General methods in proof theory for modal logic - Lecture 3

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The modal logic of provability GL

- $GL = \mathbf{K} + \Box(\Box p \supset p) \supset \Box p$ (Löb's axiom)
- characterised by the class $\mathcal{F}_{GL}$ of Kripke frames satisfying transitivity and no ∞-R-chains (finite transitive trees)
- I.e. for every formula A: $A \in GL$ iff $\mathcal{F}_{GL} \models A$
- proof omitted
- Interpreting $\Box A$ as "$\bar{A}$ is provable in Peano arithmetic" (frequently written $Bew(\bar{A})$) GL is sound and complete wrt formal provability interpretation in Peano arithmetic (Solovay, 1976).
- Hence the name provability logic
- The logic is decidable (a benefit of studying a fragment of Peano arithmetic)

A sequent calculus for *GL*

- **K**:

$$\frac{X \Rightarrow A}{\Box X \Rightarrow \Box A}\ \Box\mathrm{K}$$

- **K**4 (the 4 axiom is $\Box A \supset \Box\Box A$ and corresponds to transitivity)

$$\frac{X, \Box X \Rightarrow A}{\Box X \Rightarrow \Box A}\ \Box 4$$

- **GL** (axiomatised by addition of $\Box(\Box A \supset A) \supset \Box A$ to **K**)

$$\frac{\Box X, X, \Box A \Rightarrow A}{\Box X \Rightarrow \Box A}\ GLR$$

  (Sambin and Valentini, 1982).
  $\Box A$ is called the diagonal formula. Motivated from $\Box 4$ rule.

The sequent calculus **sGL** for **GL**

Initial sequents: $A \Rightarrow A$ for each formula A

Logical rules:

$$\frac{X \Rightarrow Y, A}{X, \neg A \Rightarrow Y}\ L\neg \qquad \frac{A, X \Rightarrow Y}{X \Rightarrow Y, \neg A}\ R\neg$$

$$\frac{A_i, X \Rightarrow Y}{A_1 \wedge A_2, X \Rightarrow Y}\ L\wedge \qquad \frac{X \Rightarrow Y, A_1 \qquad X \Rightarrow Y, A_2}{X \Rightarrow Y, A_1 \wedge A_2}\ R\wedge$$

$$\frac{A_1, X \Rightarrow Y \qquad A_2, X \Rightarrow Y}{A_1 \vee A_2, X \Rightarrow Y}\ L\vee \qquad \frac{X \Rightarrow Y, A_i}{X \Rightarrow Y, A_1 \vee A_2}\ R\vee$$

$$\frac{X \Rightarrow Y, A \qquad B, U \Rightarrow W}{A \supset B, X, U \Rightarrow Y, W}\ {\rightarrow}\mathrm{L} \qquad \frac{A, X \Rightarrow Y, B}{X \Rightarrow Y, A \supset B}\ {\rightarrow}R$$

Modal rule:

$$\frac{\Box X, X, \Box A \Rightarrow A}{\Box X \Rightarrow \Box A}\ GLR$$

Structural rules:

$$\frac{X \Rightarrow Y}{A, X \Rightarrow Y}\ LW \qquad \frac{X \Rightarrow Y}{X \Rightarrow Y, A}\ RW$$

Soundness of **sGL** wrt **K***L*

- As before soundness can be verified by taking the contrapositive of each rule and falsifying on a finite transitive irreflexive trees.
- Let us consider the rule GLR
- Omitting the context for simplicity, suppose that the conclusion of GLR is falsifiable so there is a model M s.t. $M, w_0 \not\models \Box A$. Then there exists w_1 s.t. $M, w_1 \models \neg A$. If $M, w_1 \models \Box A$ then the premise of GLR is falsified.
- If $M, w_1 \not\models \Box A$ then there exists w_2 s.t. $M, w_2 \models \neg A$. If $M, w_2 \models \Box A$ then the premise of GLR is falsified.
- . . . and so on. . .
- We cannot continue this indefinitely because the trees are finite!
- To see why transitivity is required, consider the contexts too.

