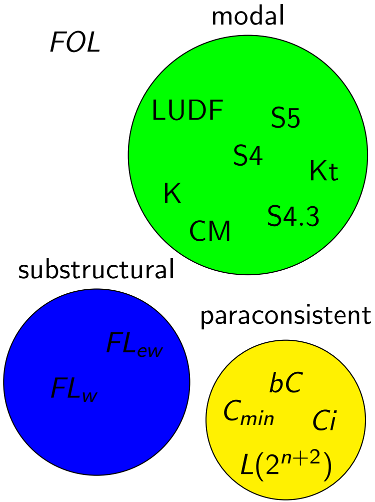
Motivation 1: The Zoo of Logics



Fact 1
There are many logics.

Fact 2
Their number is growing (almost) everyday.

Motivation 1: The Zoo of Logics



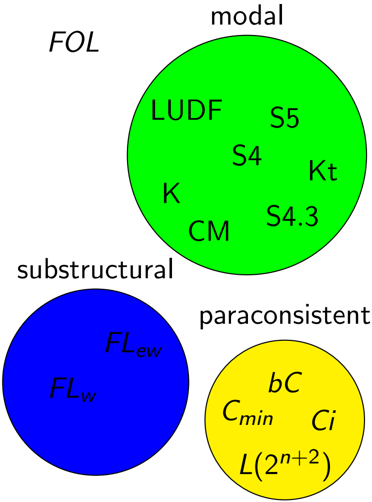
Fact 1

There are many logics.

Fact 2

Their number is growing (almost) everyday.

Motivation 1: The Zoo of Logics



Fact 1
There are many logics.

Fact 2
Their number is growing (almost) everyday.

Problem
How can we reason in these logics?

Motivation 1: The Zoo of Logics

Problem

Given the specification of a logic, construct an analytic calculus to be used in a decision procedure for it!

In the spirit of a "smart reuse of resources" we would like to have general methods to approach this problem.

Motivation 1: The Zoo of Logics

Problem

Given the specification of a logic, construct an analytic calculus to be used in a decision procedure for it!

- Assume the logic is given as a Hilbert-style axiom system.

In the spirit of a "smart reuse of resources" we would like to have general methods to approach this problem.

Motivation 1: The Zoo of Logics

Problem

Given the specification of a logic, construct an analytic calculus to be used in a decision procedure for it!

- Assume the logic is given as a Hilbert-style axiom system.
- Which framework to choose for the calculus: sequents, hypersequents, nested sequents, display, ...?
- How to construct the calculus?

In the spirit of a "smart reuse of resources" we would like to have general methods to approach this problem.

Motivation 2: The Zoo of Formalisms

Display

Nested sequents

2-sequents

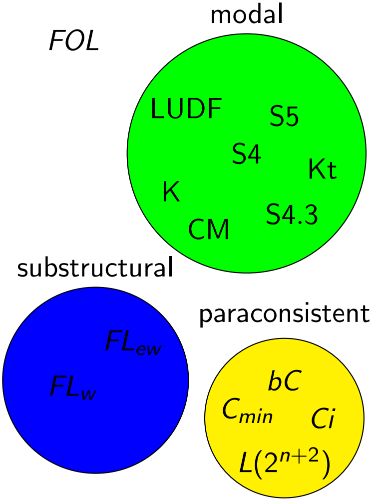
Hypersequents

Sequents

Hilbert systems / formulae

We need a general theory of derivation systems including results about which frameworks are appropriate for which logics!

