Standard Sequent Calculi for Lewis' Logics of Counterfactuals

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Abstract. We present new sequent calculi for Lewis' logics of counterfactuals. The calculi are based on Lewis' connective of comparative plausibility and modularly capture almost all logics of Lewis' family. Our calculi are standard, in the sense that each connective is handled by a finite number of rules with a fixed and finite number of premises; internal, meaning that a sequent denotes a formula in the language, and analytical. We present two equivalent versions of the calculi: in the first one, the calculi comprise simple rules; we show that for the basic case of logic $\mathbb{V}$, the calculus allows for syntactic cut-elimination, a fundamental proof-theoretical property. In the second version, the calculi comprise invertible rules, they allow for terminating proof search and semantical completeness. We finally show that our calculi can simulate the only internal (non-standard) sequent calculi previously known for these logics.

1 Introduction

In his seminal works [14], Lewis proposed a formalization of conditional logics in order to represent a kind of hypothetical reasoning that cannot be captured by the material implication of classical logic. His original motivation was to formalize counterfactuals, that is to say, conditionals of the form "if A were the case then B would be the case", where A is false. Independently from counterfactuals, conditional logics have found an interest in several fields of knowledge representation; for instance, they have been used to model belief change [10]. To this regard, a multi-agent version of Lewis' conditional logic $\mathbb{VTA}$ [2, 3] has been used to formalize epistemic change in a multi-agent setting, where the conditional operator expresses the "conditional beliefs" of an agent. In a different context, conditional logics have been used to reason about prototypical properties

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[8, 5], and to provide an axiomatic foundation of non-monotonic reasoning [11], in which a conditional $A \boxright B$ is read as "in normal circumstances, if A then B".

The family of logics studied by Lewis is semantically characterized by sphere models, a particular kind of neighbourhood models introduced by Lewis himself. In Lewis' terminology, a *sphere* denotes a set of worlds; in sphere models, each world is equipped with a nested system of such spheres. From the viewpoint of the given world, inner sets represent the "most plausible worlds", while worlds belonging only to outer sets are considered as less plausible. In order to treat the conditional operator, Lewis takes as primitive the comparative plausibility connective $\preccurlyeq$: a formula $A \preccurlyeq B$ means "A is at least as plausible as B". The conditional $A \boxright B$ can be then defined as "A is impossible" or "$A \wedge \neg B$ is less plausible than $A \wedge B$". However, the latter assertion is equivalent to the simpler one "$A \wedge \neg B$ is less plausible than A"[4].

From the point of view of proof theory and automated deduction, conditional logics do not have a state of the art comparable with, say, the one of modal logics, for which there exist well-established calculi with well-understood proof-theoretical and computational properties. Calculi for some weaker conditional logics are given, e.g., in [1, 18] and more recently in [19, 15]. Regarding Lewis' counterfactual logics, external labelled calculi have been proposed in [9] and in [16], both based on a relational reformulation of the sphere semantics. We are interested in *internal* sequent calculi, where a sequent denotes a formula of the language. Calculi of this kind have been proposed by Gent [7] and de Swart [20], and more recently in [12, 13]. They are analytical and provide a decision procedure for the respective logics; on the other hand, they comprise an infinite set of rules with a variable number of premises.

Our aim is to provide internal calculi for the whole family of Lewis' logics. We sought the calculi to display the following features: (i) they should be *standard*, i.e. each connective should be handled by a fixed finite set of rules with a fixed finite set of premises; (ii) they should be *modular*, i.e. it should be possible to obtain calculi for stronger logics adding independent rules to calculi for weaker ones; (iii) they should have good proof-theoretical properties, first they should allow a syntactic proof of cut admissibility; (iv) they should provide a decision procedure for the respective logics; finally (v) they should be of optimal complexity with respect to the known complexity of the logic. In our opinion requirement (i) is particularly important: a standard calculus could provide a self-explanatory presentation of the logic, thus a kind of proof-theoretic semantics. A first step in this direction is the calculus $\mathcal{I}_V$ presented in [17] for logic $\mathbb{V}$: it is internal and it is formulated in terms of structured sequents containing blocks encoding disjunctions of $\preccurlyeq$-formulas. The calculus provides an optimal decision procedure for $\mathbb{V}$; however, no syntactic proof of cut admissibility is known for it.

In this work we make a further step towards the objectives mentioned above, extending the results of [17]. We present internal, standard, cut-free calculi for most logics of the Lewis family, namely logics $\mathbb{V}$, $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$, $\mathbb{VC}$, $\mathbb{VA}$ and $\mathbb{VNA}$ (hereafter denoted by $\mathcal{L}$). Our calculi make use of a simplified block structure with

[4] It is worth noticing that in turn the connective $\preccurlyeq$ can be defined in terms of $\boxright$.

respect to $\mathcal{I}_V$. We first present the calculi $\mathcal{I}_{\mathcal{L}}$, containing particularly perspicuous non-invertible rules together with explicit contraction rules. As a preliminary result we provide a syntactic proof of the admissibility of the cut rule for the basic case of logic $\mathbb{V}$, obtaining, as a by-product, a syntactic proof of completeness of the calculus. We then present the calculi $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$, an alternative version of $\mathcal{I}_{\mathcal{L}}$ with invertible rules and provably admissible contraction rules. We show that calculi $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ are equivalent to $\mathcal{I}_{\mathcal{L}}$, and that they allow terminating proof-search; therefore they provide a decision procedure for the respective logics. Moreover, we also prove the semantic completeness of $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ calculi for all logics of Lewis family not including the absoluteness condition. As a final result, we show that calculi $\mathcal{I}_{\mathcal{L}}$ (whence $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$) can simulate the non-standard calculi of [12, 13]. This result is interesting in itself as it clarifies the relation between rather different proof-systems, and moreover it provides an alternative completeness proof of both $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ calculi, in particular for the missing cases of logics $\mathbb{VA}$ and $\mathbb{VNA}$. For the remaining logics of Lewis' family such as $\mathbb{VTA}$, $\mathbb{VWA}$, and $\mathbb{VCA}$ the issue of completeness of our calculi is open and will be dealt with in future research.

2 Preliminaries

We consider the *conditional logics* defined by Lewis in [14]. The set of *conditional formulae* is given by $\mathcal{F} ::= p \mid \bot \mid \mathcal{F} \to \mathcal{F} \mid \mathcal{F} \preccurlyeq \mathcal{F}$, where $p \in \mathcal{V}$ is a propositional variable. The other boolean connectives are defined in terms of $\bot, \to$ as usual. Intuitively, a formula $A \preccurlyeq B$ is interpreted as "A is at least as plausible as B".

As mentioned above, Lewis' counterfactual implication $\Box\!\!\rightarrow$ can be defined in terms of comparative plausibility $\preccurlyeq$ as $A \Box\!\!\rightarrow B \;\equiv\; (\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq A)$.

The semantics of this logic is defined by Lewis in terms of *sphere semantics*:

Definition 1. *A* sphere model *(or* model*) is a triple $\langle W, \mathsf{SP}, [\![.]\!]\rangle$, consisting of a non-empty set W of elements, called* worlds*, a mapping $\mathsf{SP} : W \to \mathcal{P}(\mathcal{P}(W))$, and a propositional valuation $[\![.]\!] : \mathcal{V} \to \mathcal{P}(W)$. Elements of $\mathsf{SP}(x)$ are called* spheres*. We assume the following conditions: for every $\alpha \in \mathsf{SP}(w)$ we have $\alpha \neq \emptyset$, and for every $\alpha, \beta \in \mathsf{SP}(w)$ we have $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. The latter condition is called* sphere nesting*.*

The valuation $[\![.]\!]$ is extended to all formulae by: $[\![\bot]\!] = \emptyset$; $[\![A \to B]\!] = (W - [\![A]\!]) \cup [\![B]\!]$; $[\![A \preccurlyeq B]\!] = \{w \in W \mid$ for all $\alpha \in \mathsf{SP}(w)$. if $[\![B]\!] \cap \alpha \neq \emptyset$, then $[\![A]\!] \cap \alpha \neq \emptyset\}$. For $w \in W$ we also write $w \Vdash A$ instead of $w \in [\![A]\!]$. As for spheres, we write $\alpha \Vdash^{\forall} A$ meaning $\forall x \in \alpha.\ x \Vdash A$ and $\alpha \Vdash^{\exists} A$ meaning $\exists x \in \alpha.\ x \Vdash A$[5]. Validity and satisfiability of formulae in a class of models are defined as usual. Conditional logic $\mathbb{V}$ *is the set of formulae valid in all sphere models.*

Extensions of $\mathbb{V}$ are semantically given by specifying additional conditions on the class of sphere models, namely:

- *normality*: for all $w \in W$ we have $\mathsf{SP}(w) \neq \emptyset$;

[5] Employing this notation, satisfiability of a $\preccurlyeq$-formula in a model becomes the following: $x \Vdash A \preccurlyeq B$ iff for all $\alpha \in \mathsf{SP}(x)$. $\alpha \Vdash^{\forall} \neg B$ or $\alpha \Vdash^{\exists} A$.