Completeness of **sGL** wrt **K***L*

- Completeness: simulate *modus ponens* with cut; eliminate cut to obtain subformula property
- An alternative semantic proof of completeness: since $\mathcal{F}_{\mathbf{GL}} \models A$ implies **sGL** derives $\vdash A$, taking the contrapositive it suffices to prove:

  if there is no derivation of $\vdash A$ in **sGL** then $\mathcal{F}_{\mathbf{GL}} \not\models A$

- Idea. Suppose that there is no derivation of $\vdash A$. Use this to build a finite tree that falsifies A at the root.
- Nonetheless, the proof of cut-elimination is interesting so let us sketch the proof.

Syntactic cut-elimination for *GL* - a brief history

- Leivant (1981) suggests a syntactic proof, counter-example by Valentini (1982)
- new proof of syntactic CE for GLS_{set} proposed by Valentini (1983) — induction on $degree \cdot \omega^2 + width \cdot \omega + cutheight$
- Subsequently Borga (1983) and Sasaki (2001) present new proofs
- Moen (2001) claimed that Valentini's proof has a gap when contractions are made explicit
- Many other proofs were subsequently presented as an alternative (e.g. Mints, Negri)
- Goré and R. (2008) show Moen's claim is incorrect, Valentini's argument is sound, and introduce new transformations to deal with contraction
- Dawson and Goré (2010) verify this argument in Isabelle/HOL

Sambin Normal Form

The interesting case is the Sambin Normal Form (SNF) where both Π and Ω are cutfree

$$\dfrac{\dfrac{\overset{\Pi}{\Box X, X, \Box B \overset{k}{\Rightarrow} B}}{\Box X \overset{k+1}{\Rightarrow} \Box B}\, GLR \qquad \dfrac{\overset{\Omega}{\Box B, B, \Box U, U, \Box D \overset{l}{\Rightarrow} D}}{\Box B, \Box U \overset{l+1}{\Rightarrow} \Box D}\, GLR}{\Box X, \Box U \Rightarrow \Box D}\, cut(\Box B)$$

cut-height is $(k+1)+(l+1)$. degree of cut-formula is $d(\Box B)$.

The principal case — a derivation in SNF

A derivation is in Sambin Normal Form when:

- the last rule is the cut rule with cutfree premises
- the cut-formula is principal by *GLR* in both premises

A naive transformation to eliminate cut:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\Pi}{\Box X, X, \Box B \overset{k}{\Rightarrow} B}}{\Box X \overset{k+1}{\Rightarrow} \Box B}\,\mathit{GLR} \qquad \dfrac{\overset{\Pi}{X, \Box X, \Box B \overset{k}{\Rightarrow} B} \qquad \overset{\Omega}{\Box B, B, \Box U, U, \Box D \overset{l}{\Rightarrow} D}}{X, \Box X, \Box B, \Box U, U, \Box D \overset{[k,l]+1}{\Rightarrow} D}\,\mathit{cut}_1}{X, \Box X, \Box X, \Box U, U, \Box D \Rightarrow D}\,\mathit{cut}_2}{X, \Box X, \Box U, U, \Box D \Rightarrow D}\,\mathit{LC}^*(\Box X)}{\Box X, \Box U \Rightarrow \Box D}\,\mathit{GL}$$

Cut-height is $k + l$ (cut_1) and $(k+1) + ([k,l]+1)$ (cut_2)
Problem with cut_2 !

A successful transformation for SNF

Transform derivation in SNF to:

$$
\dfrac{\dfrac{\dfrac{\overset{\Sigma}{\Box X, X \Rightarrow B} \qquad \dfrac{\dfrac{\overset{\Pi}{\Box X, X, \Box B \overset{k}{\Rightarrow} B}}{\Box X \overset{k+1}{\Rightarrow} \Box B}\,GLR \qquad \overset{\Omega}{\Box B, B, \Box U, U, \Box D \overset{l}{\Rightarrow} D}}{\Box X, B, \Box U, U, \Box D \Rightarrow D}\,cut_1}{\Box X, \Box X, X, \Box U, U, \Box D \Rightarrow D}\,cut_2}{\Box X, X, \Box U, U, \Box D \Rightarrow D}\,LC^*(\Box X)}{\Box X, \Box U \Rightarrow \Box D}\,GLR
$$

where Σ is some cut-free derivation.