Some first results



Display

Nested sequents

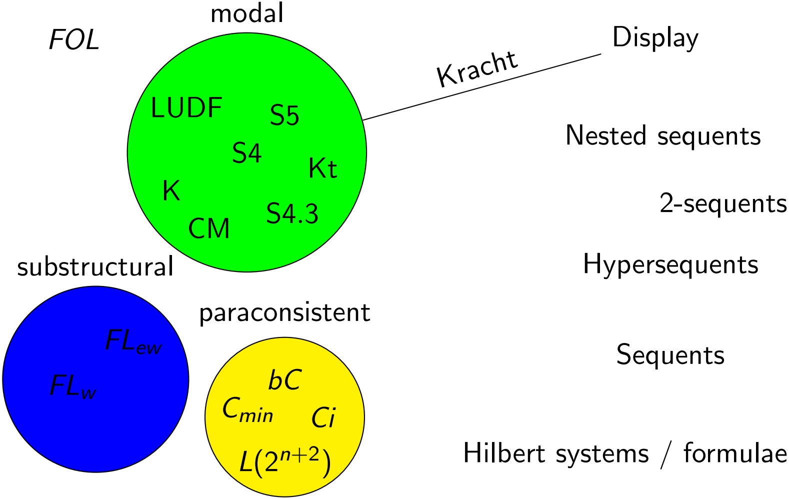
2-sequents

Hypersequents

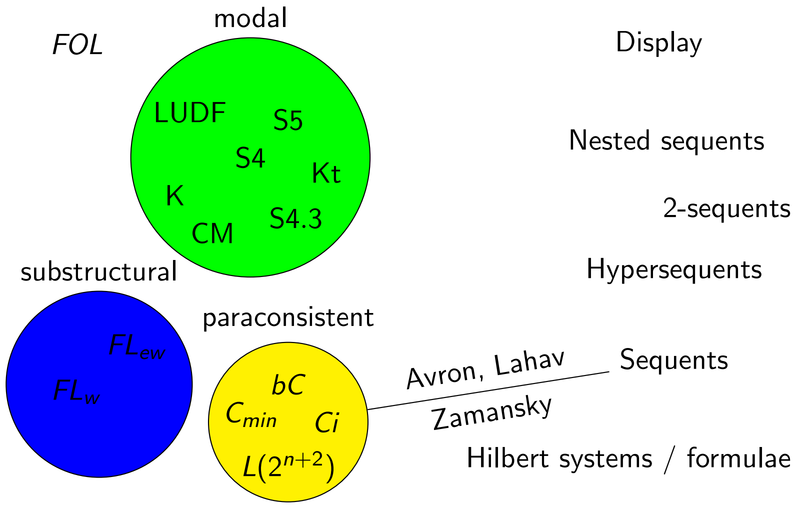
Sequents

Hilbert systems / formulae

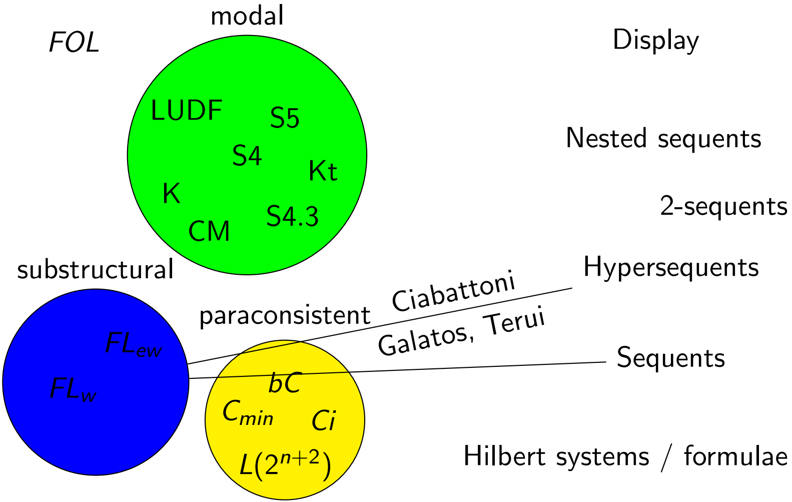
Some first results



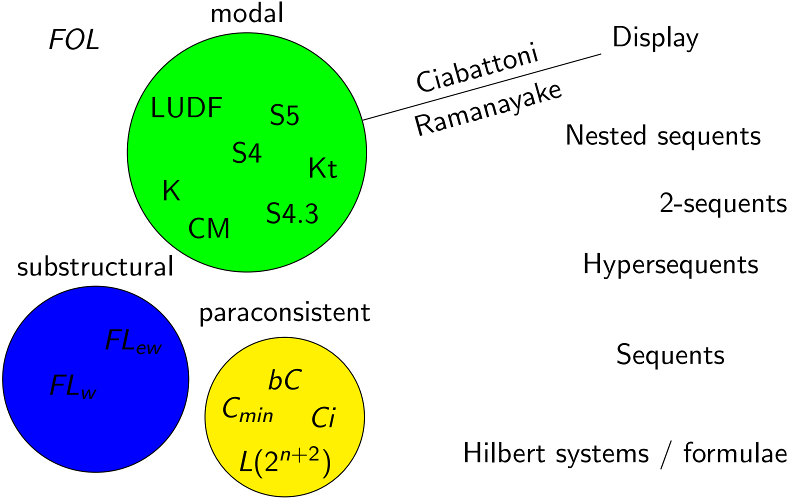
Some first results



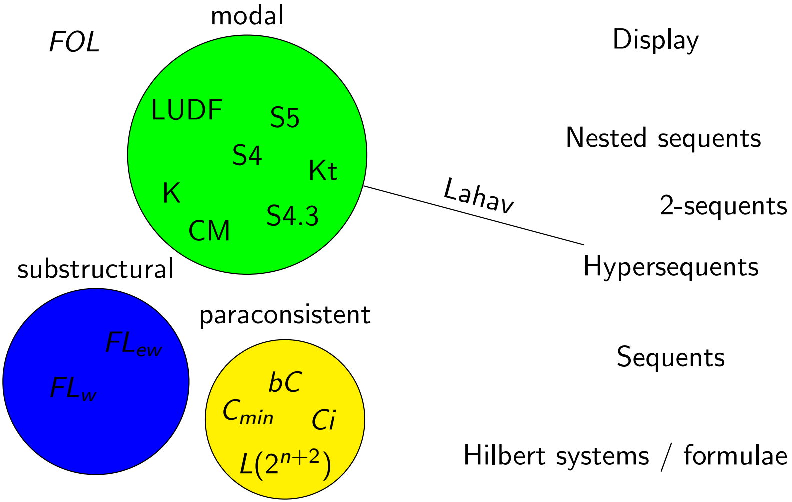
Some first results



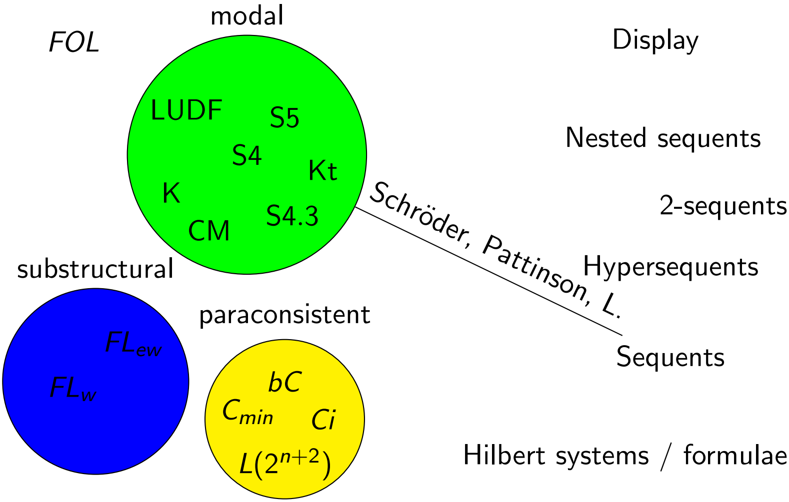
Some first results



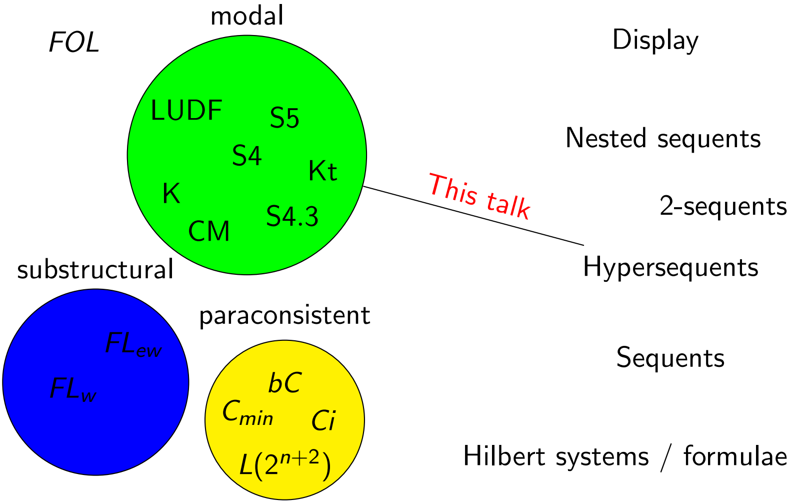
Some first results



Some first results



Some first results



Hypersequent calculi

Hypersequent Basics

The formulae of normal modal logics are given by

$$\mathcal{F} \ni \varphi ::= p_i \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \heartsuit\varphi \mid \dots$$

Sequents are tuples $\Gamma \Rightarrow \Delta$ of multisets of formulae read as $\bigwedge \Gamma \to \bigvee \Delta$, and hypersequents are multisets of sequents written

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

We consider hypersequent calculi with axioms and structural rules:

$$\frac{}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \varphi, \Delta}\ \mathsf{Ax} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{IW} \qquad \frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ \mathsf{EW}$$

$$\frac{\mathcal{G} \mid \varphi, \varphi, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \varphi, \Gamma \Rightarrow \Delta}\ \mathsf{IC_L} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi, \varphi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi}\ \mathsf{IC_R}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ \mathsf{EC} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \qquad \mathcal{G} \mid \varphi, \Sigma \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{Cut}$$

