Generic Methods in the Construction of Cut-free Sequent Systems

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Motivation

Fact:

There are a many modal logics:

$$\mathsf{K}, \mathsf{KT}, \mathsf{K4}, \mathsf{S4}, \dots, \mathbb{V}, \mathbb{VA}, \dots, \mathsf{CK}, \mathsf{CS4}, \dots, \mathsf{CL}, \mathsf{IL}, \dots$$

. . . and their number is growing every day!

For deciding these logics we often use backwards proof search or the subformula property in "good" sequent systems.

But coming up with such a "good" sequent system is not easy!

Question:

Is there a generic method of constructing "good" sequent systems?

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Question:

Is there a generic method of constructing "good" sequent systems?

- What is a sequent system for a modal logic?
- What is a good sequent system?
- How to generically construct good sequent systems?

What is a sequent system for a modal logic?

Basics

In this talk we consider intuitionistic propositional modal logics. (But everything works in the classical case as well!)

Formulae are defined as usual:

$$A_1, \dots, A_n \ni \mathcal{F} ::= \begin{array}{l} p \mid \bot \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid A_1 \to A_2 \\ \mid \Box A_1 \mid \Diamond A_1 \mid \heartsuit(A_1, \dots, A_n) \mid \dots \end{array}$$

We use asymmetric sequents $\Gamma \Rightarrow \delta$ with Γ a multiset of formulae and δ empty or a formula. Intended interpretation: $\bigwedge \Gamma \to \delta$.

Our sequent systems have axioms $\overline{\Gamma, A \Rightarrow A}$, the structural rules

$$\frac{\Gamma \Rightarrow \delta}{A, \Gamma \Rightarrow \delta(, B)}\ \mathsf{W}, \qquad \frac{\Gamma, A, A \Rightarrow \delta}{\Gamma, A \Rightarrow \delta}\ \mathsf{Con}_L, \qquad \frac{\Gamma \Rightarrow A \quad A, \Sigma \Rightarrow \pi}{\Gamma, \Sigma \Rightarrow \pi}\ \mathsf{Cut},$$

the propositional rules, the congruence rules for all operators $\heartsuit$:

$$\frac{A_1 \Rightarrow B_1 \quad B_1 \Rightarrow A_1 \quad \dots \quad A_n \Rightarrow B_n \quad B_n \Rightarrow A_n}{\Gamma, \heartsuit(A_1, \dots, A_n) \Rightarrow \heartsuit(B_1, \dots, B_n)}$$

and additional rules of a specific format.

Rules with Context Restrictions

A context restriction is a tuple $\langle F_\ell; F_r \rangle$ of sets of formulae. It restricts a sequent $\Gamma \Rightarrow \delta$ by allowing only substitution instances of formulae from F_ℓ (resp. F_r) in Γ (resp. δ).

A rule with context restrictions is of the form

$$\frac{(\Gamma_1 \Rightarrow \delta_1; \mathcal{C}_1) \quad \dots \quad (\Gamma_n \Rightarrow \delta_n; \mathcal{C}_n)}{\Sigma \Rightarrow \pi}$$

with principal formulae $\Sigma \Rightarrow \pi \in \mathit{Seq}(\mathit{Mod}(\mathit{Var}))$ and premisses $\Gamma_i \Rightarrow \delta_i \in \mathit{Seq}(\mathit{Var})$ with associated context restrictions $\mathcal{C}_i$.

In an application of such a rule a premiss with associated restriction $\mathcal{C}_i$ carries over only the context restricted according to $\mathcal{C}_i$ from the conclusion.

Examples of Rules with Context Restrictions

Our rule format captures many standard rules for modal logics, e.g. the rules for $\mathsf{CK}_\Box$ and $\mathsf{CS4}_\Box$:

$$\frac{A_1, \ldots, A_n \Rightarrow B}{\Gamma, \Box A_1, \ldots, \Box A_n \Rightarrow \Box B}\ \mathsf{K}_n \qquad \frac{(A_1, \ldots, A_n \Rightarrow B\ ; \langle \emptyset; \emptyset \rangle)}{\Box A_1, \ldots, \Box A_n \Rightarrow \Box B}$$

$$\frac{\Gamma, A \Rightarrow \delta}{\Gamma, \Box A \Rightarrow \delta}\ \mathsf{T}_\Box \qquad \frac{(A \Rightarrow\ ; \langle \{p\}; \{p\} \rangle)}{\Box A \Rightarrow}$$

$$\frac{\Box \Gamma \Rightarrow A}{\Sigma, \Box \Gamma \Rightarrow \Box A}\ 4_\Box \qquad \frac{(\Rightarrow A\ ; \langle \{\Box p\}; \emptyset \rangle)}{\Rightarrow \Box A}$$

We often use the more suggestive notation on the left.