- cut_1 has cut-height $(k+1)+l$
- cut_2 has smaller degree of cut-formula

New task: obtain a cut-free derivation of $\Box X, X \Rightarrow B$ from a derivation of $\Box X, X, \Box B \Rightarrow B$

A sketch of the proof of $\Box X, X \vdash B$ from $\Box X, X, \Box B \vdash B$

The width is the number n of occurrences of the following schema, where no GLR rule occurrences appear between GLR_1 and GLR_2

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box G, G, \Box B, B, \Box C \Rightarrow C}{\Box G, \Box B \Rightarrow \Box C}\; GLR_2}{\vdots}}{\Box X, X, \Box B \Rightarrow B}}{\Box X \vdash \Box B}\; GLR_1$$

If $n = 0$ then the $\Box B$ in $\Box X, X, \Box B \Rightarrow B$ has either been introduced by

1. $LW(\Box B)$. In this case delete the $LW(\Box B)$ rule. Or,
2. the initial sequent $\Box B \Rightarrow \Box B$. Replace with $\Box X \Rightarrow \Box B$.

In this way we obtain a derivation of $\Box X, X \vdash B$.

The width is the number n of occurrences of the following schema, where no GLR rule occurrences appear between GLR_1 and GLR_2

$$\cfrac{\cfrac{\cfrac{\cfrac{\Box G, G, \Box B, B, \Box C \Rightarrow C}{\Box G, \Box B \Rightarrow \Box C}\,GLR_2}{\vdots}}{\Box X, X, \Box B \Rightarrow B}}{\Box X \vdash \Box B}\,GLR_1$$

If $n = k + 1$, each occurrence of the above schema is deleted as follows. Replace below left by below right.

$$\cfrac{\Box G, G, \Box B, B, \Box C \Rightarrow C}{\Box G, \Box B \Rightarrow \Box C}\,GLR_2 \qquad \cfrac{\Box C \Rightarrow \Box C}{\Box G, \Box B, \Box C \Rightarrow \Box C}\,\mathrm{lw}$$

Continuing downwards we obtain a derivation of $\Box X, \Box C \vdash \Box B$ with smaller width.

Now proceed:

$$\dfrac{\Box X, \Box C \vdash \Box B \qquad \dfrac{\Box X, X, \Box B \vdash B \qquad \Box G, G, \Box B, B, \Box C \Rightarrow C}{\Box X, X, \Box G, G, \Box B, \Box C \vdash C}\,\text{cut}}{\Box X, \Box X, X, \Box G, G, \Box C, \Box C \vdash C}\,\text{cut}$$

The second cut has lesser width than before! So we obtain a cutfree derivation of $\Box X, X, \Box G, G, \Box C \vdash C$.

Now replace below left in original derivation with below right.

$$\dfrac{\Box G, G, \Box B, B, \Box C \Rightarrow C}{\Box G, \Box B \Rightarrow \Box C}\,GLR_2 \qquad \dfrac{\dfrac{\Box X, X, \Box G, G, \Box C \vdash C}{\Box X, \Box G \vdash \Box C}\,\text{GLR}}{\Box X, \Box G, \Box B \vdash \Box C}\,\text{lw}$$

We thus obtain a derivation of the following of lesser width.

$$\dfrac{\Box X, X, \Box B \Rightarrow B}{\Box X \vdash \Box B}\,GLR_1$$

GL, **Grz** and **Go**

$$L: \quad \Box(\Box p \supset p) \supset \Box p \quad \text{(Löb's axiom)}$$

$$Grz: \quad \Box(\Box(p \supset \Box p) \supset p) \supset p$$

$$Go: \quad \Box(\Box(p \supset \Box p) \supset p) \supset \Box p$$

$$\mathbf{GL}{=}\mathbf{K} + L \quad \mathbf{Go}{=}\mathbf{K} + Go \quad \mathbf{Grz}{=}\mathbf{K} + Grz$$