Hypersequent rules with restrictions examples

What could the additional rules look like?
Two (classic) examples from the literature [Avron 1996]:

$$\frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box\varphi \Rightarrow \Delta}\ \mathsf{T} \qquad\qquad \frac{\mathcal{G} \mid \Box\Gamma, \Sigma \Rightarrow \Box\Delta, \Pi}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box\Delta \mid \Sigma \Rightarrow \Pi}\ \mathsf{MS}$$

The characteristic features of these rules are:

- They might introduce one layer of connectives in the active part of the conclusion
- One active component per premiss
- Possibly more than one active component in the conclusion
- They copy a restricted part of the contexts of each component to the premisses

How can we make that precise?

Hypersequent rules with restrictions formally

A context restriction is a tuple $\langle F_\ell; F_r \rangle$ of sets of formulae. It restricts a sequent $\Gamma \Rightarrow \Delta$ by allowing only substitution instances of formulae from F_ℓ (resp. F_r) in Γ (resp. Δ).

Hypersequent rules with context restrictions are of the form

$$\frac{(\Gamma_1 \Rightarrow \Delta_1; \mathcal{C}^1_1 \dots \mathcal{C}^1_n) \quad \dots \quad (\Gamma_m \Rightarrow \Delta_m; \mathcal{C}^m_1 \dots \mathcal{C}^m_n)}{\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n}$$

with $\mathcal{C}^i_j$ context restrictions and $\Gamma_i, \Delta_i \subseteq Var$ and $\Sigma_i, \Pi_i \subseteq \Box(Var)$.
Simple rules use only $\langle \emptyset, \emptyset \rangle, \langle \{p\}, \{p\} \rangle, \langle \{\Box p\}, \emptyset \rangle$.

In an application the premiss with restriction $\mathcal{C}^i_1 \dots \mathcal{C}^i_n$ copies the context of the jth component restricted by $\mathcal{C}^i_j$.