Rules K_n and $\mathsf{T}_\Box$ are shallow: they use only restrictions $\langle \emptyset; \emptyset \rangle$ or $\langle \{p\}; \{p\} \rangle$.

What is a good sequent system for a modal logic?

Cut Elimination

The structural rules Con_L and Cut are bad for backwards proof search, since they give rise to infinite search trees. Also, Cut sabotages the subformula property.

$$\frac{\Gamma, A, A \Rightarrow \delta}{\Gamma, A \Rightarrow \delta}\ \text{Con}_L, \qquad \frac{\Gamma \Rightarrow A \quad A, \Sigma \Rightarrow \pi}{\Gamma, \Sigma \Rightarrow \pi}\ \text{Cut}$$

Thus in a good sequent system these rules should be admissible: the system should derive the same sequents if we drop these rules.

Idea:

Extract general conditions on the rule sets from the standard proofs which guarantee admissibility of Cut and Con_L.

Cut (and Contraction) as Operations on Rules

Cuts between rules:

Slogan:
Cut the conclusion, cut the premisses, be liberal on the restrictions!

Example:

$$\frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} \qquad \frac{A, B \Rightarrow C}{\Box A, \Box B \Rightarrow \Box C}$$

Contractions of rules:

Slogan: Contract the conclusion, contract the premisses!
Example:

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$$\Box\Gamma, \Box B \Rightarrow \Box C$$

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The General Conditions: Saturation

A set $\mathcal{R}$ of rules is

- principal-cut closed if cuts between rules from $\mathcal{R}$ are $\mathcal{R}$ConW-derivable;
- context-cut closed if whenever context restrictions of R and Q overlap on A (i.e. if $A \in F_r^R\sigma \cap F_\ell^Q\tau$) , then the principal formulae and all restrictions of one rule satisfy all restrictions of the other rule overlapping on A;
- mixed-cut closed if whenever a principal formula A of R satisfies a context restriction of Q then all restrictions and principal formulae of R satisfy this restriction;
- contraction closed if contractions of rules from $\mathcal{R}$ are in $\mathcal{R}$; [1]
- saturated if it is all of the above.

Examples:

The standard rule sets for the standard modal logics built from $K, T, D, 4$ and the rules for propositional logic are all saturated.

[1] Compare the closure condition in [Negri, von Plato 2001]

Generic Cut Elimination

Theorem (Generic Cut Elimination)

In saturated rule sets the cut rule can be eliminated.

Proof Sketch.

As usual eliminate multicuts $\dfrac{\Gamma \Rightarrow A \quad A^n, \Sigma \Rightarrow \delta}{\Gamma, \Sigma \Rightarrow \delta}$ by double induction on the rank and depth of the cut. E.g.

$$\dfrac{\dfrac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} R_{4\Box} \quad \dfrac{\Sigma, \Box A, \Box A, A \Rightarrow \Pi}{\Sigma, \Box A, \Box A, \Box A \Rightarrow \Pi} R_{T\Box}}{\Box\Gamma, \Sigma \Rightarrow \Pi} \text{mCut}$$

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lower depth

psc + lower rank

Decidability

Copying all formulae obeying the associated context restriction into the premisses, yields admissibility of Contraction. For tractable rule sets (codes of applicable rules / their premisses can be computed in pspace from the conclusion / the rule code) we also have

Theorem (Complexity)

For saturated and tractable sets of rules with restrictions the derivability problem is in EXPTIME*. If all rules are shallow, then the problem is in* PSPACE*.*

Idea of Proof.

1. Eliminate Cuts
2. Eliminate Contraction
3. Use subformula property in a fixed-point argument for EXPTIME
4. Use set-sequents and backwards proof search for PSPACE.

How to generically construct good sequent systems?

Constructing Cut-free Calculi

Lemma (Cuts preserve soundness)

If G2*ip* $\in \mathcal{R}$*, then cuts between rules in* $\mathcal{R}$ *are* $\mathcal{R}$ConCut*-derivable.*

This suggests the following heuristic to construct a cut-free sequent system by saturation: given a set of sequent rules

1. Saturate the rules under cuts and contractions (guarantees principal-cut closure and contraction closure)
2. check context- and mixed-cut closure and tractability

This heuristic together with a graphical tool was used e.g. in the construction of new cut-free systems for several conditional logics including $\mathbb{V}_{\preccurlyeq}$ and $\mathbb{VA}_{\preccurlyeq}$. [L., Pattinson 2012]

Constructing Cut-free Calculi

Question: How do we get the rules to start with?

