Relating Axioms and Rules for Modal Logics

Björn Lellmann
(Joint work with Dirk Pattinson)

Imperial College London

Vienna, 24 April 2013

Motivation

Fact:

There are a many modal logics:

$$\mathsf{K}, \mathsf{KT}, \mathsf{K4}, \mathsf{S4}, \dots, \mathbb{V}, \mathbb{VA}, \dots, \mathsf{CK}, \mathsf{CS4}, \dots, \mathsf{CL}, \mathsf{IL}, \dots$$

... and their number is growing every day!

For syntactically specifying these logics we have 2 big formalisms:

- axioms for a Hilbert system (good for capturing intuitions)
- rules for a sequent system (good for deciding the logic)

$$\mathsf{T}_\Box \quad \Box p \to p \qquad \frac{\Gamma, p \Rightarrow \Delta}{\Gamma, \Box p \Rightarrow \Delta}\ R_{\mathsf{T}\Box}$$

$$4_\Box \quad \Box p \to \Box\Box p \qquad \frac{\Box\Gamma \Rightarrow p}{\Sigma, \Box\Gamma \Rightarrow \Box p, \Delta}\ R_{4\Box}$$

Question:
- How do we get rules from axioms?
- Which axioms can be captured by rules?

Examples

Question:

What is the relationship between axioms and rules?

Plan:

- What is a sequent rule?
- General properties of such systems: Cut elimination and decidability
- What is the relation between axioms and sequent rules?
- What are the applications?

What is a sequent system for a modal logic?

Basics

In this talk we consider intuitionistic propositional modal logics with unary monotone modalities.
(But everything works in the classical / n-ary case as well!)

Formulae are defined as usual:

$$A, B \ni \mathcal{F} ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \to B \mid \Box A \mid \heartsuit A \mid \dots$$

We use asymmetric sequents $\Gamma \Rightarrow \delta$ with Γ a multiset of formulae and δ empty or a formula. Intended interpretation: $\bigwedge \Gamma \to \delta$.

Our sequent systems have axioms $\overline{\Gamma, A \Rightarrow A}$, the structural rules

$$\frac{\Gamma \Rightarrow \delta}{A, \Gamma \Rightarrow \delta}\ \mathsf{W}, \qquad \frac{\Gamma, A, A \Rightarrow \delta}{\Gamma, A \Rightarrow \delta}\ \mathsf{Con}_L, \qquad \frac{\Gamma \Rightarrow A \quad A, \Sigma \Rightarrow \pi}{\Gamma, \Sigma \Rightarrow \pi}\ \mathsf{Cut},$$

the propositional rules, the monotonicity rules for all operators $\heartsuit$:

$$\frac{A \Rightarrow B}{\Gamma, \heartsuit A \Rightarrow \heartsuit B}$$

and additional rules of a specific format.

Rules with Context Restrictions

A context restriction is a tuple $\langle F_\ell; F_r \rangle$ of sets of formulae. It restricts a sequent $\Gamma \Rightarrow \delta$ by allowing only substitution instances of formulae from F_ℓ (resp. F_r) in Γ (resp. δ).

A rule with context restrictions is of the form

$$\frac{(\Gamma_1 \Rightarrow \delta_1; \mathcal{C}_1) \quad \dots \quad (\Gamma_n \Rightarrow \delta_n; \mathcal{C}_n)}{\Sigma \Rightarrow \pi}$$

with principal formulae $\Sigma \Rightarrow \pi \in \mathit{Seq}(\mathit{Mod}(\mathit{Var}))$ and premisses $\Gamma_i \Rightarrow \delta_i \in \mathit{Seq}(\mathit{Var})$ with associated context restrictions $\mathcal{C}_i$.

In an application of such a rule a premiss with associated restriction $\mathcal{C}_i$ carries over only the context restricted according to $\mathcal{C}_i$ from the conclusion.