Example:

$$\frac{(\Rightarrow ; \langle \{\Box p\}; \{\Box p\} \rangle \; \langle \{p\}; \{p\} \rangle)}{\Rightarrow \mid \Rightarrow} \quad \rightsquigarrow \quad \frac{\mathcal{G} \mid \Box\Gamma, \Sigma \Rightarrow \Box\Delta, \Pi}{\mathcal{G} \mid \Omega, \Box\Gamma \Rightarrow \Box\Delta, \Theta \mid \Sigma \Rightarrow \Pi}$$

Cut elimination

Theorem

Every

set of rules with

restrictions has cut elimination.

Proof idea: Adapt the proof of [Ciabattoni, Metcalfe, Montagna 2010]: push cuts up to the left, then right.

$$\dfrac{\dfrac{\dfrac{.\Rightarrow .,\varphi \mid \Gamma \Rightarrow \Delta}{.\Rightarrow .,\varphi \mid .\Rightarrow .,\varphi}\,R}{.\Rightarrow .,\varphi}\,\mathrm{EC} \qquad \dfrac{\dfrac{\Sigma \Rightarrow \Pi}{\varphi, .\Rightarrow . \mid \varphi, .\Rightarrow .}\,Q}{\varphi, .\Rightarrow .}\,\mathrm{EC}}{., .\Rightarrow ., .}\,\mathrm{Cut}$$

Cut elimination

Theorem

Every

set of rules with

restrictions has cut elimination.

Proof idea: Adapt the proof of [Ciabattoni, Metcalfe, Montagna 2010]: push cuts up to the left, then right.

$$\dfrac{\dfrac{\dfrac{. \Rightarrow ., \varphi \mid \Gamma \Rightarrow \Delta}{. \Rightarrow ., \varphi \mid . \Rightarrow ., \varphi}\,R \qquad \dfrac{\dfrac{\Sigma \Rightarrow \Pi}{\varphi, . \Rightarrow . \mid \varphi, . \Rightarrow .}\,Q}{\varphi, . \Rightarrow .}\,\mathrm{EC}}{., . \Rightarrow ., . \mid ., . \Rightarrow ., .}\,IH}{., . \Rightarrow ., .}\,\mathrm{EC}$$

Cut elimination

Theorem

Every right-substitutive,

set of rules with

restrictions has cut elimination.

Proof idea: Adapt the proof of [Ciabattoni, Metcalfe, Montagna 2010]: push cuts up to the left, then right.

$$
\dfrac{\dfrac{\dfrac{. \Rightarrow ., \varphi \mid \Gamma \Rightarrow \Delta}{. \Rightarrow ., \varphi \mid . \Rightarrow ., \varphi}\; R \qquad \dfrac{\dfrac{\Sigma \Rightarrow \Pi}{\varphi, . \Rightarrow . \mid \varphi, . \Rightarrow .}\; Q}{\varphi, . \Rightarrow .}\; \mathrm{EC}}{., . \Rightarrow ., . \mid ., . \Rightarrow ., .}\; IH}{., . \Rightarrow ., .}\; \mathrm{EC}
$$

Cut elimination

Theorem

Every right-substitutive,

set of rules with

restrictions has cut elimination.

Proof idea: Adapt the proof of [Ciabattoni, Metcalfe, Montagna 2010]: push cuts up to the left, then right.

$$\dfrac{\dfrac{\dfrac{\dfrac{. \Rightarrow ., \varphi \mid \Gamma \Rightarrow \Delta \qquad \varphi, . \Rightarrow .}{., . \Rightarrow ., . \mid \Gamma \Rightarrow \Delta}\ \text{Cut}}{., . \Rightarrow ., . \mid . \Rightarrow ., \varphi}\ R \qquad \dfrac{\dfrac{\Sigma \Rightarrow \Pi}{\varphi, . \Rightarrow . \mid \varphi, . \Rightarrow .}\ Q}{\varphi, . \Rightarrow .}\ \text{EC}}{., . \Rightarrow ., . \mid ., . \Rightarrow ., .}\ \text{Cut}}{., . \Rightarrow ., .}\ \text{EC}$$

Cut elimination

Theorem

Every right-substitutive, single-conclusion right, right-contraction closed, set of rules with restrictions has cut elimination.

Proof idea: Adapt the proof of [Ciabattoni, Metcalfe, Montagna 2010]: push cuts up to the left, then right.

$$
\dfrac{
\dfrac{
\dfrac{. \Rightarrow .,\varphi \mid \Gamma \Rightarrow \Delta \qquad \varphi, . \Rightarrow .}{.,. \Rightarrow .,. \mid \Gamma \Rightarrow \Delta}\ \text{Cut}
}{.,. \Rightarrow .,. \mid . \Rightarrow .,\varphi}\ R
\qquad
\dfrac{\dfrac{\Sigma \Rightarrow \Pi}{\varphi, . \Rightarrow . \mid \varphi, . \Rightarrow .}\ Q}{\varphi, . \Rightarrow .}\ \text{EC}
}{
\dfrac{.,. \Rightarrow .,. \mid .,. \Rightarrow .,.}{.,. \Rightarrow .,.}\ \text{EC}
}\ \text{Cut}
$$

Cut elimination

Theorem

Every right-substitutive, single-conclusion right, right-contraction closed, set of rules with restrictions has cut elimination.