CPR $\dfrac{\vdash B \to A}{\vdash A \preccurlyeq B}$	CPA $(A \preccurlyeq A \lor B) \lor (B \preccurlyeq A \lor B)$
TR $(A \preccurlyeq B) \land (B \preccurlyeq C) \to (A \preccurlyeq C)$	CO $(A \preccurlyeq B) \lor (B \preccurlyeq A)$
N $\neg(\bot \preccurlyeq \top)$	W $A \to (A \preccurlyeq \top)$
T $(\bot \preccurlyeq \neg A) \to A$	A1 $(A \preccurlyeq B) \to (\bot \preccurlyeq \neg(A \preccurlyeq B))$
C $(A \preccurlyeq \top) \to A$	A2 $\neg(A \preccurlyeq B) \to (\bot \preccurlyeq (A \preccurlyeq B))$

$\mathcal{A}_{\mathbb{V}} := \{\mathsf{CPR}, \mathsf{CPA}, \mathsf{TR}, \mathsf{CO}\}$

$\mathcal{A}_{\mathbb{VN}} := \mathcal{A}_{\mathbb{V}} \cup \{\mathsf{N}\}$	$\mathcal{A}_{\mathbb{VT}} := \mathcal{A}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}\}$	$\mathcal{A}_{\mathbb{VW}} := \mathcal{A}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}, \mathsf{W}\}$
$\mathcal{A}_{\mathbb{VC}} := \mathcal{A}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}, \mathsf{W}, \mathsf{C}\}$	$\mathcal{A}_{\mathbb{VA}} := \mathcal{A}_{\mathbb{V}} \cup \{\mathsf{A1}, \mathsf{A2}\}$	$\mathcal{A}_{\mathbb{VNA}} := \mathcal{A}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{A1}, \mathsf{A2}\}$

Table 1. Lewis' logics and axioms.

- *total reflexivity*: for all $w \in W$ we have $w \in \bigcup \mathsf{SP}(w)$;
- *weak centering*: normality holds and for all $\alpha \in \mathsf{SP}(w)$ we have $w \in \alpha$;
- *centering*: for all $w \in W$ we have $\{w\} \in \mathsf{SP}(w)$;
- *absoluteness*: for all $w, v \in W$ we have $\mathsf{SP}(w) = \mathsf{SP}(v)$[6].

Extensions of $\mathbb{V}$ are denoted by concatenating the letters for these properties: $\mathbb{N}$ for normality, $\mathbb{T}$ for total reflexivity, $\mathbb{W}$ for weak centering, $\mathbb{C}$ for centering, and $\mathbb{A}$ for absoluteness. All the above logics can be characterized by axioms in a Hilbert-style system [14, Chp. 6]. The modal axioms formulated in the language with only the comparative plausibility operator are presented in Table 1 (where $\lor$ and $\land$ bind stronger than $\preccurlyeq$). The propositional axioms and rules are standard.

3 A sequent calculus for Lewis' logic and extensions

We propose internal sequent calculi for the basic Lewis' logic $\mathbb{V}$ as well as for some extensions. Our calculi are based on a modification of the sequent format from [17]. To make contraction explicit we consider sequents based on multisets, and write Γ, Δ for multiset union and A^n for the multiset containing n copies of the formula A. The basic constituent of sequents are *blocks* of the form $[A_1, \ldots, A_m \lhd A]$, with $A_1, \ldots, A_m, A$ formulas, representing disjunctions of $\preccurlyeq$-formulas.

Definition 2. *A* block *is a tuple consisting of a multiset Σ of formulae and a single formula A, written $[\Sigma \lhd A]$. A* sequent *is a tuple $\Gamma \Rightarrow \Delta$, where Γ is a multiset of conditional formulae, and Δ is a multiset of conditional formulae and blocks. The* formula interpretation *of a sequent is given by (all blocks shown):*

$$\iota(\Gamma \Rightarrow \Delta', [\Sigma_1 \lhd A_1], \ldots, [\Sigma_n \lhd A_n]) := \bigwedge \Gamma \to \bigvee \Delta' \lor \bigvee_{1 \le i \le n} \bigvee_{B \in \Sigma_i} (B \preccurlyeq A_i)$$

Table 2 presents non-invertible calculi for logic $\mathbb{V}$ and its extensions, including rules for contraction both on the sequent level and inside blocks[7]. We write $[\Theta, \Sigma \lhd A]$ for $[(\Theta, \Sigma) \lhd A]$, with Θ, Σ standing for multiset union.

[6] Lewis' original presentation in [14] is slightly different: he did not assume the general condition on sphere models that for every $\alpha \in \mathsf{SP}(w)$: $\alpha \neq \emptyset$, and formulated normality as $\forall w \in W : \bigcup \mathsf{SP}(w) \neq \emptyset$ and weak centering as normality plus $\forall w \in W\ \alpha \in \mathsf{SP}(w)$, if $\alpha \neq \emptyset$ then $w \in \alpha$. Furthermore, note that absoluteness can be equally stated as *local absoluteness*: $\forall w \in W \forall v \in \bigcup \mathsf{SP}(w)\ \mathsf{SP}(w) = \mathsf{SP}(v)$.

[7] Actually, the rules Con_S and Con_B are not needed for completeness (refer to Sct. 6); we have included them in our official formulation of the calculi for technical convenience.

$$
\dfrac{}{\Gamma, \bot \Rightarrow \Delta}\ \bot_L \qquad \dfrac{}{\Gamma, p \Rightarrow \Delta, p}\ \mathsf{init} \qquad \dfrac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}\ \to_L \qquad \dfrac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ \to_R
$$

$$
\dfrac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B}\ \preccurlyeq_R \qquad \dfrac{\Gamma \Rightarrow \Delta, [D, \Sigma \lhd A] \quad \Gamma \Rightarrow \Delta, [\Sigma \lhd C]}{\Gamma, C \preccurlyeq D \Rightarrow \Delta, [\Sigma \lhd A]}\ \preccurlyeq_L
$$

$$
\dfrac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A] \quad \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}\ \mathsf{com} \qquad \dfrac{A \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{jump}
$$

$$
\dfrac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \mathsf{Con}_L \qquad \dfrac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \mathsf{Con}_R \qquad \dfrac{\Gamma \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd A]}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{Con}_S
$$

$$
\dfrac{\Gamma \Rightarrow \Delta, [\Sigma, A, A \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma, A \lhd B]}\ \mathsf{Con}_B \qquad \dfrac{\Gamma \Rightarrow \Delta, [\bot \lhd \top]}{\Gamma \Rightarrow \Delta}\ \mathsf{N} \qquad \dfrac{\Gamma \Rightarrow \Delta, B \qquad \Gamma \Rightarrow \Delta, [\bot \lhd A]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta}\ \mathsf{T}
$$

$$
\dfrac{\Gamma \Rightarrow \Delta, \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{W} \qquad \dfrac{\Gamma, C \Rightarrow \Delta \qquad \Gamma \Rightarrow D, \Delta}{\Gamma, C \preccurlyeq D \Rightarrow \Delta}\ \mathsf{C} \qquad \dfrac{\Gamma^{\preccurlyeq}, B \Rightarrow \Delta^{\preccurlyeq}, \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd B]}\ \mathsf{A}
$$

Here $\Gamma^{\preccurlyeq} \Rightarrow \Delta^{\preccurlyeq}$ is $\Gamma \Rightarrow \Delta$ restricted to formulae of the form $C \preccurlyeq D$ and blocks.

$$
\mathcal{I}_{\mathbb{V}} \ := \ \{\bot_L, \mathsf{init}, \to_L, \to_R, \preccurlyeq_R, \preccurlyeq_L, \mathsf{com}, \mathsf{jump}, \mathsf{Con}_R, \mathsf{Con}_L, \mathsf{Con}_S\}
$$

$$
\begin{array}{lll}
\mathcal{I}_{\mathbb{VN}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}\} & \mathcal{I}_{\mathbb{VW}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}, \mathsf{W}\} & \mathcal{I}_{\mathbb{VA}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{A}\} \\
\mathcal{I}_{\mathbb{VT}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}\} & \mathcal{I}_{\mathbb{VC}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}, \mathsf{W}, \mathsf{C}\} & \mathcal{I}_{\mathbb{VNA}} := \mathcal{I}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{A}\}
\end{array}
$$

Table 2. The calculus $\mathcal{I}_{\mathbb{V}}$ and its extensions

For notational convenience in the following we take $\mathcal{L}$ to range over the logics $\mathbb{V}, \mathbb{VN}, \mathbb{VT}, \mathbb{VW}, \mathbb{VC}, \mathbb{VA}, \mathbb{VNA}$, unless specified otherwise. As usual, given a formula $G \in \mathcal{L}$, in order to check whether G is valid we look for a derivation of $\Rightarrow G$. Given a sequent $\Gamma \Rightarrow \Delta$, we say that it is derivable, written $\mathcal{I}_{\mathcal{L}} \vdash \Gamma \Rightarrow \Delta$, if it admits a *derivation*, namely a tree where the root is $\Gamma \Rightarrow \Delta$, every leaf is an instance of axioms init or $\bot_L$, and every non-leaf node is (an instance of) the conclusion of a rule having (an instance of) the premises of the rule as children.