Examples of Rules with Context Restrictions

Our rule format captures many standard rules for modal logics, e.g. the rules for $\mathsf{CK}_\Box$ and $\mathsf{CS4}_\Box$:

$$\frac{A_1, \ldots, A_n \Rightarrow B}{\Gamma, \Box A_1, \ldots, \Box A_n \Rightarrow \Box B} \; R_{\mathsf{K}_n} \qquad \frac{(p_1, \ldots, p_n \Rightarrow q \; ; \langle \emptyset ; \emptyset \rangle)}{\Box p_1, \ldots, \Box p_n \Rightarrow \Box q}$$

$$\frac{\Gamma, A \Rightarrow \delta}{\Gamma, \Box A \Rightarrow \delta} \; R_{\mathsf{T}\Box} \qquad \frac{(p \Rightarrow \; ; \langle \{q\} ; \{q\} \rangle)}{\Box p \Rightarrow}$$

$$\frac{\Box \Gamma \Rightarrow A}{\Sigma, \Box \Gamma \Rightarrow \Box A} \; R_{4\Box} \qquad \frac{(\Rightarrow p \; ; \langle \{\Box q\} ; \emptyset \rangle)}{\Rightarrow \Box p}$$

We often use the more suggestive notation on the left.

Rules K_n and $\mathsf{T}_\Box$ are shallow: they use only restrictions $\langle \emptyset ; \emptyset \rangle$ or $\langle \{q\} ; \{q\} \rangle$. Rule K_n is one-step: it only uses restriction $\langle \emptyset ; \emptyset \rangle$.

General properties of such sequent systems

Cut Elimination

The structural rules Con_L and Cut are bad for backwards proof search, since they give rise to infinite search trees. Also, Cut sabotages the subformula property.

$$\frac{\Gamma, A, A \Rightarrow \delta}{\Gamma, A \Rightarrow \delta}\ \text{Con}_L, \qquad \frac{\Gamma \Rightarrow A \quad A, \Sigma \Rightarrow \pi}{\Gamma, \Sigma \Rightarrow \pi}\ \text{Cut}$$

Thus in a good sequent system these rules should be admissible: the system should derive the same sequents if we drop these rules.

Idea:
Extract general conditions on the rule sets from the standard proofs which guarantee admissibility of Cut and Con_L.

The General Conditions: Saturation

A set $\mathcal{R}$ of rules is

- principal-cut closed if "cuts between rules" from $\mathcal{R}$ are $\mathcal{R}$ConW-derivable;
- context-cut closed if whenever context restrictions of R and Q overlap on A (i.e. if $A \in F_r^R\sigma \cap F_\ell^Q\tau$) , then the principal formulae and all restrictions of one rule satisfy all restrictions of the other rule overlapping on A;
- mixed-cut closed if whenever a principal formula A of R satisfies a context restriction of Q then all restrictions and principal formulae of R satisfy this restriction;
- contraction closed if "contractions of rules" from $\mathcal{R}$ are in $\mathcal{R}$;
- saturated if it is all of the above.

Examples:

The standard rule sets for the standard modal logics built from $K, T, D, 4$ and the rules for propositional logic are all saturated.

Generic Cut Elimination by Permutation of Rules

Theorem (Generic Cut Elimination)

In saturated rule sets the cut rule can be eliminated.

Proof Sketch.

As usual eliminate multicuts $\dfrac{\Gamma \Rightarrow A \quad A^n, \Sigma \Rightarrow \delta}{\Gamma, \Sigma \Rightarrow \delta}$ by double induction on the rank and depth of the cut. E.g.

$$\dfrac{\dfrac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} R_{4\Box} \qquad \dfrac{\Sigma, \Box A, \Box A, A \Rightarrow \Pi}{\Sigma, \Box A, \Box A, \Box A \Rightarrow \Pi} R_{T\Box}}{\Box\Gamma, \Sigma \Rightarrow \Pi} \text{mCut}$$

mcc

$$\rightsquigarrow \dfrac{\dfrac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} R_{4\Box} \qquad \dfrac{\dfrac{\dfrac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} R_{4\Box} \qquad \Sigma, \Box A, \Box A, A \Rightarrow \Pi}{\Box\Gamma, \Sigma, A \Rightarrow \Pi} \text{mCut}}{\Box\Gamma, \Sigma, \Box A \Rightarrow \Pi} R_{T\Box}}{\Box\Gamma, \Box\Gamma, \Sigma \Rightarrow \Pi} \text{mCut}$$

Generic Cut Elimination by Permutation of Rules

Theorem (Generic Cut Elimination)

In saturated rule sets the cut rule can be eliminated.

Proof Sketch.