Proof idea: Adapt the proof of [Ciabattoni, Metcalfe, Montagna 2010]: push cuts up to the left, then right.

$$\dfrac{\dfrac{\dfrac{.,. \Rightarrow .,. \mid \Gamma \Rightarrow \Delta}{.,. \Rightarrow .,. \mid . \Rightarrow .,\varphi}\,R \qquad \dfrac{\Sigma \Rightarrow \Pi}{\varphi,. \Rightarrow . \mid \varphi,. \Rightarrow .}\,Q}{.,. \Rightarrow .,. \mid .,. \Rightarrow .,. \mid .,. \Rightarrow .,. \mid .,. \Rightarrow .,.}\,\text{IH}}{.,. \Rightarrow .,.}\,\text{EC}$$

Cut elimination

Theorem

Every right-substitutive, single-conclusion right, right-contraction closed, mixed-cut permuting, set of rules with restrictions has cut elimination.

Proof idea: Adapt the proof of [Ciabattoni, Metcalfe, Montagna 2010]: push cuts up to the left, then right.

$$
\dfrac{
\dfrac{\cdot,\cdot \Rightarrow \cdot,\cdot \mid \Gamma \Rightarrow \Delta}{\cdot,\cdot \Rightarrow \cdot,\cdot \mid \cdot \Rightarrow \cdot,\varphi}\, R
\qquad
\dfrac{\Sigma \Rightarrow \Pi}{\varphi,\cdot \Rightarrow \cdot \mid \varphi,\cdot \Rightarrow \cdot}\, Q
}{
\cdot,\cdot \Rightarrow \cdot,\cdot \mid \cdot,\cdot \Rightarrow \cdot,\cdot \mid \cdot,\cdot \Rightarrow \cdot,\cdot \mid \cdot,\cdot \Rightarrow \cdot,\cdot
}\, \mathrm{IH}
$$

$$
\dfrac{\cdot,\cdot \Rightarrow \cdot,\cdot \mid \cdot,\cdot \Rightarrow \cdot,\cdot \mid \cdot,\cdot \Rightarrow \cdot,\cdot \mid \cdot,\cdot \Rightarrow \cdot,\cdot}{\cdot,\cdot \Rightarrow \cdot,\cdot}\, \mathrm{EC}
$$

Cut elimination

Theorem

Every right-substitutive, single-conclusion right, right-contraction closed, mixed-cut permuting, set of rules with restrictions has cut elimination.

Proof idea: Adapt the proof of [Ciabattoni, Metcalfe, Montagna 2010]: push cuts up to the left, then right.

$$\dfrac{\dfrac{\dfrac{.,. \Rightarrow .,. \mid \Gamma \Rightarrow \Delta}{.,. \Rightarrow .,. \mid . \Rightarrow .,\varphi}\,R \qquad \dfrac{\dfrac{\dfrac{.,. \Rightarrow .,. \mid \Gamma \Rightarrow \Delta}{.,. \Rightarrow .,. \mid . \Rightarrow .,\varphi}\,R \qquad \dfrac{\Sigma \Rightarrow \Pi}{\varphi,. \Rightarrow . \mid \varphi,. \Rightarrow .}\,Q}{\varphi,. \Rightarrow . \mid .,. \Rightarrow .,. \mid .,. \Rightarrow .,.}\,\text{Cut}}{.,. \Rightarrow .,. \mid .,. \Rightarrow .,. \mid .,. \Rightarrow .,. \mid .,. \Rightarrow .,.}\,\text{Cut}}{.,. \Rightarrow .,.}\,\text{EC}$$

Cut elimination

Theorem

Every right-substitutive, single-conclusion right, right-contraction closed, mixed-cut permuting, principal cut closed set of rules with restrictions has cut elimination.

Proof idea: Adapt the proof of [Ciabattoni, Metcalfe, Montagna 2010]: push cuts up to the left, then right.

$$\dfrac{\dfrac{\cdot,\cdot \Rightarrow \cdot,\cdot \mid \Gamma \Rightarrow \Delta}{\cdot,\cdot \Rightarrow \cdot,\cdot \mid \cdot \Rightarrow \cdot,\varphi}\,R \qquad \dfrac{\dfrac{\cdot,\cdot \Rightarrow \cdot,\cdot \mid \Gamma \Rightarrow \Delta}{\cdot,\cdot \Rightarrow \cdot,\cdot \mid \cdot \Rightarrow \cdot,\varphi}\,R \quad \dfrac{\Sigma \Rightarrow \Pi}{\varphi,\cdot \Rightarrow \cdot \mid \varphi,\cdot \Rightarrow \cdot}\,Q}{\varphi,\cdot \Rightarrow \cdot \mid \cdot,\cdot \Rightarrow \cdot,\cdot \mid \cdot,\cdot \Rightarrow \cdot,\cdot}\,\mathrm{Cut}}{\dfrac{\cdot,\cdot \Rightarrow \cdot,\cdot \mid \cdot,\cdot \Rightarrow \cdot,\cdot \mid \cdot,\cdot \Rightarrow \cdot,\cdot \mid \cdot,\cdot \Rightarrow \cdot,\cdot}{\cdot,\cdot \Rightarrow \cdot,\cdot}\,\mathrm{EC}}\,\mathrm{Cut}$$

Cut elimination

Theorem

Every right-substitutive, single-conclusion right, right-contraction closed, mixed-cut permuting, principal cut closed set of rules with restrictions has cut elimination.