Given the definition of $\Box\!\!\!\rightarrow$ in terms of $\preccurlyeq$, rules for counterfactual implication can be explicitly stated as follows:

$$
\dfrac{\bot \preccurlyeq A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, [A \wedge \neg B \lhd A]}{A \Box\!\!\!\rightarrow B, \Gamma \Rightarrow \Delta}\ \Box\!\!\!\rightarrow_L \qquad \dfrac{(A \wedge \neg B) \preccurlyeq A, \Gamma \Rightarrow \Delta, [\bot \lhd A]}{\Gamma \Rightarrow \Delta, A \Box\!\!\!\rightarrow B}\ \Box\!\!\!\rightarrow_R
$$

Theorem 3 (Soundness). *If $\mathcal{I}_{\mathcal{L}} \vdash \Gamma \Rightarrow \Delta$, then $\iota(\Gamma \Rightarrow \Delta)$ is a theorem of $\mathcal{L}$.*

Example 4. To illustrate the use of the calculus we show a derivation of the characteristic axiom $(\bot \preccurlyeq \neg A) \to A$ for logic $\mathbb{VT}$ in the calculus $\mathcal{I}_{\mathbb{VW}}$ and a derivation of it in the calculus $\mathbb{VC}$ (where $\neg A = (A \to \bot)$):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A, \bot, \bot}\ \mathsf{init}}{\Rightarrow A, A \to \bot, \bot}\ \to_R}{\Rightarrow A, [(A \to \bot), \bot \lhd \top]}\ \mathsf{W} \qquad \dfrac{\dfrac{}{\bot \Rightarrow \bot}\ \bot_L}{\Rightarrow A, [\bot \lhd \bot]}\ \mathsf{jump}}{\bot \preccurlyeq (A \to \bot) \Rightarrow A, [\bot \lhd \top]}\ \preccurlyeq_L}{\bot \preccurlyeq (A \to \bot) \Rightarrow A}\ \mathsf{N}}{\Rightarrow (\bot \preccurlyeq (A \to \bot)) \to A}\ \to_R
\qquad
\dfrac{\dfrac{}{\bot \Rightarrow A}\ \bot_L \qquad \dfrac{\dfrac{}{A \Rightarrow A, \bot}\ \mathsf{init}}{\Rightarrow A, A \to \bot}\ \to_R}{\bot \preccurlyeq (A \to \bot) \Rightarrow A}\ \mathsf{C}
$$

Therefore, rule T could be omitted in the rule sets $\mathcal{I}_{\mathbb{VW}}$ and $\mathcal{I}_{\mathbb{VC}}$.

Completeness of the calculi is shown in next section. We now provide the cut elimination proof in presence of the contraction rules (Con_L, Con_R, Con_S and Con_B). The general strategy, adapted from the hypersequent setting [4], consists of eliminating topmost applications of cut of maximal complexity by first permuting them into the left premise until we reach an occurrence of the cut formula which is principal, and then permuting them into the right one. The cut rules are:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{cut}_1 \qquad \frac{\Gamma \Rightarrow \Delta, [\Omega \lhd A] \quad \Sigma \Rightarrow \Pi, [A, \Theta \lhd B]}{\Gamma, \Sigma \Rightarrow \Delta, \Pi\ [\Omega, \Theta \lhd B]}\ \mathsf{cut}_2$$

Definition 5. *We write $\mathcal{I}_{\mathcal{L}}\mathsf{Cut}$ for the calculus $\mathcal{I}_{\mathcal{L}}$ extended with the cut rules cut_1 and cut_2. The* complexity *of an application of cut_1 or cut_2 is the complexity of the cut formula, i.e., the number $|A|$ of symbols of the cut formula A. Given a derivation $\mathcal{D}$ in $\mathcal{I}_{\mathcal{L}}\mathsf{Cut}$, its* formula cut rank *$\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D})$ is the maximal complexity of an application of cut_1 in it. Analogously, its* structural cut rank *$\mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D})$ is the maximal complexity of an application of cut_2 in it. The* height *of a derivation is the number of nodes of its longest branch minus one. Thus, a derivation of height 0 is an axiom. We write $\mathcal{I}_{\mathcal{L}} \vdash_n \Gamma \Rightarrow \Delta$ if there exists a derivation of height n in $\mathcal{I}_{\mathcal{L}}$ with endsequent $\Gamma \Rightarrow \Delta$. Similarly for $\mathcal{I}_{\mathcal{L}}\mathsf{Cut}$.*

By straightforward induction on the height of the derivation we obtain:

Lemma 6. *The* weakening rules *are height-preserving admissible in $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{L}}\mathsf{Cut}$, i.e. (using the uniform notation $\mathcal{I}_{\mathcal{L}}(\mathsf{Cut})$ for both cases): If $\mathcal{I}_{\mathcal{L}}(\mathsf{Cut}) \vdash_n \Gamma \Rightarrow \Delta$, then $\mathcal{I}_{\mathcal{L}}(\mathsf{Cut}) \vdash_n \Gamma, \Sigma \Rightarrow \Delta, \Pi$ and if $\mathcal{I}_{\mathcal{L}}(\mathsf{Cut}) \vdash_n \Gamma \Rightarrow \Delta, [\Sigma \lhd A]$, then $\mathcal{I}_{\mathcal{L}}(\mathsf{Cut}) \vdash_n \Gamma \Rightarrow \Delta, [\Sigma, \Omega \lhd A]$. Moreover, both the formula cut rank and the structural cut rank are preserved.*

Lemma 7 (cut_1-reduction). *Suppose $\mathcal{I}_{\mathbb{V}}\mathsf{Cut} \vdash \Gamma \Rightarrow \Delta, A^n$ and $\mathcal{I}_{\mathbb{V}}\mathsf{Cut} \vdash A^m, \Sigma \Rightarrow \Pi$ by derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ with $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}_1) < |A| > \mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}_2)$ and $\mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D}_1) < |A| > \mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D}_2)$, where A^n and A^m are n and m occurrences of A. Then there is a derivation $\mathcal{D}$ in $\mathcal{I}_{\mathbb{V}}\mathsf{Cut}$ of $\Gamma, \Sigma \Rightarrow \Delta, \Pi$ with $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}) < |A| > \mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D})$.*

Proof. By induction on the sum of the heights of $\mathcal{D}_1$ and $\mathcal{D}_2$. We write R_1 and R_2 for the last rules in $\mathcal{D}_1$ resp. $\mathcal{D}_2$, and count the atom p in init and the contracted formula in the contraction rules as principal. If none of the occurrences of A is principal in R_1, we apply the induction hypothesis on the premise(s) of R_1 followed by R_1. Otherwise, if none of the occurrences of A is principal in R_2, we apply the induction hypothesis to the premise(s) of R_2 followed by R_2.