As usual eliminate multicuts $\dfrac{\Gamma \Rightarrow A \quad A^n, \Sigma \Rightarrow \delta}{\Gamma, \Sigma \Rightarrow \delta}$ by double induction on the rank and depth of the cut. E.g.

$$\dfrac{\dfrac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\, R_{4\Box} \qquad \dfrac{\Sigma, \Box A, \Box A, A \Rightarrow \Pi}{\Sigma, \Box A, \Box A, \Box A \Rightarrow \Pi}\, R_{T\Box}}{\Box\Gamma, \Sigma \Rightarrow \Pi}\, \text{mCut}$$

mcc

$$\rightsquigarrow \dfrac{\dfrac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\, R_{4\Box} \qquad \dfrac{\dfrac{\dfrac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\, R_{4\Box} \qquad \Sigma, \Box A, \Box A, A \Rightarrow \Pi}{\Box\Gamma, \Sigma, A \Rightarrow \Pi}\, \text{mCut}}{\Box\Gamma, \Sigma, \Box A \Rightarrow \Pi}\, R_{T\Box}}{\Box\Gamma, \Box\Gamma, \Sigma \Rightarrow \Pi}\, \text{mCut}$$

lower depth

Generic Cut Elimination by Permutation of Rules

Theorem (Generic Cut Elimination)

In saturated rule sets the cut rule can be eliminated.

Proof Sketch.

As usual eliminate multicuts $\dfrac{\Gamma \Rightarrow A \quad A^n, \Sigma \Rightarrow \delta}{\Gamma, \Sigma \Rightarrow \delta}$ by double induction on the rank and depth of the cut. E.g.

$$\dfrac{\dfrac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\, R_{4\Box} \qquad \dfrac{\Sigma, \Box A, \Box A, A \Rightarrow \Pi}{\Sigma, \Box A, \Box A, \Box A \Rightarrow \Pi}\, R_{T\Box}}{\Box\Gamma, \Sigma \Rightarrow \Pi}\, \text{mCut}$$

mcc

$$\rightsquigarrow \quad \dfrac{\dfrac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\, R_{4\Box} \qquad \dfrac{\dfrac{\dfrac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\, R_{4\Box} \qquad \Sigma, \Box A, \Box A, A \Rightarrow \Pi}{\Box\Gamma, \Sigma, A \Rightarrow \Pi}\, \text{mCut}}{\Box\Gamma, \Sigma, \Box A \Rightarrow \Pi}\, R_{T\Box}}{\Box\Gamma, \Box\Gamma, \Sigma \Rightarrow \Pi}\, \text{mCut}$$

lower depth

prc + lower rank

Decidability

Copying all formulae obeying the associated context restriction into the premisses, yields admissibility of Contraction. For tractable rule sets (codes of applicable rules / their premisses can be computed in pspace from the conclusion / the rule code) we also have

Theorem (Complexity)

For saturated and tractable sets of rules with restrictions the derivability problem is in EXPTIME. *If all rules are shallow, then the problem is in* PSPACE.

Idea of Proof.

1. Eliminate Cuts
2. Eliminate Contraction
3. Use subformula property in a fixed-point argument for EXPTIME
4. Use set-sequents and backwards proof search for PSPACE.

What is the relation between axioms and rules?

Translating Axioms

Often modal logics are given as a Hilbert-system, i.e. a set $\mathcal{A}$ of axioms closed under modus ponens and uniform substitution:

$$\frac{\vdash A \quad \vdash A \to B}{\vdash B}\; MP \qquad \frac{\vdash A}{\vdash A\left[\frac{B}{p}\right]}\; US$$

Examples:

- $\mathcal{A}_{\mathsf{CK}_\Box} = \mathsf{IL} \cup \{\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q),\ \Box\top\}$
- $\mathcal{A}_{\mathsf{CS4}_\Box} = \mathcal{A}_{\mathsf{CK}} \cup \{\Box p \to p,\ \Box p \to \Box\Box p\}$

Question: How can we turn these axioms into sequent rules?

Idea:

- find a translation
- extract criteria ensuring translatability of axioms
- show that the criteria are necessary by translating rules

Translating Axioms: An Example

Consider axiom 4 $\Box p \to \Box\Box p$.

Key observation: $\Box p$ occurs on top level and under exactly one $\Box$.