A sequent calculus for **Grz** is obtained by adding the rules below left and center. For **Go** add rule below right.

$$\frac{B, X \Rightarrow Y}{\Box B, X \Rightarrow Y} \qquad \frac{\Box X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B} \qquad \frac{\Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\, GoR$$

- **sGrz** has cut-elimination (Borga and Gentilini, 1986). Reflexivity rule above left simplifies argument.
- Cut-elimination for **sGo** (Goré and R., 2013).
- The proof requires a deeper study of the derivation (not just the GoR_2 rule instance). Extends Valentini's argument for **sGL** and uses a quaternary induction measure

Extending the sequent calculus to present more logics

- The sequent calculus is simple to work with
- However, it is hard to extend the proofs of cut-elimination for axiomatic extensions. . .
- The addition of a new rule typically breaks cut-elimination
- This motivates the extension of the sequent calculus to yield modular extensions (see next page!)

Labelled Sequents

A very general method for constructing sequent calculi from frame conditions was developed e.g. in (Viganò, 2000), (Negri, 2005 and 2011)

Main idea: Explicitly include the Kripke semantics in the calculus

Definition

Let $u, v, w, \ldots$ be a countably infinite set of labels.

- A labelled modal formula has the form $w : A$ for a label w and a modal formula A.
- A relational term has the form wRv for labels w, v.
- A labelled sequent is a sequent consisting of labelled modal formulae and relational terms.

The calculus G3K

The modal rules of the labelled sequent calculus G3K for modal logic **K** are

$$\frac{\Gamma, wRv \vdash \Delta, v : A}{\Gamma \vdash \Delta, w : \Box A}\; R\Box \qquad\qquad \frac{\Gamma, v : A, w : \Box A, wRv \vdash \Delta}{\Gamma, w : \Box A, wRv \vdash \Delta}\; L\Box$$

(v does not occur in Γ, Δ)

Intuition behind the rules:

- $R\Box$ is equivalent to the condition

$$\forall v.\, (wRv \implies v : A) \implies w : \Box A$$

- $L\Box$ is equivalent to the condition

$$w : \Box A \text{ and } wRv \implies v : A$$

The calculus G3K - propositional part

The propositional rules of G3K are essentially the standard ones extended with labels:

$$\frac{}{\Gamma, w : \bot \vdash \Delta}\ L\bot$$

$$\frac{}{\Gamma, w : p \vdash w : p, \Delta} \qquad \frac{}{\Gamma, wRv \vdash wRv, \Delta}$$

$$\frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta}\ L\wedge \qquad \frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta}\ R\wedge$$

$$\frac{\Gamma, w : A \vdash \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \vee B \vdash \Delta}\ L\vee \qquad \frac{\Gamma \vdash w : A, w : B\Delta}{\Gamma \vdash w : A \vee B\Delta}\ R\vee$$

$$\frac{\Gamma, w : B \to \Delta \quad \Gamma \vdash w : A, \Delta}{\Gamma, w : A \to B \vdash \Delta}\ L \to \qquad \frac{\Gamma, w : A \to w : B, \Delta}{\Gamma \vdash w : A \to B, \Delta}\ R \to$$

The calculus G3K

Example

The axiom $\Box(p \to q) \to (\Box p \to \Box q)$ is derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, v : q, v : p \vdash v : q}\ \text{init} \qquad \dfrac{}{\Gamma, v : p \vdash v : p, v : q}\ \text{init}}{w : \Box(p \to q), w : \Box p, wRv, v : p \to q, v : p \vdash v : q}\ L\to}{w : \Box(p \to q), w : \Box p, wRv \vdash v : q}\ L\Box}{w : \Box(p \to q), w : \Box p \vdash w : \Box q}\ R\Box}{w : \Box(p \to q) \to (\Box p \to \Box q)}\ R\to$$

The calculus G3K - useful properties

Proposition

The following properties can all be established by standard methods (mostly induction on the depth of the derivation):

- The sequent $\Gamma, w : A \vdash w : A, \Delta$ is derivable for every A
- Substitution of labels $\dfrac{\Gamma \vdash \Delta}{\Gamma(v/w) \vdash \Delta(v/w)}$ is depth-preserving admissible.
- Weakening is depth-preserving admissible.
- The labelled necessitation rule $\dfrac{\vdash w : A}{\vdash w : \Box A}$ is derivable.
- The rules of G3K are depth-preserving invertible.
- Contraction is depth-preserving admissible.