Proof idea: Adapt the proof of [Ciabattoni, Metcalfe, Montagna 2010]: push cuts up to the left, then right.

$$
\cfrac{\cfrac{\cfrac{.,. \Rightarrow .,. \mid \Gamma \Rightarrow \Delta}{.,. \Rightarrow .,. \mid . \Rightarrow .,\varphi}\ R \qquad \cfrac{\cfrac{.,. \Rightarrow .,. \mid \Gamma \Rightarrow \Delta \quad \Sigma \Rightarrow \Pi}{.,. \Rightarrow .,. \mid \Gamma', \Sigma' \Rightarrow \Delta', \Pi'}\ \text{Cut}}{\varphi, . \Rightarrow . \mid .,. \Rightarrow .,. \mid .,. \Rightarrow .,.}\ \text{cut}(R, Q)}{.,. \Rightarrow .,. \mid .,. \Rightarrow .,. \mid .,. \Rightarrow .,. \mid .,. \Rightarrow .,.}\ \text{Cut}}{.,. \Rightarrow .,.}\ \text{EC}
$$

Decidability and Complexity

Theorem

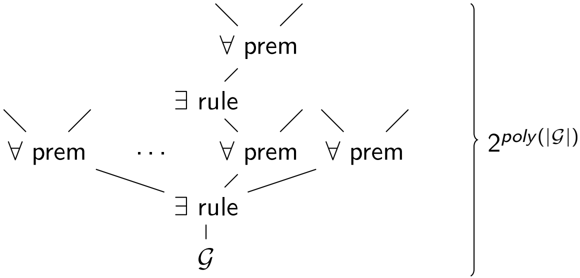
Derivability in a cut-free, contraction-closed, bounded conclusion and tractable rule set is decidable in EXPSPACE.

Proof idea: Modify the rules to make contraction admissible, e.g.:

$$\frac{\mathcal{G} \mid \Box\Gamma', \Sigma \Rightarrow p}{\mathcal{G} \mid \Gamma, \Box\Gamma', \Box\Sigma \Rightarrow \Box p, \Delta}\ 4$$

$$\rightsquigarrow \quad \frac{\mathcal{G} \mid \Gamma, \Box\Gamma', \Box\Sigma \Rightarrow \Box p, \Delta \mid \Box\Gamma', \Box\Sigma, \Sigma \Rightarrow p}{\mathcal{G} \mid \Gamma, \Box\Gamma', \Box\Sigma \Rightarrow \Box p, \Delta}$$

and perform backwards proof search:



Axioms and Rules

Axioms and Interpretations

We assume that the specification of a logic is given as a Hilbert system, i.e. by a set $\mathcal{A}$ of axioms and the rules

$$\frac{\vdash \varphi}{\vdash \varphi\sigma}\ \text{US} \qquad \frac{\vdash \varphi \quad \vdash \varphi \to \psi}{\vdash \psi}\ \text{MP} \qquad \frac{\vdash \varphi \to \psi}{\vdash \heartsuit\varphi \to \heartsuit\psi}\ \text{Mon}$$

We want to interpret a hypersequent as a formula – but the interpretation for $|$ is not clear! So let's make it a parameter:

An interpretation for a logic $\mathcal{L}$ is a set $\{\varphi_n(p_1, \ldots, p_n) : n \in \mathbb{N}\}$ of formulae such that $\models_{\mathcal{L}} \psi$ iff $\models_{\mathcal{L}} \varphi_1(\psi)$ (regularity) and which respects the structural rules:

- $\models_{\mathcal{L}} \varphi_n(\xi_1, \xi_2, \vec{\chi})$ iff $\models_{\mathcal{L}} \varphi_n(\xi_2, \xi_1, \vec{\chi})$
- If $\models_{\mathcal{L}} \varphi_n(\vec{\chi})$ then $\models_{\mathcal{L}} \varphi_{n+1}(\xi, \vec{\chi})$
- similarly for external contraction, cut, etc.

Example: $\iota_{\Box} = \{\bigvee_{i \leq n} \Box p_i : n \in \mathbb{N}\}$ for reflexive normal modal logics or $\iota_{\boxplus} = \{\bigvee_{i \leq n} (p_i \wedge \Box p_i) : n \in \mathbb{N}\}$ for normal modal logics.

Axioms vs Rules

Consider the axiom for S4.3:

$$(.3)\ \ \Box(\Box p \to q) \vee \Box(\Box q \to p)$$

The $\iota_\Box$-simple axioms corresponding to simple hypersequent rules for $\iota_\Box = \{\bigvee_{i \leq n} \Box p_i : n \in \mathbb{N}\}$ are given by the following grammar:

$$S ::= \varphi_n(L \to R, \ldots, L \to R)$$

$$L ::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

$$P_\ell ::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p_i \mid \bot \mid \top$$

with $\heartsuit \in \Lambda \cup \{\epsilon\}$

Axioms vs Rules

Consider the axiom for S4.3:

$$(.3)\ \Box(\Box p \to q) \vee \Box(\Box q \to p) \quad \rightsquigarrow \quad \frac{}{\Box p \Rightarrow q \mid \quad \Box q \Rightarrow p}$$