We first resolve propositional logic.

$$\frac{}{\Box p \Rightarrow \Box\Box p}$$

Translating Axioms: An Example

Consider axiom 4 $\Box p \to \Box\Box p$.

Key observation: $\Box p$ occurs on top level and under exactly one $\Box$.

We first resolve propositional logic. Take the occurrence of $\Box p$ under $\Box$,

$$\frac{}{\Box p \Rightarrow \Box\Box p}$$

Translating Axioms: An Example

Consider axiom 4 $\Box p \to \Box\Box p$.

Key observation: $\Box p$ occurs on top level and under exactly one $\Box$.

We first resolve propositional logic. Take the occurrence of $\Box p$ under $\Box$, substitute a fresh variable q for this

$$\frac{}{\Box p \Rightarrow \Box\Box p} \quad \leadsto \quad \frac{\Box p \Rightarrow q \quad q \Rightarrow \Box p}{\Box p \Rightarrow \Box q}$$

Translating Axioms: An Example

Consider axiom 4 $\Box p \to \Box\Box p$.

Key observation: $\Box p$ occurs on top level and under exactly one $\Box$.

We first resolve propositional logic. Take the occurrence of $\Box p$ under $\Box$, substitute a fresh variable q for this and use monotonicity to delete one premiss.

$$\frac{}{\Box p \Rightarrow \Box\Box p} \quad \rightsquigarrow \quad \frac{\Box p \Rightarrow q \quad q \Rightarrow \Box p}{\Box p \Rightarrow \Box q} \quad \rightsquigarrow \quad \frac{\Box p \Rightarrow q}{\Box p \Rightarrow \Box q}$$

Translating Axioms: An Example

Consider axiom 4 $\Box p \to \Box\Box p$.

Key observation: $\Box p$ occurs on top level and under exactly one $\Box$.

We first resolve propositional logic. Take the occurrence of $\Box p$ under $\Box$, substitute a fresh variable q for this and use monotonicity to delete one premiss.

$$\frac{}{\Box p \Rightarrow \Box\Box p} \quad \rightsquigarrow \quad \frac{\Box p \Rightarrow q \quad q \Rightarrow \Box p}{\Box p \Rightarrow \Box q} \quad \rightsquigarrow \quad \frac{\Box p \Rightarrow q}{\Box p \Rightarrow \Box q}$$

But $\Box p$ is normal for CK: we have $\bigwedge_{i \leq n} \Box p_i \leftrightarrow \Box \bigwedge_{i \leq n} p_i$.
Thus replacing $\Box p$ with context $\Box\Gamma$ gives the well-known rule $R_{4\Box}$:

$$\frac{\Box\Gamma \Rightarrow q}{\Box\Gamma \Rightarrow \Box q}\ R_{4\Box} \qquad\qquad \frac{\Gamma, p \Rightarrow \delta}{\Gamma, \Box p \Rightarrow \delta}\ R_{\mathsf{T}\Box}$$

Similarly, axiom T $\Box p \to p$ translates into rule $R_{\mathsf{T}\Box}$.
This gives a purely syntactic construction of the rules for CS4.

Translating Axioms: Translatable Axioms

A propositional formula A is in $\mathcal{F}_r$ (resp. $\mathcal{F}_\ell$) iff the sequent $\Rightarrow A$ (resp. $A \Rightarrow$) is resolvable into atomic sequents. A variable of A is purely positive or pp (resp. purely negative or pn) iff it occurs only on the RHS (resp. LHS) in the sequent resolution of $\Rightarrow A$.

Let $A = \mathbf{v}^\ell \wedge \mathbf{c}^\ell \wedge \mathbf{p}^\ell \to \mathbf{v}^r \vee \mathbf{c}^r \vee \mathbf{p}^r \in \mathcal{F}_r$ and $P_p(\bar{v}, \bar{c})$ propositional formulae for $p \in \mathbf{p}^\ell \cup \mathbf{p}^r$. Then $(A, (\heartsuit_p P_p)_p)$ is fit for translation if

1. $P_{p^\ell} \in \mathcal{F}_r$ and $P_{p^r} \in \mathcal{F}_\ell$
2. every c occurs in at least one P
3. for $c \in \mathbf{c}^\ell$ (resp. $\mathbf{c}^r$): $c \notin P_{p^\ell}$ or c pn (resp. pp) in P_{p^ℓ}
4. for $c \in \mathbf{c}^\ell$ (resp. $\mathbf{c}^r$): $c \notin P_{p^r}$ or c pn (resp. pp) in $P_{p^r} \to \bot$.