Soundness and completeness

The cut rule in the labelled sequent framework, written cut_ℓ, comes in two shapes, depending on the shape of the cut formula:

$$\frac{\Gamma \vdash \Delta, w : A \quad w : A, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi} \qquad \frac{\Gamma \vdash \Delta, wRv \quad wRv, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}$$

Theorem

The calculus $\mathsf{G3Kcut}_\ell$ *is sound and complete for modal logic* **K**, *i.e., for every formula A:*

$$A \textit{ is a theorem of } \mathbf{K} \quad \textit{iff} \quad \vdash w : A \textit{ is derivable in } \mathsf{G3Kcut}_\ell\ .$$

Sketch of proof.

Since the labelled necessitation rule is admissible, deriving the axioms of **K** and simulating modus ponens using cut_ℓ is enough.

Cut Elimination for G3K

The cut elimination proof is essentially the standard one, using a double induction on the size of the cut formula and the height of the cut (the sum of the depths of the derivations of its premisses).

The interesting case:

$$\dfrac{\dfrac{\Gamma, wRx \vdash \Delta, x : A}{\Gamma \vdash \Delta, w : \Box A}\,R\Box \qquad \dfrac{w : \Box A, wRv, v : A, \Sigma \vdash \Pi}{w : \Box A, wRv, \Sigma \vdash \Pi}\,L\Box}{\Gamma, wRv, \Sigma \vdash \Delta, \Pi}\,\mathsf{cut}_\ell$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\Gamma, wRx \vdash \Delta, x : A}{\Gamma, wRv \vdash \Delta, v : A}\,sb \qquad \dfrac{\dfrac{\Gamma, wRx \vdash \Delta, x : A}{\Gamma \vdash \Delta, w : \Box A}\,R\Box \quad w : \Box A, wRv, v : A, \Sigma \vdash \Pi}{\Gamma, v : A, wRv, \Sigma \vdash \Delta, \Pi}\,\mathsf{cut}_\ell}{\Gamma, wRv, \Gamma, wRv, \Sigma \vdash \Delta, \Delta, \Pi}\,\mathsf{cut}_\ell}{\Gamma, wRv, \Sigma \vdash \Delta, \Pi}\,\mathsf{Con}$$

Cut Elimination for G3K

The cut elimination proof is essentially the standard one, using a double induction on the size of the cut formula and the height of the cut (the sum of the depths of the derivations of its premisses).

Theorem

The labelled cut rule is admissible in G3K*. Hence the calculus* G3K *is cut-free complete for modal logic* **K***, i.e.:*

If A is a theorem of **K** *then* $\vdash w : A$ *is derivable in* G3K .

Converting frame conditions into rules

Definition

A geometric axiom is a formula of the form

$$\forall \vec{x}(P \to \exists \vec{y_1} M_1 \lor \cdots \lor \exists \vec{y_n} M_n)$$

where

- the M_j and P are conjunctions of relational terms
- the variables $\vec{y_j}$ are not free in P.

Examples

- $\forall x\, xRx$ for reflexivity
- $\forall x, y, z\, (xRy \land yRz \to xRz)$ for transitivity
- $\forall x, y\, (xRy \to yRx)$ for symmetry
- $\forall x, y, z\, (xRy \land xRz \to \exists w\, (yRw \land zRw))$ for directedness

Converting frame conditions into rules

Definition

A geometric axiom is a formula of the form

$$\forall \vec{x}(P \to \exists \vec{y_1} M_1 \vee \cdots \vee \exists \vec{y_n} M_n)$$

where

- the M_j and P are conjunctions of relational terms
- the variables $\vec{y_j}$ are not free in P.