The $\iota_\Box$-simple axioms corresponding to simple hypersequent rules for $\iota_\Box = \{\bigvee_{i \le n} \Box p_i : n \in \mathbb{N}\}$ are given by the following grammar:

$$S ::= \varphi_n(L \to R, \ldots, L \to R)$$

$$L ::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

$$P_\ell ::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p_i \mid \bot \mid \top$$

with $\heartsuit \in \Lambda \cup \{\epsilon\}$

Axioms vs Rules

Consider the axiom for S4.3:

$$(.3)\ \ \Box(\Box p \to q) \vee \Box(\Box q \to p) \quad \rightsquigarrow \quad \frac{}{\Box p \Rightarrow q \mid \quad \Box q \Rightarrow p}$$

The $\iota_\Box$-simple axioms corresponding to simple hypersequent rules for $\iota_\Box = \{\bigvee_{i \leq n} \Box p_i : n \in \mathbb{N}\}$ are given by the following grammar:

$$S ::= \varphi_n(L \to R, \ldots, L \to R)$$

$$L ::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

$$P_\ell ::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p_i \mid \bot \mid \top$$

with $\heartsuit \in \Lambda \cup \{\epsilon\}$

Axioms vs Rules

Consider the axiom for S4.3:

$$(.3)\ \Box(\Box p \to q) \vee \Box(\Box q \to p) \quad \rightsquigarrow \quad \frac{q \Rightarrow r \qquad p \Rightarrow s}{\Box p \Rightarrow r \mid \quad \Box q \Rightarrow s}$$

The $\iota_\Box$-simple axioms corresponding to simple hypersequent rules for $\iota_\Box = \{\bigvee_{i \leq n} \Box p_i : n \in \mathbb{N}\}$ are given by the following grammar:

$$S ::= \varphi_n(L \to R, \ldots, L \to R)$$

$$L ::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

$$P_\ell ::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p_i \mid \bot \mid \top$$

with $\heartsuit \in \Lambda \cup \{\epsilon\}$

Axioms vs Rules

Consider the axiom for S4.3:

$$(.3)\ \Box(\Box p \to q) \vee \Box(\Box q \to p) \quad \rightsquigarrow \quad \frac{\Gamma, q \Rightarrow \Delta \qquad \Sigma, p \Rightarrow \Pi}{\Gamma, \Box p \Rightarrow \Delta \mid \Sigma, \Box q \Rightarrow \Pi}$$

The $\iota_\Box$-simple axioms corresponding to simple hypersequent rules for $\iota_\Box = \{\bigvee_{i \leq n} \Box p_i : n \in \mathbb{N}\}$ are given by the following grammar:

$$S ::= \varphi_n(L \to R, \ldots, L \to R)$$

$$L ::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

$$P_\ell ::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p_i \mid \bot \mid \top$$

with $\heartsuit \in \Lambda \cup \{\epsilon\}$

Axioms vs Rules

Consider the axiom for S4.3:

$$(.3)\ \ \Box(\Box p \to q) \vee \Box(\Box q \to p) \quad \rightsquigarrow \quad \frac{\Gamma, \Omega \Rightarrow \Delta \qquad \Sigma, \Theta \Rightarrow \Pi}{\Gamma, \Box\Theta \Rightarrow \Delta \mid \Sigma, \Box\Omega \Rightarrow \Pi}$$

The $\iota_\Box$-simple axioms corresponding to simple hypersequent rules for $\iota_\Box = \{\bigvee_{i \leq n} \Box p_i : n \in \mathbb{N}\}$ are given by the following grammar:

$$S ::= \varphi_n(L \to R, \ldots, L \to R)$$

$$L ::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

$$P_\ell ::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p_i \mid \bot \mid \top$$

with $\heartsuit \in \Lambda \cup \{\epsilon\}$

Axioms vs Rules

Consider the axiom for S4.3:

$$(.3)\ \Box(\Box p \to q) \vee \Box(\Box q \to p) \quad \rightsquigarrow \quad \frac{\Gamma, \Omega \Rightarrow \Delta \qquad \Sigma, \Theta \Rightarrow \Pi}{\Gamma, \Box\Theta \Rightarrow \Delta \mid \Sigma, \Box\Omega \Rightarrow \Pi}$$

The $\iota_\Box$-simple axioms corresponding to simple hypersequent rules for $\iota_\Box = \{\bigvee_{i \leq n} \Box p_i : n \in \mathbb{N}\}$ are given by the following grammar:

$$S ::= \varphi_n(L \to R, \dots, L \to R)$$

$$L ::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot$$

$$P_r ::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p_i \mid \bot \mid \top$$

$$P_\ell ::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p_i \mid \bot \mid \top$$

with $\heartsuit \in \Lambda \cup \{\epsilon\}$ and $\psi_\ell \in \{q_i, \Box q_i : i \in \mathbb{N}\}$, $\psi_r \in \{r_i : i \in \mathbb{N}\}$ such that every ψ_ℓ, ψ_r occurs under φ_n once on the top level and at least once under a modality.

Caveat

The translations between axioms and rules use the rules for K:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow q}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box q}\ R_{\mathsf{K}}$$

But soundness of these is not necessarily preserved in extensions!