Often the logics are given as a Hilbert-system, i.e. a set $\mathcal{A}$ of axioms closed under modus ponens and uniform substitution:

$$\frac{\vdash A \quad \vdash A \to B}{\vdash B}\ MP \qquad \frac{\vdash A}{\vdash A\left[\frac{B}{p}\right]}\ US$$

Examples:

- $\mathcal{A}_{\mathsf{CK}_\Box} = \mathsf{IL} \cup \{\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q),\ \Box\top\}$
- $\mathcal{A}_{\mathsf{CS4}_\Box} = \mathcal{A}_{\mathsf{CK}} \cup \{\Box p \to p,\ \Box p \to \Box\Box p\}$

Idea:

Follow a similar approach as for cut elimination: find criteria guaranteeing translatability of axioms into rules with restrictions.

Translating Axioms: Nesting Depth 1

Consider $\mathrm{CS4}_{\Box} = \mathrm{CK}_{\Box} + (\Box p \to p) + (\Box p \to \Box\Box p)$.
(With standard rules for $\mathrm{CK}_{\Box}$.)

First take axiom $\Box p \to p$.

We take the axiom ...

$\Box p \to p$

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$$\Box p \to p \quad \rightsquigarrow \quad \frac{}{\Rightarrow \Box p \to p}$$

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First take axiom $\Box p \to p$.

We take the axiom, turn it into a zero-premiss rule,
resolve propositional logic and turn top-level propositional variables into contextual premisses

$$\Box p \to p \quad \rightsquigarrow \quad \frac{}{\Rightarrow \Box p \to p} \quad \rightsquigarrow \quad \frac{}{\Box p \Rightarrow p} \quad \rightsquigarrow \quad \frac{\Gamma, p \Rightarrow \delta}{\Gamma, \Box p \Rightarrow \delta}\ \mathsf{T}_\Box$$

introducing the restriction $\langle \{p\}, \{p\} \rangle$ in the last step.

Translating Axioms: Higher Nesting Depth

Now consider axiom $\Box p \to \Box\Box p$.

Key observation: $\Box p$ occurs on top level and under exactly one $\Box$.

Again first resolve propositional logic.

$$\frac{}{\Box p \Rightarrow \Box\Box p}$$

Translating Axioms: Higher Nesting Depth

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Again first resolve propositional logic. Take the occurrence of $\Box p$ under $\Box$,

$$\frac{}{\Box p \Rightarrow \Box{\color{red}\Box p}}$$

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Again first resolve propositional logic. Take the occurrence of $\Box p$ under $\Box$, substitute a fresh variable q for this

$$\frac{}{\Box p \Rightarrow \Box\Box p} \quad \rightsquigarrow \quad \frac{\Box p \Rightarrow q \quad q \Rightarrow \Box p}{\Box p \Rightarrow \Box q}$$

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Now computing multiple cuts with instances of the K-rule gives:

$$\frac{\Box p_1 \Rightarrow q_1}{\Box p_1 \Rightarrow \Box q_1}, \quad \frac{q_1, \ldots, q_n \Rightarrow r}{\Box q_1, \ldots, \Box q_n \Rightarrow \Box r} \quad \rightsquigarrow \quad \frac{\Box p_1, q_2, \ldots, q_n \Rightarrow r}{\Box p_1, \Box q_2, \ldots, \Box q_n \Rightarrow \Box r}$$

$$\rightsquigarrow \quad \ldots \quad \rightsquigarrow \frac{\Box p_1, \ldots, \Box p_n \Rightarrow r}{\Box p_1, \ldots, \Box p_n \Rightarrow \Box r} \quad \rightsquigarrow \quad \frac{\Box\Gamma \Rightarrow r}{\Box\Gamma \Rightarrow \Box r}\ 4_\Box$$

This gives a purely syntactic construction of the rules for CS4.

Translating Axioms: Translatable Axioms

A propositional formula A is in $\mathcal{F}_r$ (resp. $\mathcal{F}_\ell$) iff the sequent $\Rightarrow A$ (resp. $A \Rightarrow$) is resolvable into atomic sequents. A variable of A is purely positive or pp (resp. purely negative or pn) iff it occurs only on the RHS (resp. LHS) in the sequent resolution of $\Rightarrow A$.