Theorem

The geometric axiom above is equivalent to the geometric rule

$$\frac{\Gamma, \bar{P}, \bar{M_1}(z_1/y_1) \vdash \Delta \quad \ldots \quad \Gamma, \bar{P}, \bar{M_n}(z_n/y_n) \vdash \Delta}{\Gamma, \bar{P} \vdash \Delta}$$

with $\bar{M_i}$ and $\bar{P}$ the multisets of relational atoms in M_i resp. P, and $z_1, \ldots, z_n$ not in the conclusion.

Converting frame conditions into rules: Examples

- Reflexivity $\forall x\, xRx$ is converted to

$$\frac{\Gamma, yRy \vdash \Delta}{\Gamma \vdash \Delta}$$

- Transitivity $\forall x, y, z\, (xRy \wedge yRz \to xRz)$ is converted to

$$\frac{\Gamma, xRy, yRz, xRz \vdash \Delta}{\Gamma, xRy, yRz \vdash \Delta}$$

- Symmetry $\forall x, y\, (xRy \to yRx)$ is converted to

$$\frac{\Gamma, xRy, yRz \vdash \Delta}{\Gamma, xRy \vdash \Delta}$$

- Directedness $\forall x, y, z\, (xRy \wedge xRz \to \exists w\, (yRw \wedge zRw))$ gives

$$\frac{\Gamma, xRy, xRz, yRv, zRv \vdash \Delta}{\Gamma, xRy, xRz \vdash \Delta}\ v \text{ not in conclusion}$$

Converting frame conditions into rules: Contraction

To obtain the nice structural properties for extensions of G3K with geometric rules we need to close the rule set under contraction:

Definition

A geometric rule set satisfies the closure condition if for every rule

$$\frac{\Gamma, \bar{P}, Q, R, \bar{M}_1(z_1/y_1) \vdash \Delta \quad \ldots \quad \Gamma, \bar{P}, Q, R, \bar{M}_n(z_n/y_n) \vdash \Delta}{\Gamma, \bar{P}, Q, R \vdash \Delta}$$

and injective renaming σ with $Q\sigma = R\sigma = Q$ it also includes

$$\frac{\Gamma, \bar{P}\sigma, Q, \bar{M}_1\sigma(z_1/y_1\sigma) \vdash \Delta \quad \ldots \quad \Gamma, \bar{P}\sigma, Q, \bar{M}_n\sigma(z_n/y_n\sigma) \vdash \Delta}{\Gamma, \bar{P}\sigma, Q \vdash \Delta}$$

Lemma

Contraction is admissible in extensions of G3K *with geometric rules satisfying the closure condition.*

Converting frame conditions into rules: Contraction

To obtain the nice structural properties for extensions of G3K with geometric rules we need to close the rule set under contraction:

Example

For directedness

$$\frac{\Gamma, xRy, xRz, yRv, zRv \vdash \Delta}{\Gamma, xRy, xRz \vdash \Delta}\ v \text{ not in conclusion}$$

we need to add the rule which identifies y and z and contracts the two occurrences of xRy:

$$\frac{\Gamma, xRy, yRv, yRv \vdash \Delta}{\Gamma, xRy \vdash \Delta}\ v \text{ not in conclusion}$$

Remark: Closing a rule set under contraction only demands the addition of finitely many rules and thus is unproblematic!

Cut elimination for extended calculi

The so constructed geometric rules

$$\frac{\Gamma, \bar{P}, \bar{M_1}(z_1/y_1) \vdash \Delta \quad \ldots \quad \Gamma, \bar{P}, \bar{M_n}(z_n/y_n) \vdash \Delta}{\Gamma, \bar{P} \vdash \Delta}$$

have nice properties: all their active parts

- occur on the left hand side only
- consist of relational terms only
- occur in the premisses if they occur in the conclusion.

Hence we can add them to G3K without harming cut elimination!

Cut elimination for extended calculi

Theorem

If G3K* *is an extension of* G3K *by finitely many geometric rules satisfying the closure condition, then* cut_ℓ *is admissible in* G3K*.*

Proof.