If at least one occurrence of A was principal both in R_1 and R_2, we apply the induction hypothesis to the premise(s) of R_1 and the conclusion of R_2 and vice versa to delete the occurrences of A in the context. If either of the rules was a contraction rule we are done, otherwise apply cut_1 or cut_2 on formulae of smaller complexity. The propositional cases are standard, the case where $A = C \preccurlyeq D$ is straightforward. Applying contraction rules then yields the result. □

Lemma 8 (Shift-right). *Suppose for $k_1, \ldots, k_n \geq 1$ we have $\mathcal{I}_{\mathbb{V}}\mathsf{Cut}$-derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\Gamma \Rightarrow \Delta, [\Omega \lhd A]$ and $\Sigma \Rightarrow \Pi, \left[A^{k_1}, \Theta_1 \lhd B_1\right], \ldots, \left[A^{k_n}, \Theta_n \lhd B_n\right]$*

respectively with $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}_1) \leq |A| \geq \mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}_2)$ *and* $\mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D}_1) < |A| > \mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D}_2)$ *such that the last applied rule in* $\mathcal{D}_1$ *is* jump*. Then there is a derivation* $\mathcal{D}$ *in* $\mathcal{I}_{\mathbb{V}}\mathsf{Cut}$ *with* $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}) \leq |A| > \mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D})$ *of the sequent*

$$\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega, \Theta_1 \lhd B_1], \ldots, [\Omega, \Theta_n \lhd B_n]$$

Proof. By induction on the height of $\mathcal{D}_2$, distinguishing cases according to the last applied rule R. If R is a rule other than $\mathsf{jump}, \mathsf{com}$ we apply the induction hypothesis to the premise(s) of R, followed by R if necessary. In particular, the general induction hypothesis immediately takes care of Con_S and Con_B. If R is jump, we apply cut_1 several times to the occurrence of A in the premise of the application of jump in $\mathcal{D}_1$ and the occurrences of A in the premise of R, followed by applications of Con_L and an application of jump. These new cuts have complexity $|A|$. If R is com, again we apply the induction hypothesis on the premises of R, but now we might need to apply weakening inside a block before applying com again. □

Lemma 9 (cut_2-reduction). *Suppose we have* $\mathcal{I}_{\mathbb{V}}$*-derivations* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *of* $\Gamma \Rightarrow \Delta, [\Omega_1 \lhd A], \ldots, [\Omega_n \lhd A]$ *and* $\Sigma \Rightarrow \Pi, [A, \Theta \lhd B]$ *with* $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}_1) \leq |A| \geq \mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}_2)$ *and* $\mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D}_1) < |A| > \mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D}_2)$*. Then there is a derivation* $\mathcal{D}$ *in* $\mathcal{I}_{\mathbb{V}}\mathsf{Cut}$ *with* $\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}) \leq |A| > \mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D})$ *of the sequent*

$$\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega_1, \Theta \lhd B], \ldots, [\Omega_n, \Theta \lhd B]$$

Proof. By induction on the height of $\mathcal{D}_1$, distinguishing cases according to the last applied rule R. If none of the occurrences of A in the conclusion of R is in an active block we apply the induction hypothesis to the premise(s) of R followed by an application of R. Suppose A occurs in an active block. If R is com or $\preccurlyeq_L$ we apply the induction hypothesis on the premises, followed possibly by admissibility of Weakening (Lem. 6) and finally an application of R. If R is Con_B, we simply apply the induction hypothesis to its premise. If R is jump, we apply Lem. 8. □

Theorem 10 (Cut Elimination). *If* $\mathcal{I}_{\mathbb{V}}\mathsf{Cut} \vdash \Gamma \Rightarrow \Delta$*, then* $\mathcal{I}_{\mathbb{V}} \vdash \Gamma \Rightarrow \Delta$*. In particular, there is a procedure to eliminate cuts from a derivation in* $\mathcal{I}_{\mathbb{V}}\mathsf{Cut}$*.*

Proof. We show how to convert an $\mathcal{I}_{\mathbb{V}}\mathsf{Cut}$-derivation $\mathcal{D}$ into a cut-free derivation with same conclusion by induction on the tuples $\langle \mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}), \#_{\mathsf{cut}_2}(\mathcal{D}), \#_{\mathsf{cut}_1}(\mathcal{D})\rangle$ in the lexicographic ordering, where $\#_{\mathsf{cut}_1}(\mathcal{D})$ is the number of applications of cut_1 in $\mathcal{D}$ with cut formula of complexity $\max\{\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}), \mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D})\}$, and analogous for $\#_{\mathsf{cut}_2}(\mathcal{D})$ with respect to cut_2. A topmost application of cut_1 with complexity $\max\{\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}), \mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D})\}$ is eliminated using Lem. 7. A topmost application of cut_2 with complexity $\max\{\mathsf{rk}_{\mathsf{cut}_1}(\mathcal{D}), \mathsf{rk}_{\mathsf{cut}_2}(\mathcal{D})\}$ is eliminated using Lem. 9. It follows from the lemmas that in both cases the induction measure decreases. □

As a consequence of the admissibility of cut, we can provide a syntactical proof of completeness of logic $\mathbb{V}$:

Corollary 11 (Completeness via cut elimination). *If a formula* F *is valid in* $\mathbb{V}$*, then there is a derivation of* $\Rightarrow F$ *in* $\mathcal{I}_{\mathbb{V}}$*.*

$$\frac{}{\Gamma, \bot \Rightarrow \Delta}\ \bot_L \qquad \frac{}{\Gamma, p \Rightarrow \Delta, p}\ \mathsf{init} \qquad \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}\ \to_L \qquad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ \to_R$$

$$\frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B}\ \preccurlyeq_R \qquad \frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd A], [\Sigma \lhd C]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C]}\ \preccurlyeq^{\mathsf{i}}_L$$

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}\ \mathsf{com}^{\mathsf{i}}$$

$$\frac{A \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{jump} \qquad \frac{\Gamma \Rightarrow \Delta, [\bot \lhd \top]}{\Gamma \Rightarrow \Delta}\ \mathsf{N}$$

$$\frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, B \qquad \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\bot \lhd A]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta}\ \mathsf{T}^{\mathsf{i}} \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma \lhd A], \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd A]}\ \mathsf{W}^{\mathsf{i}}$$

$$\frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, B \quad \Gamma, A \preccurlyeq B, A \Rightarrow \Delta}{\Gamma, A \preccurlyeq B \Rightarrow \Delta}\ \mathsf{C}^{\mathsf{i}} \qquad \frac{\Gamma^{\preccurlyeq}, B \Rightarrow \Delta^{\preccurlyeq}, [\Sigma \lhd B], \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd B]}\ \mathsf{A}^{\mathsf{i}}$$

Here $\Gamma^{\preccurlyeq} \Rightarrow \Delta^{\preccurlyeq}$ is $\Gamma \Rightarrow \Delta$ restricted to formulae of the form $C \preccurlyeq D$ and blocks.

$$\mathcal{I}^{\mathsf{i}}_{\mathbb{V}} \ := \ \{\bot_L, \mathsf{init}, \to_L, \to_R, \preccurlyeq_R, \preccurlyeq^{\mathsf{i}}_L, \mathsf{com}^{\mathsf{i}}, \mathsf{jump}\}$$

$$\mathcal{I}^{\mathsf{i}}_{\mathbb{VN}} := \mathcal{I}^{\mathsf{i}}_{\mathbb{V}} \cup \{\mathsf{N}\} \qquad \mathcal{I}^{\mathsf{i}}_{\mathbb{VW}} := \mathcal{I}^{\mathsf{i}}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}^{\mathsf{i}}, \mathsf{W}^{\mathsf{i}}\} \qquad \mathcal{I}^{\mathsf{i}}_{\mathbb{VA}} := \mathcal{I}^{\mathsf{i}}_{\mathbb{V}} \cup \{\mathsf{A}^{\mathsf{i}}\}$$
$$\mathcal{I}^{\mathsf{i}}_{\mathbb{VT}} := \mathcal{I}^{\mathsf{i}}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}^{\mathsf{i}}\} \qquad \mathcal{I}^{\mathsf{i}}_{\mathbb{VC}} := \mathcal{I}^{\mathsf{i}}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{T}^{\mathsf{i}}, \mathsf{W}^{\mathsf{i}}, \mathsf{C}^{\mathsf{i}}\} \qquad \mathcal{I}^{\mathsf{i}}_{\mathbb{VNA}} := \mathcal{I}^{\mathsf{i}}_{\mathbb{V}} \cup \{\mathsf{N}, \mathsf{A}^{\mathsf{i}}\}$$

Table 3. The invertible calculus $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ and its extensions

Proof. By deriving the rules and axioms of the Hilbert-calculus for $\mathbb{V}$ (Tab. 1) in $\mathcal{I}_{\mathbb{V}}\mathsf{Cut}$ and using Thm. 10. For rule CPR from $\Rightarrow B \to A$ by propositional rules and cut_1 we obtain $B \Rightarrow A$, and applications of jump and $\preccurlyeq_R$ yield $\Rightarrow A \preccurlyeq B$. □

4 The invertible calculus

In Table 3 we present fully invertible calculi for Lewis' logics. The equivalence between $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ is proved via admissibility of weakening and contraction; furthermore, we shall use $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ to semantically prove completeness of logics $\mathbb{V}$, $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$ and $\mathbb{VC}$. It can be shown that weakening is height preserving admissible in $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ and its extensions, and that all the rules are invertible, with the exception of jump and A^{i}. Given these properties, we can prove that:

Lemma 12 (Adm. of Contraction). *1. Rules Con_L and Con_R are admissible in $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$; 2. Rule Con_S is admissible in $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$; 3. Rule Con_B is admissible in $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$.*