Let $A = \mathbf{v}^\ell \wedge \mathbf{c}^\ell \wedge \mathbf{p}^\ell \to \mathbf{v}^r \vee \mathbf{c}^r \vee \mathbf{p}^r \in \mathcal{F}_r$ and $P_p(\bar{v}, \bar{c})$ propositional formulae for $p \in \mathbf{p}^\ell \cup \mathbf{p}^r$. Then $(A, (\heartsuit_p P_p)_p)$ is fit for translation if

1. $P_{p^\ell} \in \mathcal{F}_r$ and $P_{p^r} \in \mathcal{F}_\ell$
2. $\mathbf{c}^\ell \cup \mathbf{c}^r = \emptyset$ and every c occurs in at least one P
3. for $c \in \mathbf{c}^\ell$ (resp. $\mathbf{c}^r$): $c \notin P_{p^\ell}$ or c pn (resp. pp) in P_{p^ℓ}
4. for $c \in \mathbf{c}^\ell$ (resp. $\mathbf{c}^r$): $c \notin P_{p^r}$ or c pn (resp. pp) in $P_{p^r} \to \bot$.

Theorem

If $(A, (\heartsuit_p P_p)_p)$ is fit for translation, the $\heartsuit_p$ are monotone and C_{c^ℓ} normal, then $A[\frac{\heartsuit_\mathbf{p} P_\mathbf{p}}{\mathbf{p}}][\frac{C_\mathbf{c}}{\mathbf{c}}]$ is equivalent to a rule with restrictions.

Where C is normal if $\bigwedge C(\mathbf{q}) \equiv C(\mathbf{D}(\mathbf{q}))$.

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Examples:

$$(\Box p \to \Box\Box p) \;\equiv\; (c^\ell \to p^r)\left[\frac{\Box c^\ell}{p^r}\right]\left[\frac{\Box p}{c^\ell}\right]$$

$$\bigcirc(p \to \bigcirc q) \to (p \to \bigcirc q) \;\equiv\; (p^\ell \wedge c^\ell \to c^r)\left[\frac{\bigcirc(c^\ell \to c^r)}{p^\ell}\right]\left[\frac{p \quad \bigcirc q}{c^\ell \quad c^r}\right]$$

Byproduct: Correspondence between axioms and rules

Translating Rules into Axioms

Theorem

For monotone modalities: Every rule R with restrictions is equivalent to a set of translatable axioms. If all restrictions of R are normal, then R is equivalent to a single translatable axiom

Idea of Proof.

1. take context instance $\widehat{R}$ (fixed number of context formulae)
2. turn premisses and conclusion of $\widehat{R}$ into formulae $\varphi_{\widehat{R}}$ and $\psi_{\widehat{R}}$
3. construct substitution σ witnessing projectivity of $\varphi_{\widehat{R}}$ [Ghilardi 1999]:
   - $\vdash_{\mathsf{Gi}} \Rightarrow \varphi_{\widehat{R}}\sigma$
   - $\vdash_{\mathsf{Gi}} \varphi_{\widehat{R}} \Rightarrow p \leftrightarrow p\sigma$ for all p
4. then $\widehat{R}$ is equivalent to $\psi_{\widehat{R}}\sigma$

Correspondence between Axioms and Rules

This gives correspondences for logics with monotone modalities:

translatable scheme	$\longleftrightarrow$	rule with restrictions
normal translatable	$\longleftrightarrow$	rule with normal restrictions
translatable non-nested	$\longleftrightarrow$	shallow rule
translatable rank-1	$\longleftrightarrow$	one-step rule

where a translatable scheme is a set

$$\left\{ A\left[\frac{\heartsuit_{\mathbf{p}} P_{\mathbf{p}}}{\mathbf{p}}\right]\left[\frac{\bigwedge_{i\le n_{\mathbf{c}\ell}} C_{\mathbf{c}\ell} \quad \bigvee_{i\le n_{\mathbf{c}r}} C_{\mathbf{c}r}}{\mathbf{c}^{\ell} \qquad\qquad \mathbf{c}^{r}}\right] \mid n_{\mathbf{c}} \ge 0 \right\}$$

of axioms with $(A, (\heartsuit_p P_p)_p)$ fit for translation.

Corollary (classically)

- *$\Box p \to p$ is not equivalent to a set of one-step rules*
- *$\Box p \to \Box\Box p$ is not equivalent to a set of shallow rules*
- L *and* MA *are not equivalent to a set of shallow rules*
- *$\Box\Box p \to \Box\Box\Box p$ is not equivalent to a saturated set of rules*

Summary

- A rule format capturing most standard systems
- General (sufficient) conditions for Cut Elimination
- Correspondences between classes of axioms and rules
- All results for both classical and intuitionistic frameworks

Thank you!