Theorem

If $(A, (\heartsuit_p P_p)_p)$ is fit for translation and C_{c^ℓ} normal (for $\mathcal{R}$), then $A[\frac{\heartsuit_\mathbf{p} P_\mathbf{p}}{\mathbf{p}}][\frac{C_\mathbf{c}}{\mathbf{c}}]$ is equivalent (over $\mathcal{R}$) to a rule with restrictions.

Where C is normal if $\bigwedge C(\mathbf{q}) \equiv C(\mathbf{D}(\mathbf{q}))$.

Translating Axioms: Translatable Axioms

A propositional formula A is in $\mathcal{F}_r$ (resp. $\mathcal{F}_\ell$) iff the sequent $\Rightarrow A$ (resp. $A \Rightarrow$) is resolvable into atomic sequents. A variable of A is purely positive or pp (resp. purely negative or pn) iff it occurs only on the RHS (resp. LHS) in the sequent resolution of $\Rightarrow A$.

Let $A = \mathbf{v}^\ell \wedge \mathbf{c}^\ell \wedge \mathbf{p}^\ell \to \mathbf{v}^r \vee \mathbf{c}^r \vee \mathbf{p}^r \in \mathcal{F}_r$ and $P_p(\bar{v}, \bar{c})$ propositional formulae for $p \in \mathbf{p}^\ell \cup \mathbf{p}^r$. Then $(A, (\heartsuit_p P_p)_p)$ is fit for translation if

1. $P_{p^\ell} \in \mathcal{F}_r$ and $P_{p^r} \in \mathcal{F}_\ell$
2. every c occurs in at least one P
3. for $c \in \mathbf{c}^\ell$ (resp. $\mathbf{c}^r$): $c \notin P_{p^\ell}$ or c pn (resp. pp) in P_{p^ℓ}
4. for $c \in \mathbf{c}^\ell$ (resp. $\mathbf{c}^r$): $c \notin P_{p^r}$ or c pn (resp. pp) in $P_{p^r} \to \bot$.

Examples:

$$(\Box p \to \Box\Box p) \;\equiv\; (c^\ell \to p^r) \left[\frac{\Box c^\ell}{p^r}\right] \left[\frac{\Box p}{c^\ell}\right]$$

$$\bigcirc(p \to \bigcirc q) \to (p \to \bigcirc q) \;\equiv\; (p^\ell \wedge c^\ell \to c^r) \left[\frac{\bigcirc(c^\ell \to c^r)}{p^\ell}\right] \left[\frac{p \quad \bigcirc q}{c^\ell \quad c^r}\right]$$

Translating Rules into Axioms

Theorem

For monotone modalities: Every rule R with restrictions is equivalent to a set of translatable axioms. If all restrictions of R are normal, then R is equivalent to a single translatable axiom

Idea of Proof.

1. take a context instance $\widehat{R}$ (fixed number of context formulae)
2. turn premisses and conclusion of $\widehat{R}$ into formulae $\varphi_{\widehat{R}}$ and $\psi_{\widehat{R}}$
3. construct substitution σ witnessing projectivity of $\varphi_{\widehat{R}}$ [Ghilardi 1999]:
   - $\vdash_{\mathsf{Gi}} \Rightarrow \varphi_{\widehat{R}}\sigma$
   - $\vdash_{\mathsf{Gi}} \varphi_{\widehat{R}} \Rightarrow p \leftrightarrow p\sigma$ for all p
4. then $\widehat{R}$ is equivalent to $\psi_{\widehat{R}}\sigma$
5. thus R is equivalent to $\{\psi_{\widehat{R}}\sigma \mid \widehat{R}$ context instance of $R\}$

Translating Rules into Axioms

Theorem

For monotone modalities: Every rule R with restrictions is equivalent to a set of translatable axioms. If all restrictions of R are normal, then R is equivalent to a single translatable axiom

Example:

The context instance of $R_{4\Box}$ below gives formulae $\varphi_{\widehat{R_{4\Box}}}$ and $\psi_{\widehat{R_{4\Box}}}$:

$$\frac{\Box c_1, \Box c_2 \Rightarrow p}{\Box c_1, \Box c_2 \Rightarrow \Box p}\ \widehat{R_{4\Box}} \qquad \begin{array}{ll} \varphi_{\widehat{R_{4\Box}}}: & \Box c_1 \wedge \Box c_2 \rightarrow p \\ \psi_{\widehat{R_{4\Box}}}: & \Box c_1 \wedge \Box c_2 \rightarrow \Box p \end{array}$$

For substitution σ with $\sigma(p) = \varphi_{\widehat{R_{4\Box}}} \rightarrow p$ and $\sigma(c_i) = c_i$ we get:

$$\begin{array}{rcl} \psi_{\widehat{R_{4\Box}}}\sigma & = & \Box c_1 \wedge \Box c_2 \rightarrow \Box((\Box c_1 \wedge \Box c_2 \rightarrow p) \rightarrow p) \\ & \equiv & \Box c_1 \wedge \Box c_2 \rightarrow \Box((\Box c_1 \wedge \Box c_2) \vee p) \end{array}$$

which as axiom is equivalent to $\Box c_1 \wedge \Box c_2 \rightarrow \Box(\Box c_1 \wedge \Box c_2)$.
Since $\Box c$ is normal this is equivalent to axiom 4 $\Box c \rightarrow \Box\Box c$.

Correspondence between Axioms and Rules

This gives correspondences for logics with monotone modalities:

translatable rank-1	$\longleftrightarrow$	one-step rule
translatable non-nested	$\longleftrightarrow$	shallow rule
normal translatable	$\longleftrightarrow$	rule with normal restrictions
translatable scheme	$\longleftrightarrow$	rule with restrictions

where a translatable scheme is a set

$$\left\{ A \left[\frac{\heartsuit_{\mathbf{p}} P_{\mathbf{p}}}{\mathbf{p}} \right] \left[\frac{\bigwedge_{i \leq n_{\mathbf{c}^\ell}} C_{\mathbf{c}^\ell} \quad \bigvee_{i \leq n_{\mathbf{c}^r}} C_{\mathbf{c}^r}}{\mathbf{c}^\ell \qquad\qquad \mathbf{c}^r} \right] \mid n_{\mathbf{c}} \geq 0 \right\}$$

of axioms with $(A, (\heartsuit_p P_p)_p)$ fit for translation.

What are the applications?

Applications

Use the translation to construct sequent systems from axioms (e.g. for conditional logics based on $\mathbb{V}$, [L.-Pattinson, 2012])

Use the correspondence to show limitative results for rule formats:

Corollary (classically)

- T $\Box p \to p$ *is not equivalent over* K *to a set of one-step rules*
- 4 $\Box p \to \Box\Box p$ *is not equivalent over* K *to a set of shallow rules*

Question: Do we get limitative results for cut-free sequent systems?

Perhaps - so far we only have results for sequent systems admitting a proof of cut-elimination by permutation of rules...

Applications: S5 and Cut Elimination

S5 is K plus the axioms $\mathsf{T}, 4$ and 5 $\neg\Box p \to \Box\neg\Box p$.
The standard set of rules with restrictions for this is

$$\frac{\Gamma, p \Rightarrow \Delta}{\Gamma, \Box p \Rightarrow \Delta}\ R_{\mathsf{T}} \qquad\qquad \frac{\Box\Gamma \Rightarrow p, \Box\Delta}{\Sigma, \Box\Gamma \Rightarrow \Box p, \Box\Delta, \Pi}\ R_5$$

Theorem

There is no set $\mathcal{R}$ of rules such that $\mathsf{Gc}\mathcal{R}$ is mixed-cut closed and sound and cut-free complete for S5.

Idea of Proof.

- show that the translations of such rules have the form

$$\bigwedge \vec{p} \wedge c_1^{\ell} \wedge \Box c_2^{\ell} \wedge \bigwedge_{i \leq n} \Box P_i \to \bigvee_{j \leq m} \Box Q_j \vee c^r \vee \bigvee \vec{q}$$

- show that such axioms are not strong enough to capture S5.

Summary

- A rule format capturing most standard systems
- General (sufficient) conditions for Cut Elimination
- Correspondences between classes of axioms and rules
- All results for both classical and intuitionistic frameworks
- There is no rule set for S5 with "standard" proof of cut elimination

Thank you!