Theorem 13 (Equivalence). *For A arbitrary formula, A is derivable in the calculus $\mathcal{I}_{\mathcal{L}}$ iff A is derivable in the invertible calculus $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$.*

Proof. Both directions are proved by easy induction on the height of the derivation, modulo weakening and contraction. Note that for the **[if]** direction application of weakening is justified, since the rule is admissible in the calculus $\mathcal{I}_{\mathcal{L}}$, and for direction **[only if]** applications of weakening and contraction are legitimate since both rules are admissible in $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$. □

Standard reasoning shows that the calculi $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ can be used in a decision procedure for the logic $\mathcal{L}$ as follows. Since contractions and weakenings are admissible we may assume that a derivation of a duplication-free sequent (containing duplicates neither of formulae nor of blocks) only contains duplication-free sequents:

whenever a (backwards) application of a rule introduces a duplicate of a formula already in the sequent, it is immediately deleted in the next step using a backwards application of weakening. While officially our calculi do not contain the weakening rules, the proof of admissibility of weakening yields a procedure to transform a derivation with these rules into one without. Since all rules have the subformula property, the number of duplication-free sequents possibly relevant to a derivation of a sequent is bounded in the number of subformulae of that sequent, and hence enumerating all possible loop-free derivations of the above form yields a decision procedure for the logic. This argument is sufficient to show termination; however, it is clear that the complexity of the resulting procedure is far from the optimal PSPACE or coNP complexities of the logics [6, 20].

Theorem 14. *Proof search for a sequent $\Gamma \Rightarrow \Delta$ in calculus $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$ always comes to an end in a finite number of steps.*

5 Semantic Completeness

In this section we prove the semantic completeness of $\mathcal{I}^{\mathsf{i}}_{\mathcal{L}}$. In order to simplify the proof we adopt a cumulative version of rules $\to_L$, $\to_R$, $\preccurlyeq_R$ and $\mathsf{com}^{\mathsf{i}}$. This allows us to consider only the upper sequent of each derivation branch, instead of taking into account whole branches of the derivation.

$$\frac{\Gamma, A \to B, B \Rightarrow \Delta \quad \Gamma, A \to B \Rightarrow \Delta, B}{\Gamma, A \to B \Rightarrow \Delta} \to^{\mathsf{c}}_L$$

$$\frac{\Gamma, A \Rightarrow \Delta, A \to B, B}{\Gamma \Rightarrow \Delta, A \to B} \to^{\mathsf{c}}_R \qquad \frac{\Gamma \Rightarrow \Delta, A \preccurlyeq B, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B} \preccurlyeq^{\mathsf{c}}_R$$

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_1 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd B]\, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]} \mathsf{com}^{\mathsf{c}}$$

Definition 15. *The modal degree of a formula resp. sequent is defined as follows: $md(\bot) = md(P) = 0$, for P atomic formula; $md(A \to B) = max(md(A), md(B))$; $md(A \preccurlyeq B) = max(md(A), md(B)) + 1$; $md([\Sigma \lhd A]) = max(md(\Sigma), md(A)) + 1$; $md(\Gamma \Rightarrow \Delta) = max\{md(G) \mid G \in \Gamma \cup \Delta,\ G$ formula or block$\}$.*

Proposition 16. *All rules of $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$ preserve the modal degree: the premises of the rule have a modal degree no greater than the one of the respective conclusion.*

Observe that jump is the only rule which decreases the modal degree. Furthermore, an application of a rule is said to be *redundant* if the conclusion of the rule can be derived from one of its premises by weakening or contraction. If a sequent is derivable it has a non redundant derivation, since the redundant applications of the rules can be removed without affecting the correctness of the derivation. If an application of $\mathsf{com}^{\mathsf{c}}$ is non redundant, then it must respect the restriction $(*)$ $\Sigma_1 \nsubseteq \Sigma_2$ and $\Sigma_2 \nsubseteq \Sigma_1$. To see this: if $(*)$ is not respected then either $\Sigma_1 \subseteq \Sigma_2$ or $\Sigma_2 \subseteq \Sigma_1$; in both cases we get a redundant application of $\mathsf{com}^{\mathsf{c}}$.

Definition 17. *A sequent is saturated if it has the form* $\Pi_1 \Rightarrow \Pi_2, [\Sigma_1 \lhd C_1], ..., [\Sigma_n \lhd C_n]$ *where* Π_1, Π_2 *are a multi-set of formulas such that* (init) $\Pi_1 \cap \Pi_2 = \emptyset$*;* $(\bot_L)$ $\bot \notin \Pi_1$ *and* $\top \notin \Pi_2$*;* $(\rightarrow^{\mathsf{c}}_L)$ *if* $A \rightarrow B \in \Pi_1$ *then either* $A \in \Pi_2$ *or* $B \in \Pi_1$*;* $(\rightarrow^{\mathsf{c}}_R)$ *if* $A \rightarrow B \in \Pi_2$ *then* $A \in \Pi_1$ *and* $B \in \Pi_2$*;* (com$^{\mathsf{c}}$) *for every* $[\Sigma_i \lhd C_i]$, $[\Sigma_j \lhd C_j]$ *it holds that either* $\Sigma_i \subseteq \Sigma_j$ *or* $\Sigma_j \subseteq \Sigma_i$*;* $(\preccurlyeq^{\mathsf{c}}_R)$ *for every* $A \preccurlyeq B \in \Pi_2$ *it holds that* $[A \lhd B] \in \{[\Sigma_1 \lhd C_1], ..., [\Sigma_n \lhd C_n]\}$*;* $(\preccurlyeq^{\mathsf{i}}_L)$ *for every* $A \preccurlyeq B \in \Pi_1$ *and for every* $[\Sigma_i \lhd C_i]$*, where* $1 \leqslant i \leqslant n$*, it holds that either* $B \in \Sigma_i$ *or there exists* $[\Pi, \Sigma \lhd A] \in \{[\Sigma_1 \lhd C_1], ..., [\Sigma_n \lhd C_n]\}$*;* (N) *either* $\Gamma \Rightarrow \Delta$ *has the form* $\bot \Rightarrow \top$ *or* $[\bot \lhd \top]$ *belongs to* Δ*;* (T$^{\mathsf{i}}$) *for every* $A \preccurlyeq B$ *in* Π_1*, it holds that either* $B \in \Pi_2$ *or* $[\bot \lhd A] \in \{[\Sigma_1 \lhd C_1], ..., [\Sigma_n \lhd C_n]\}$*;* (W$^{\mathsf{i}}$) *for every block* $[\Sigma \lhd A]$*, it holds that* $\Sigma \subseteq \Pi_2$*;* (C$^{\mathsf{i}}$) *for every* $A \preccurlyeq B$ *in* Π_1*, it holds that either* $B \in \Pi_2$ *or* $A \in \Pi_1$*. For each logic* $\mathcal{L}$*, the definition of saturated sequent takes into account only the saturation conditions of the rules of the corresponding calculus.*

All the blocks $[\Sigma_1 \lhd C_1], ..., [\Sigma_n \lhd C_n]$ of a saturated sequent can be considered as ordered with respect to set inclusion[8]. We call *static* all the rules except for jump and A$^{\mathsf{i}}$. By *finished* sequent we mean a sequent for which every further static rule application is redundant. Note that a finished sequent is saturated.

Proposition 18. *After finitely many non redundant static rule applications we reach an axiom or a finished sequent.*

Proof. Let $\Gamma \Rightarrow \Delta$ be the root sequent of a derivation. We consider any branch of a derivation (i) without applications of jump or A$^{\mathsf{i}}$, (ii) without redundant applications of rules. Observe that each rule application *must* add at least one formula or block to each premise, and the number of formulas or blocks (each one is finite in itself) that can occur within a sequent is finite. Thus the branch must be finite: if not, then it would not contain axioms and some formula or block would be added infinitely many times by eventually redundant applications of a rule. Moreover, once a rule (R) has been applied to a formula or block, the saturation condition with respect to the rule (R) and the involved formulas or blocks will be satisfied by the premises of (R). Thus the last node of the branch, if it is not an axiom, must be finished. □

Corollary 19. *Given a sequent* $\Gamma \Rightarrow \Delta$*, every branch of any derivation tree starting with* $\Gamma \Rightarrow \Delta$ *ends in a finite number of steps with a saturated sequent of no greater modal degree than that of* $\Gamma \Rightarrow \Delta$*.*

Theorem 20. *If a sequent* $\Gamma_0 \Rightarrow \Delta_0$ *is valid, then it is derivable in* $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$*.*

Proof. We first prove completeness for $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$, then show how to extend the proof to $\mathcal{I}^{\mathsf{i}}_{\mathbb{VN}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VT}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VW}}$, $\mathcal{I}^{\mathsf{i}}_{\mathbb{VC}}$[9]. The proof strategy is the same in all cases, and it

[8] A quick argument: once all non redundant com$^{\mathsf{c}}$ have been applied, it holds that either $\Sigma_i \subseteq \Sigma_j$ or $\Sigma_j \subseteq \Sigma_i$; we then order the blocks: $\Sigma_1 \subseteq \Sigma_2 \subseteq ... \subseteq \Sigma_n$.