As for G3K, possibly renaming variables. E.g. for directedness:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, v : A}{\Gamma \vdash \Delta, w : \Box A}\,R\Box \qquad \dfrac{w : \Box A, \Sigma, xRy, xRz, yRv, zRv \vdash \Pi}{w : \Box A, \Sigma, xRy, xRz \vdash \Pi}\,\mathit{dir}}{\Gamma, \Sigma, xRy, xRz \vdash \Delta, \Pi}\,\mathrm{cut}_\ell$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \vdash \Delta, v : A}{\Gamma \vdash \Delta, w : \Box A}\,R\Box \qquad \dfrac{w : \Box A, \Sigma, xRy, xRz, yRv, zRv \vdash \Pi}{w : \Box A, \Sigma, xRy, xRz, yRu, zRu \vdash \Pi}\,\mathit{sub}}{\Gamma, \Sigma, xRy, xRz, yRu, zRu \vdash \Delta, \Pi}\,\mathrm{cut}_\ell}{\Gamma, \Sigma, xRy, xRz \vdash \Delta, \Pi}\,\mathit{dir}$$

where u does not occur in $\Gamma, \Sigma, xRy, xRz \vdash \Delta, \Pi$. $\Box$

Where's the catch?

So, labelled sequent calculi seem ideal to treat modal logics.

However, there are some issues:

- Decidability results need to be shown for every single logic.
- since the method is based heavily on Kripke semantics, the modification for non-normal modal logics is not immediately clear (see however (Gilbert and Maffezioli, 2015) and recent work by Negri).
- The calculi are not fully internal: there seems not to be a formula translation of a labelled sequent.

Recovering labelled sequents with a formula translation

- Following (Fitting 2012) and (Goré and R. 2012), let us see how the labelled sequents might be restricted to those which support a formula translation.
- First of all, let us treat formulae in negation normal form (pushing all negations inwards onto the propositional variables)
- This preserves equivalence because in every extension of **K**:

$$\begin{aligned} \neg\Box A &= \Diamond\neg A & \qquad \neg\Diamond A &= \Box\neg A \\ \neg(A \wedge B) &= \neg A \vee \neg B & \qquad \neg(A \vee B) &= \neg A \wedge \neg B \\ \neg(A \to B) &= A \wedge \neg B & & \end{aligned}$$

- In fact, while we are at it, let us eliminate $A \to B$ in favour of $\neg A \vee B$
- Only a small apology for changing notation at this (late) stage: notation is notation, choose what works best

- With these changes, G3K can be written as follows:

$$\frac{}{\mathcal{R}, x : p, x : \overline{p}, \Gamma}\ \text{init} \qquad \frac{\mathcal{R}, x : A, x : B, \Gamma}{\mathcal{R}, x : A \vee B, \Gamma}\ \vee$$

$$\frac{\mathcal{R}, x : A, \Gamma \qquad \mathcal{R}, x : B, \Gamma}{\mathcal{R}, x : A \wedge B, \Gamma}\ \wedge \qquad \frac{\mathcal{R}, Rxy, y : A, \Gamma}{\mathcal{R}, x : \Box A, \Gamma}\ \Box^{*}$$

$$\frac{\mathcal{R}, Rxy, y : A, x : \Diamond A, \Gamma}{\mathcal{R}, Rxy, x : \Diamond A, \Gamma}\ \Diamond$$

*eigenvariable y does not occur in conclusion

- Here $\mathcal{R}$ consists of relational terms Rxy (possibly empty)
- Interpreting each Rxy as an edge (x, y), we naturally obtain a graph from $\mathcal{R}$
- So the labelled sequent $\mathcal{R}, \Gamma$ is a labelled graph

Labelled tree sequents = nested sequents

Definition

A labelled tree sequent (or LTS) is a labelled sequent $\mathcal{R}, \Gamma$ where $\mathcal{R}$ defines a tree

- A LTS calculus is a labelled sequent calculus where every sequent is a LTS
- Since a labelled tree sequent is a labelled tree, we can define its grammar:

$$\Gamma := A_1, \ldots, A_n, [\Gamma], \ldots, [\Gamma]$$

- With the added constraints: finite and non-empty
- This object is precisely a nested sequent; these have been investigated independently since (Kashima, 1994) and independently rediscovered by (Poggiolessi, 2009) and (Brünnler, 2009).