Theorem

If $\mathsf{K} \subseteq \mathcal{L}$ *and* ι *is a regular interpretation for* $\mathcal{L}$*, then:*

$$R_{\mathsf{K}} \text{ is sound for } (\mathcal{L}, \iota) \quad \text{iff} \quad \frac{\iota(\mathcal{G}, \varphi)}{\iota(\mathcal{G}, \Box\varphi)} \text{ is admissible in } \mathcal{L}$$

and both hold for $\iota = \iota_{\Box}$ *if* $\mathcal{L}$ *is transitive or extensible.*

(Here $\mathcal{L}$ is extensible if given by a class of frames closed under "adding a predecessor to everyone").

This seems to suggest that hypersequent calculi are mainly suited for transitive or extensible logics!

Applications

Applications: Simple Frame Properties

All the calculi for logics given by simple frame properties based on K or K4 in [Lahav:2013] fit our framework and satisfy the criteria for cut elimination and decidability. E.g. linearity:

$$\forall w_1, w_2 \exists u (w_1 R u \wedge w_2 = u) \vee (w_2 R u \wedge w_1 = u)$$

$$\rightsquigarrow \quad \frac{\mathcal{G} \mid \Sigma, \Gamma' \Rightarrow \Pi \qquad \mathcal{G} \mid \Gamma, \Sigma' \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box\Gamma' \Rightarrow \Delta \mid \Sigma, \Box\Sigma' \Rightarrow \Pi}$$

Thus we purely syntactically reprove cut elimination and have

Theorem

Logics given by simple frame properties are decidable in EXPSPACE.

The correspondence between simple rules and $\iota_\Box$-simple axioms also gives (under some conditions) Hilbert systems for these logics.

Applications: S5

Modal logic S5 is given by the axioms for KT and the simple axiom

$$(5)\ \Diamond p \to \Box\Diamond p \quad \equiv \quad \Box p \lor \Box\neg\Box p \quad \rightsquigarrow \quad \frac{\mathcal{G} \mid p, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Box p \Rightarrow \mid \Gamma \Rightarrow \Delta}$$

Adding contraction-absorbing contexts and dropping derivable rules yields the rules from [Restall:2007]:

$$\frac{\mathcal{G} \mid \Sigma, \Box p \Rightarrow \Pi \mid p, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Sigma, \Box p \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta}\ 5 \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Box p, \Delta \mid \Rightarrow p}{\mathcal{G} \mid \Gamma \Rightarrow \Box p, \Delta}\ \text{nec}$$

Applications: S5

Modal logic S5 is given by the axioms for KT and the simple axiom

$$(5)\ \Diamond p \to \Box\Diamond p \quad \equiv \quad \Box p \vee \Box\neg\Box p \quad \rightsquigarrow \quad \frac{\mathcal{G} \mid p, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Box p \Rightarrow \mid \Gamma \Rightarrow \Delta}$$

Adding contraction-absorbing contexts and dropping derivable rules yields the rules from [Restall:2007]:

$$\frac{\mathcal{G} \mid \Sigma, \Box p \Rightarrow \Pi \mid p, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Sigma, \Box p \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta}\ 5 \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Box p, \Delta \mid \Rightarrow p}{\mathcal{G} \mid \Gamma \Rightarrow \Box p, \Delta}\ \text{nec}$$

Theorem

Backwards proof search in this calculus runs in coNP*.*

Algorithm:

Work on set-based hypersequents; apply nec to obtain a "genuinely new" component; apply Prop and 5 all possible ways (universally guessing the premiss); repeat until you hit axiom or no new components are found.

Applications: K4.2

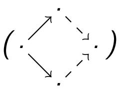
Modal logic K4.2 is K plus (4) $\Box p \to \Box\Box p$ plus (.2) $\Diamond\Box p \to \Box\Diamond p$.

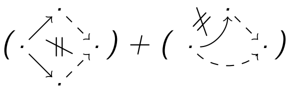
Bad News: $\iota_{\Box}$ is not regular for K4.2...

Good News: ... but $\iota_{\boxplus} = \{\bigvee_{i \leq n}(p_i \wedge \Box p_i) : n \geq 1\}$ is.

Lemma

The following are frame equivalent over transitive frames:
$\Diamond\Box p \to \Box\Diamond p$ *and* $\boxplus(\neg p \vee \neg\Box q) \vee \boxplus(p \vee \neg\Box\neg q)$.





Translating the latter axiom gives the rules:

$$\frac{\mathcal{G} \mid \Omega, \Box\Gamma, \Theta, \Box\Delta \Rightarrow \Xi, \Upsilon \qquad \mathcal{G} \mid \Box\Gamma, \Sigma, \Box\Delta, \Pi \Rightarrow}{\mathcal{G} \mid \Omega, \Box\Gamma, \Box\Sigma \Rightarrow \Xi \mid \Theta, \Box\Delta, \Box\Pi \Rightarrow \Upsilon}\ 2$$

Theorem

The calculus K4 + 2 *is cut-free complete and sound for* (K4.2, $\iota_{\boxplus}$).

Summing Up

We have

- identified a general format of rules in a hypersequent calculus for modal logics
- general syntactic criteria for uniform cut elimination and decidability / complexity results
- identified a class of Hilbert axioms corresponding to such rules
- applied these results in the construction of analytic calculi for several logics.

Summing Up

We have

- identified a general format of rules in a hypersequent calculus for modal logics
- general syntactic criteria for uniform cut elimination and decidability / complexity results
- identified a class of Hilbert axioms corresponding to such rules
- applied these results in the construction of analytic calculi for several logics.

Thank you for your attention!