[9] The proof uses in an essential way the fact that a backwards application of jump reduces the modal degree of a sequent. Although rule A$^{\mathsf{i}}$ plays a similar role as jump, it does not reduce the modal degree when applied backwards. Thus we need another argument for handling logics including $\mathbb{A}$; this is object of further investigation.

proceeds by induction on the modal degree of the sequent. If $md(\Gamma_0 \Rightarrow \Delta_0) = 0$, $\Gamma_0 \Rightarrow \Delta_0$ is composed only of propositional formulas, and its completeness can be proved from the completeness of sequent calculus for propositional logic. If $md(\Gamma_0 \Rightarrow \Delta_0) > 0$, by Proposition 16 and Proposition 19 we have that $\Gamma_0 \Rightarrow \Delta_0$ can be derived from a set of saturated sequents $\Gamma_k \Rightarrow \Delta_k$ of no greater modal degree. Since all the rules are invertible, except jump, and since by hypothesis $\Gamma_0 \Rightarrow \Delta_0$ is valid, also all saturated sequents $\Gamma_k \Rightarrow \Delta_k$ are valid. Thus, either *i*) $\Gamma_k \Rightarrow \Delta_k$ is an axiom, or *ii*) it must have been obtained by jump from a valid sequent $\Gamma_{k+1} \Rightarrow \Delta_{k+1}$. In the first case the theorem is trivially proved. We shall prove *ii*): if $\Gamma_k \Rightarrow \Delta_k$ is valid and saturated, and it is not an axiom, there exists a valid sequent $\Gamma_{k+1} \Rightarrow \Delta_{k+1}$ from which $\Gamma_k \Rightarrow \Delta_k$ is obtained by jump. We shall prove the statement by contraposition. Let $\Gamma_k \Rightarrow \Delta_k$ be the saturated sequent $\Pi_1 \Rightarrow \Pi_2, [\Sigma_1 \lhd C_1], ..., [\Sigma_k \lhd C_k]$. Suppose that none of the sequents $C_1 \Rightarrow \Sigma_1, ..., C_k \Rightarrow \Sigma_k$ is valid. We prove that the sequent $\Gamma_k \Rightarrow \Delta_k$ is not valid.

By hypothesis there are models $\mathcal{M}_1, ..., \mathcal{M}_k$ which falsify the sequents $C_1 \Rightarrow \Sigma_1, ..., C_k \Rightarrow \Sigma_k$. For $1 \leqslant j \leqslant k$, let $\mathcal{M}_j = \langle W_j, \mathsf{SP}^j, [\![.]\!]_j \rangle$ and for some elements $x_j \in W_j$ let $\mathcal{M}_j, x_j \Vdash C_j$ and $\mathcal{M}_j, x_j \nVdash S$ for all $S \in \Sigma_j$. Suppose all W_j are disjoint, i.e. $W_j \cap W_{j'} = \emptyset$. From these models we build a new model $\mathcal{M} = \langle W, \mathsf{SP}, [\![.]\!] \rangle$ as follows: $W = \cup W_l \cup \{x\}$, for x new; $\mathsf{SP}(z) = \mathsf{SP}^j(z)$, if $z \in W_j$; $\mathsf{SP}(x) = \{\alpha_1, ..., \alpha_k\}$, where $\alpha_k = \{x_k\}$; $\alpha_{k-1} = \{x_k, x_{k-1}\}$, ... , $\alpha_1 = \{x_k, ..., x_1\}$; $[\![P]\!] = \cup [\![P]\!]_j$, for P atomic and $P \in \Pi_2$; $[\![P]\!] = \cup [\![P]\!]_j \cup \{x\}$, for P atomic and $P \in \Pi_1$. One can easily check that for E arbitrary formula or block, it holds that if $\mathcal{M}_j, x_j \Vdash E$, then $\mathcal{M}, x_j \Vdash E$, for $1 \leqslant j \leqslant k$.

To complete the proof we show that $\mathcal{M}$ falsifies each formula or block occurring in $\Gamma_k \Rightarrow \Delta_k$. Thus, we have to prove that *a*) if $G \in \Gamma_k$, then $\mathcal{M}, x \Vdash G$, for G formula; *b*) if $G \in \Delta_k$, then $\mathcal{M}, x \nVdash G$, for G formula; *c*) if $[\Sigma_j \lhd A_j] \in \Delta_k$, then $\mathcal{M}, x \nVdash [\Sigma_j \lhd A_j]$. The proof proceeds by induction on the modal degree of formulas. The base case and the inductive step for the propositional cases are immediate. *Proof of a.* Let $G = C \preccurlyeq D$. For the saturation conditions ($\mathsf{com}^{\mathsf{c}}$) and ($\preccurlyeq^{\mathsf{c}}_L$), it holds that for all blocks $[\Sigma_j \lhd A_j]$ in the saturated sequent, either $D \in \Sigma_j$ or there exists in the saturated sequent a block $[\Pi, \Sigma_l \lhd C]$, for $l \leqslant j$. Consider an arbitrary sphere $\alpha_j = \{x_k, ..., x_j\}$ and the corresponding block $[\Sigma_j \lhd A_j]$. There are two cases to consider: if i) $D \in \Sigma_j$, by construction of the model it holds that $\alpha_j \nVdash^{\exists} D$, i.e. $\alpha_j \Vdash^{\forall} \neg D$. Suppose that ii) there exists a block $[\Pi, \Sigma_l \lhd C]$ belonging to the saturated sequent $\Gamma_k \Rightarrow \Delta_k$. By construction of the model, we have that there exists a world x_l such that $x_l \Vdash C$; thus, $\alpha_l \Vdash^{\exists} C$. However, since the spheres are incremental, $\alpha_l \subseteq \alpha_j$; thus, $\alpha_j \Vdash^{\exists} C$. We have that for α_j arbitrary block, either $\alpha_j \Vdash^{\forall} \neg D$ or $\alpha_j \Vdash^{\exists} C$; thus, $\mathcal{M}, x \Vdash C \preccurlyeq D$. *Proof of b.* Let $G = C \preccurlyeq D$. By the saturation condition ($\preccurlyeq^{\mathsf{c}}_R$) there exists a block $[\Sigma_j \lhd A_j]$ belonging to $\Gamma_k \Rightarrow \Delta_k$ such that $C \in \Sigma_j$ and $D = A_j$. Let us consider $\alpha_j = \{x_k, ..., x_j\}$. We have that $C \in \Sigma_{j+1}, ..., C \in \Sigma_k$. By construction, $x_j \nVdash C$; therefore, $x_j \nVdash C, ..., x_k \nVdash C$. Furthermore, $x_j \Vdash A_j$; thus $x_j \Vdash D$. There exists $\alpha_j \in \mathsf{SP}(x)$ such that $\alpha_j \nVdash^{\forall} \neg D$ and $\alpha_j \nVdash^{\exists} C$; thus, $\mathcal{M}, x \nVdash C \preccurlyeq D$. The proof of *c*) is the same as in the previous case.

We have thus proven that if $\Gamma_k \Rightarrow \Delta_k$ is valid and saturated, and it is not an axiom, then there exists a valid sequent $\Gamma_{k+1} \Rightarrow \Delta_{k+1}$ from which $\Gamma_k \Rightarrow \Delta_k$ is obtained by jump. Since $md(\Gamma_{k+1} \Rightarrow \Delta_{k+1}) < md(\Gamma_k \Rightarrow \Delta_k)$, by inductive hypothesis we have that $\Gamma_{k+1} \Rightarrow \Delta_{k+1}$ is derivable; therefore, $\Gamma_k \Rightarrow \Delta_k$ is derivable as well, by the jump rule.

Completeness of $\mathcal{I}^{\mathsf{i}}_{\mathbb{VN}}$. If $md(\Gamma_0 \Rightarrow \Delta_0) = 0$, then any saturated sequent derived from it will have the form $\Gamma_k \Rightarrow \Delta_k, [\bot \lhd \top]$, where Γ_k and Δ_k are composed only of propositional formulas. If $\Gamma_k \Rightarrow \Delta_k$ is an axiom, we are done. If $\Gamma_k \Rightarrow \Delta_k$ is not an axiom, it has a propositional countermodel. Associate this countermodel to a world x, and build a model with $W = \{x\}$ and $\mathsf{SP}(x) = \{\{x\}\}$. The reader can easily check that the model satisfies N. If $md(\Gamma_0 \Rightarrow \Delta_0) > 0$, the proof proceeds in the same way as for $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$. Notice that by inductive hypothesis all the models $\mathcal{M}_i$ involved in the construction satisfy N.