Nested sequent calculus/LTS calculus for **K**

- Notation: $\Gamma\{\Delta\}$ refers to an occurrence of the sequent Δ inside Γ. $\Gamma\{\ \}$ is called a context

$$\frac{}{\Gamma\{p, \overline{p}\}}\ \text{init} \qquad \frac{\Gamma\{A\} \qquad \Gamma\{B\}}{\Gamma\{A \wedge B\}}\ (\wedge) \qquad \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}\ (\vee)$$

$$\frac{\Gamma\{[\Delta, A], \Diamond A\}}{\Gamma\{[\Delta], \Diamond A\}}\ (\Diamond) \qquad \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}\ (\Box)$$

- NS calculi (equivalently LTS calculi) have been presented for many modal logics, intuitionistic modal logics and constructive modal logics.
- Note: in general we cannot use the structural rule extensions of G3K (to present axiomatic extensions of **K**) because they are not LTS rules. Non-structural rules are typically required.

- In these systems, a nested sequent Γ below left has the formula interpretation $\mathcal{I}(\Gamma)$ below right

  $A_1, \ldots, A_n, [\Gamma_1], \ldots, [\Gamma_m] \quad A_1 \vee \ldots \vee A_n \vee \Box\mathcal{I}(\Gamma_1) \vee \ldots \vee \Box\mathcal{I}(\Gamma_m)$

- The claim that NS calculi are more 'internal'/preferred over LS calculi because they support a formula interpretation is misleading
- More accurate: NS calculi and some LS calculi (in particular LTS calculi) support a formula interpretation. Some LS calculi seem not to.
- (Fitting 2015) extended the NS formalism to indexed nested sequents in order to give cutfree proof systems for logics like $K + \Diamond\Box p \to \Box\Diamond p$. The notational variant labelled formalism is LTS with equality (R. 2016). It is not clear if it is possible to interpret the sequents as formulae.

One final extension: the display calculus for tense logic **Kt**

- The nested sequent had a single type of nesting. Following (Goré *et al.* 2011) define a display sequent with two types of nesting $\circ[\,]$ and $\bullet[\,]$:

$$\Gamma := A_1, \ldots, A_n, \circ[\Gamma], \ldots, \circ[\Gamma], \bullet[\Gamma], \ldots, \bullet[\Gamma]$$

$$\frac{}{\Gamma, p, \overline{p}}\ \text{init} \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\ \vee \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ \wedge$$

$$\frac{\Gamma, \Delta, \Delta}{\Gamma, \Delta}\ \text{c} \qquad \frac{\Gamma}{\Gamma, \Delta}\ \text{w} \qquad \frac{\Gamma, \circ[\Delta]}{\bullet[\Gamma], \Delta}\ \text{rf} \qquad \frac{\Gamma, \bullet[\Delta]}{\circ[\Gamma], \Delta}\ \text{rp}$$

$$\frac{\Gamma, \bullet[A]}{\Gamma, \blacksquare A}\ \blacksquare \qquad \frac{\Gamma, \circ[A]}{\Gamma, \Box A}\ \Box \qquad \frac{\Gamma, \bullet[\Delta, A], \blacklozenge A}{\Gamma, \bullet[\Delta], \blacklozenge A}\ \blacklozenge \qquad \frac{\Gamma, \circ[\Delta, A], \Diamond A}{\Gamma, \circ[\Delta], \Diamond A}\ \Diamond$$

- (Kracht 1996) uses the structural rule below for a display calculus for $\mathbf{Kt} + \Diamond^h \Box^i p \to \Box^j \Diamond^k p = \mathbf{Kt} + \blacklozenge^h \Diamond^j p \to \Diamond^i \blacklozenge^k p$.

$$\frac{\Gamma, \circ^i\{\bullet^k\{\Delta\}\}}{\Gamma, \bullet^h\{\circ^j\{\Delta\}\}}\ d(h, i, j, k)$$

- The computation of these rules from axioms has a nice algorithm! Limitative results by (Kracht 1996) for tense logics (Display Theorem I), modal logic case open.