Completeness of $\mathcal{I}^{\mathsf{i}}_{\mathbb{VT}}$. We modify the definition of $\mathsf{SP}(x)$ in the model $\mathcal{M}$ by adding a new sphere α_0, in order to account for total reflexivity. Thus, $\mathsf{SP}(x) = \{\alpha_0, \alpha_1, \alpha_2, ..., \alpha_k\}$, where $\alpha_k = \{x_k\}$, $\alpha_{k-1} = \{x_k, x_{k-1}\}$, ..., $\alpha_1 = \{x_k, ..., x_1\}$, $\alpha_0 = \alpha_1 \cup \{x\}$. Cases b) and c) remain the same as in the completeness proof for $\mathcal{I}^{\mathsf{i}}_{\mathbb{V}}$. As for a), consider $\mathsf{SP}(x) = \{\alpha_0, \alpha_1, \alpha_2, ..., \alpha_k\}$. For spheres $\alpha_k, ..., \alpha_1$ a) holds; we have to prove that also for α_0 either $\alpha_0 \Vdash^{\forall} \neg D$ or $\alpha_0 \Vdash^{\exists} C$. We know that either i) $\alpha_1 \Vdash^{\forall} \neg D$ or ii) $\alpha_1 \Vdash^{\exists} C$. If i) holds, the theorem is proved, since $\alpha_0 \Vdash^{\exists} C$. If it holds that $(*)$ $\alpha_1 \nVdash^{\forall} \neg D$ then ii) holds. By absurd, suppose $\alpha_0 \nVdash^{\forall} \neg D$; thus, $(**)$ $x \Vdash D$ (since all the other worlds did not satisfy D). By saturation condition (T^{i}), we have that either $D \in \Delta$ or $[\bot \lhd C] \in \Delta$. There are two cases to consider. If $D \in \Delta$, since $md(D) < md(C \preccurlyeq D)$, by inductive hypothesis we have $x \nVdash D$, against $(**)$. If $[\bot \lhd C] \in \Delta$, there exists a block $[\Sigma_u \lhd A_u]$ in the saturated sequent $\Gamma_k \Rightarrow \Delta_k$ such that $A_u = C$. Thus, by construction $\alpha_u \Vdash^{\exists} C$, and $x_u \Vdash C$ for some $x_u \in \alpha_u$. By construction $x_u \in \alpha_1$; thus, $\alpha_1 \Vdash^{\exists} C$ against $(*)$. We reached a contradiction; thus, also for α_0 it holds that $\alpha_0 \Vdash^{\forall} \neg D$ or $\alpha_0 \Vdash^{\exists} C$, and $\mathcal{M}, x \Vdash C \preccurlyeq D$.

Completeness of $\mathcal{I}^{\mathsf{i}}_{\mathbb{VW}}$. We modify $\mathsf{SP}(x)$ in order to account for weak centering by adding world x to each sphere, as follows: $\mathsf{SP}(x) = \{\alpha_1, \alpha_2, ..., \alpha_k\}$, where $\alpha_k = \{x_k, x\}$; $\alpha_{k-1} = \{x_k, x_{k-1}, x\}$, ..., $\alpha_1 = \{x_k, ..., x_1, x\}$. We have to prove that conditions a), b) and c) hold. The proof makes an essential use of the saturation condition (W^{i}), and it is omitted for space reasons.

Completeness of $\mathcal{I}^{\mathsf{i}}_{\mathbb{VC}}$. For centering, we modify $\mathsf{SP}(x)$ by adding a new sphere α_{k+1}, which contains only x. Namely: $\alpha_{k+1} = \{x\}$; $\alpha_k = \{x_k, x\}$; $\alpha_{k-1} = \{x_k, x_{k-1}, x\}$,..., $\alpha_1 = \{x_k, ..., x_1, x\}$. Conditions b) and c) are as in the proof for $\mathcal{I}^{\mathsf{i}}_{\mathbb{VW}}$; case a) is slightly different and employs the saturation condition (C^{i}). □

6 Completeness via translation

We can give quick alternative completeness proofs for the proposed calculi by simulating derivations in the corresponding sequent calculi from [13, 12], shown in Tab. 4. The main difficulty is to simulate the rules for $\preccurlyeq$.

Theorem 21. *Every rule of $\mathcal{R}_{\mathcal{L}}$ is derivable in $\mathcal{I}_{\mathcal{L}} \setminus \{\mathsf{Con}_S, \mathsf{Con}_B\}$. Hence $\mathcal{I}_{\mathcal{L}} \setminus \{\mathsf{Con}_S, \mathsf{Con}_B\}$ is cut-free complete for $\mathcal{L}$.*

$$\frac{\{B_k \Rightarrow D_1,\dots,D_m,A_1,\dots,A_n \mid 1 \le k \le n\} \cup \{C_k \Rightarrow D_1,\dots,D_{k-1},A_1,\dots,A_n \mid 1 \le k \le m\}}{\Gamma, C_1 \preccurlyeq D_1,\dots,C_m \preccurlyeq D_m \Rightarrow A_1 \preccurlyeq B_1,\dots,A_n \preccurlyeq B_n,\Delta}\; R_{m,n}$$

$$\frac{\{C_k \Rightarrow D_1,\dots,D_{k-1} \mid 1 \le k \le m\} \cup \{\Gamma \Rightarrow D_1,\dots,D_m,\Delta\}}{\Gamma, C_1 \preccurlyeq D_1,\dots,C_m \preccurlyeq D_m \Rightarrow \Delta}\; T_m \qquad \frac{\Gamma, C \Rightarrow \Delta \quad \Gamma \Rightarrow D, \Delta}{\Gamma, C \preccurlyeq D \Rightarrow \Delta}\; C2$$

$$\frac{\{B_k \Rightarrow D_1,\dots,D_m,A_1,\dots,A_n \mid 1 \le k \le n\} \cup \{\Gamma \Rightarrow D_1,\dots,D_m,A_1,\dots,A_n,\Delta\}}{\Gamma, C_1 \preccurlyeq D_1,\dots,C_m \preccurlyeq D_m \Rightarrow A_1 \preccurlyeq B_1,\dots,A_n \preccurlyeq B_n,\Delta}\; W_{m,n} \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \preccurlyeq B, \Delta}\; W2$$

$$\frac{\left\{\Gamma^{\preccurlyeq}, B_k \Rightarrow D_1,\dots,D_m,A_1,\dots,A_n,\Delta^{\preccurlyeq} \mid 1 \le k \le n\right\} \cup \left\{\Gamma^{\preccurlyeq}, C_k \Rightarrow D_1,\dots,D_{k-1},A_1,\dots,A_n,\Delta^{\preccurlyeq} \mid 1 \le k \le m\right\}}{\Gamma, C_1 \preccurlyeq D_1,\dots,C_m \preccurlyeq D_m \Rightarrow A_1 \preccurlyeq B_1,\dots,A_n \preccurlyeq B_n,\Delta}\; A_{m,n}$$

$\Gamma^{\preccurlyeq}$ is the restriction of Γ to formulae of the form $A \preccurlyeq B$; $\mathcal{R}_{\mathbb{V}_{\preccurlyeq}} := \{\mathsf{R}_{m,n} \mid m \ge 0, n \ge 1\}$;

$\mathcal{R}_{\mathbb{VN}_{\preccurlyeq}} := \{\mathsf{R}_{m,n} \mid m+n \ge 1\}$
$\mathcal{R}_{\mathbb{VT}_{\preccurlyeq}} := \mathcal{R}_{\mathbb{V}_{\preccurlyeq}} \cup \{\mathsf{T}_m \mid m \ge 1\}$
$\mathcal{R}_{\mathbb{VW}_{\preccurlyeq}} := \mathcal{R}_{\mathbb{V}_{\preccurlyeq}} \cup \{\mathsf{W}_{m,n} \mid m+n \ge 1\}$
$\mathcal{R}_{\mathbb{VC}_{\preccurlyeq}} := \mathcal{R}_{\mathbb{V}_{\preccurlyeq}} \cup \{\mathsf{R}_{\mathsf{W2}}, \mathsf{R}_{\mathsf{C2}}\}$
$\mathcal{R}_{\mathbb{VA}_{\preccurlyeq}} := \{\mathsf{A}_{m,n} \mid m \ge 0,\ n \ge 1\}$
$\mathcal{R}_{\mathbb{VNA}_{\preccurlyeq}} := \{\mathsf{A}_{m,n} \mid m+n \ge 1\}$

Table 4. The rules and rule sets for extensions of $\mathbb{V}_{\preccurlyeq}$.

Proof. We only consider the rules for $\preccurlyeq$, the remaining rules are straightforward. For the sake of readability for $k < \ell$ we abbreviate $C_k \preccurlyeq D_k,\dots,C_\ell \preccurlyeq D_\ell$ by $\boldsymbol{(C \preccurlyeq D)_k^\ell}$. Similarly, we write $\boldsymbol{A_k^\ell}$ for $A_k,\dots,A_\ell$, and $\boldsymbol{D_k^\ell}$ for $D_k,\dots,D_\ell$. To simulate rule $R_{m,n}$, for every $k \le n$ we have the following derivation:

$$\dfrac{\dfrac{\dfrac{B_k \Rightarrow \boldsymbol{A_1^n}, \boldsymbol{D_1^m}}{\Gamma \Rightarrow \Delta, [\boldsymbol{A_1^n}, \boldsymbol{D_1^m} \lhd B_k]}\,\mathsf{jump} \quad \dfrac{C_m \Rightarrow \boldsymbol{A_1^n}, \boldsymbol{D_1^{m-1}}}{\Gamma \Rightarrow \Delta, \left[\boldsymbol{A_1^n}, \boldsymbol{D_1^{m-1}} \lhd C_m\right]}\,\mathsf{jump}}{\Gamma, C_m \preccurlyeq D_m \Rightarrow \Delta, \left[\boldsymbol{A_1^n}, \boldsymbol{D_1^{m-1}} \lhd B_k\right]}\,\preccurlyeq_L \\ \vdots \\ \Gamma, \boldsymbol{(C \preccurlyeq D)_2^m} \Rightarrow \Delta, [\boldsymbol{A_1^n}, D_1 \lhd B_k] \quad \dfrac{C_1 \Rightarrow \boldsymbol{A_1^n}}{\Gamma, \boldsymbol{(C \preccurlyeq D)_2^m} \Rightarrow \Delta, [\boldsymbol{A_1^n} \lhd C_1]}\,\mathsf{jump}}{\Gamma, \boldsymbol{(C \preccurlyeq D)_1^m} \Rightarrow \Delta, [\boldsymbol{A_1^n} \lhd B_k]}\,\preccurlyeq_L$$

The conclusion is obtained by weakening (Lem. 6) and multiple applications of com to these sequents, followed by the derivation

$$\frac{\Gamma, \boldsymbol{(C \preccurlyeq D)_1^m} \Rightarrow \Delta, [A_1 \lhd B_1],\dots,[A_n \lhd B_n]}{\Gamma, C_1 \preccurlyeq D_1,\dots,C_m \preccurlyeq D_m \Rightarrow \Delta, A_1 \preccurlyeq B_1,\dots,A_n \preccurlyeq B_n}\; \preccurlyeq_R$$

The simulations for the remaining rules apart from T_m are only slight modifications. For instance, to simulate $R_{m,0}$ we would have the rule N instead of the blocks of $\preccurlyeq_R$ and com at the bottom, for $W_{m,n}$ with $n \ge 1$ we replace the top leftmost application of jump by an application of W, for $W_{m,0}$ we apply N at the bottom, and for $A_{m,n}$ we replace all applications of jump by A. Rule $C2$ is simulated straightforwardly by W followed by $\preccurlyeq_R$. For rule T_m finally, we first construct for $\ell, k \ge 0$ derivations $\mathcal{D}_{\ell,\ell+k+1}$ of the sequents

$$\Omega, \boldsymbol{(C \preccurlyeq D)_1^\ell} \Rightarrow \Theta, \left[\bot, \boldsymbol{D_{\ell+1}^{\ell+k}}, \Sigma \lhd C_{\ell+k+1}\right]$$

for arbitrary Ω, Θ, Σ from the premises $\{C_i \Rightarrow \boldsymbol{D}_{\boldsymbol{1}}^{\boldsymbol{i-1}} \mid 1 \leq i \leq \ell + k + 1\}$ as follows. The derivation $\mathcal{D}_{0,k+1}$ is straightforward using the rules of weakening (Lem. 6) and jump. The derivation $\mathcal{D}_{\ell+1,\ell+1+k+1}$ is obtained by

$$\dfrac{\Omega, (\boldsymbol{C} \preccurlyeq \boldsymbol{D})_{\boldsymbol{1}}^{\boldsymbol{\ell}} \Rightarrow \Theta, \left[\bot, \boldsymbol{D}_{\boldsymbol{\ell+1}}^{\boldsymbol{\ell+1+k}}, \Sigma \lhd C_{\ell+1+k+1}\right] \quad \Omega, (\boldsymbol{C} \preccurlyeq \boldsymbol{D})_{\boldsymbol{1}}^{\boldsymbol{\ell}} \Rightarrow \Theta, \left[\bot, \boldsymbol{D}_{\boldsymbol{\ell+2}}^{\boldsymbol{\ell+1+k}}, \Sigma \lhd C_{\ell+1}\right]}{\Omega, (\boldsymbol{C} \preccurlyeq \boldsymbol{D})_{\boldsymbol{1}}^{\boldsymbol{\ell+1}} \Rightarrow \Theta, \left[\bot, \boldsymbol{D}_{\boldsymbol{\ell+2}}^{\boldsymbol{\ell+1+k}}, \Sigma \lhd C_{\ell+1+k+1}\right]} \preccurlyeq_L$$

where the premises are derived by $\mathcal{D}_{\ell,\ell+1+k+1}$ and $\mathcal{D}_{\ell,\ell+1}$. We obtain T_m as:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \boldsymbol{D}_{\boldsymbol{1}}^{\boldsymbol{m}} \quad \Gamma \Rightarrow \Delta, \boldsymbol{D}_{\boldsymbol{2}}^{\boldsymbol{m}}, [\bot \lhd C_1]}{\Gamma, C_1 \preccurlyeq D_1 \Rightarrow \Delta, \boldsymbol{D}_{\boldsymbol{2}}^{\boldsymbol{m}}} \mathsf{T} \quad \Gamma, C_1 \preccurlyeq D_1 \Rightarrow \Delta, \boldsymbol{D}_{\boldsymbol{3}}^{\boldsymbol{m}}, [\bot \lhd C_2]}{\Gamma, (\boldsymbol{C} \preccurlyeq \boldsymbol{D})_{\boldsymbol{1}}^{\boldsymbol{2}} \Rightarrow \Delta, \boldsymbol{D}_{\boldsymbol{3}}^{\boldsymbol{m}}} \mathsf{T}}{\vdots}$$

$$\dfrac{\Gamma, (\boldsymbol{C} \preccurlyeq \boldsymbol{D})_{\boldsymbol{1}}^{\boldsymbol{m-1}} \Rightarrow \Delta, D_m \quad \Gamma, (\boldsymbol{C} \preccurlyeq \boldsymbol{D})_{\boldsymbol{1}}^{\boldsymbol{m-1}} \Rightarrow \Delta, [\bot \lhd C_m]}{\Gamma, (\boldsymbol{C} \preccurlyeq \boldsymbol{D})_{\boldsymbol{1}}^{\boldsymbol{m}} \Rightarrow \Delta} \mathsf{T}$$

where the premises are derived using $\mathcal{D}_{0,1}, \mathcal{D}_{1,2}, \ldots, \mathcal{D}_{m-1,m}$. Note that none of the simulations uses Con_S. □

Corollary 22. *Let $\mathcal{L} \in \{\mathbb{V}, \mathbb{VN}, \mathbb{VT}, \mathbb{VW}, \mathbb{VA}, \mathbb{VNA}\}$. Then the calculus $\mathcal{I}_{\mathcal{L}}^{\mathsf{i}}$ is complete for $\mathcal{L}$.* □

7 Conclusions

We have introduced internal, standard, cut-free calculi for Lewis' logics $\mathbb{V}$, $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$, $\mathbb{VC}$, $\mathbb{VA}$ and $\mathbb{VNA}$, extending the basic ideas of the calculi proposed in [17] for the basic system $\mathbb{V}$. The same logics have been considered in [12, 13], where calculi comprising an infinite set of rules with a variable number of premises are introduced, whereas the calculi we have introduced here are standard in the sense that each connective is handled by a fixed finite set of rules with a fixed finite set of premises. As far as we know, these are the first standard and internal calculi covering most, if not all, logics of the Lewis' family.

In future research we aim at extending the proof of cut elimination to extensions of $\mathbb{V}$. Moreover, we aim at providing a semantic completeness proof also for the logics with the absoluteness condition. Finally we shall study how to obtain optimal decision procedures for the respective logics based on our calculi.

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