Linear Nested Sequents, 2-Sequents and Hypersequents*

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Abstract. We introduce the framework of linear nested sequent calculi by restricting nested sequents to linear structures. We show the close connection between this framework and that of 2-sequents, and provide linear nested sequent calculi for a number of modal logics as well as for intuitionistic logic. Furthermore, we explore connections to backwards proof search for sequent calculi and to the hypersequent framework, including a reinterpretation of various hypersequent calculi for modal logic S5 in the linear nested sequent framework.

1 Introduction

One of the major enterprises in proof theory of modal logics is the development of generalisations of the sequent framework permitting the formulation of analytic calculi for large classes of modal logics in a satisfactory way. Apart from cut admissibility, among the main desiderata for such calculi are separate left and right introduction rules for the modal connectives, and that calculi for extensions of the base logic should be obtained by a modular addition of rules to the base calculus [27,21]. This was realised e.g. in the framework of *nested sequents* resp. *tree-hypersequents* [3,20] and the related framework of *labelled sequents* [18].

However, from a philosophical and computational point of view it is interesting to find the *simplest* generalisation of the sequent framework permitting good calculi for such classes of logics, i.e., to establish just how much additional structure is needed for capturing these logics. A reasonably simple extension of the sequent framework, that of *2-sequents*, was introduced by Masini to capture modal logic KD and several constructive modal logics [15,16,14]. The resulting calculi satisfy many of the desiderata such as separate left and right introduction rules for $\Box$, a direct formula translation for every structure, cut elimination and the subformula property. For the constructive logics the calculi also serve as a stepping stone towards natural deduction systems and Curry-Howard-style correspondences [14]. Despite these advantages, the framework of 2-sequents seems not to have attracted the attention it deserves. One reason might have been that it seems not to have been clear how to adapt the original calculus for KD to other modal logics based on classical propositional logic, notably basic modal logic K, see e.g. [27, Sec.2.2] or [21, p.55].

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In the following we connect this framework with that of nested sequents by making precise the idea that 2-sequents can be seen as *linear nested sequents*, i.e., nested sequents in linear instead of tree shape (Sec. 3). This observation suggests linear adaptations of standard nested sequent calculi for various modal logics (Sec. 4.1), thus answering the question on how to extend the 2-sequent framework to other logics and demonstrating that these logics do not require the full machinery of nested sequents. Of course the full nested sequent framework might still capture more modal logics, and it seems to provide better modularity for logics including the axiom 5 [13]. We also obtain linear nested sequent calculi for propositional and first-order intuitionistic logic from the calculi in [7] (Sec. 4.2). In all these cases the established completeness proofs for the full nested calculi use the tree structure of nested sequents and hence fail in the linear setting. However, we obtain quick completeness proofs by exploiting connections to standard sequent calculi. A fortiori, this also shows completeness for the full nested calculi.

Another successful generalisation of the sequent framework is that of *hypersequents*, permitting e.g. several calculi for modal logic S5. The observation that hypersequents have the same structure as linear nested sequents suggests investigations into the relation between the two frameworks, in particular a reinterpretation of hypersequent calculi for S5 in terms of linear nested sequents, and the construction of hypersequent calculi from linear nested calculi (Sec. 5).

Relation to other frameworks. By the translations in [6,8] the linear nested framework induces corresponding restrictions in the frameworks of *prefixed tableaux* and *labelled sequents*. E.g., we obtain completeness results for calculi using what could be called *labelled line sequents*, i.e., labelled sequents [18] where the relational atoms spell out the structure of a line (compare [8]). Since cut elimination for labelled sequents does not preserve this property, these are non-trivial results. An analogue of linear nested sequents in the unlabelled tableaux framework has been considered in [5] under the name of *path-hypertableau* for intermediate logics.

2 Preliminaries: Nested Sequents and 2-Sequents

As usual, *modal formulae* are built from variables $p, q, \dots$ using the propositional connectives $\bot, \wedge, \vee, \to$ and the (unary) modal connective $\Box$ with the standard conventions for omitting parentheses. We write $\top$ for $\bot \to \bot$, abbreviate $A \to \bot$ to $\neg A$ and write $\Diamond A$ for $\neg\Box\neg A$. Modal logic K is axiomatised by classical propositional logic, the axiom K and the rule Nec, and we also consider extensions of K with axioms from Fig. 1. Theoremhood in a logic $\mathcal{L}$ is written $\models_{\mathcal{L}}$. For more on modal logics see [2]. We consider extensions of the *sequent framework*, where a *sequent* is a tuple of multisets of formulae, written $\Gamma \Rightarrow \Delta$, and interpreted as $\bigwedge \Gamma \to \bigvee \Delta$, see e.g. [26]. We write $\Gamma \cup \Delta$ or Γ, Δ for multiset sum and $\Gamma \subseteq \Delta$ for multiset inclusion (respecting multiplicities) and denote the empty multiset with $\emptyset$. For $\mathcal{C}$ one of the calculi below we write $\vdash_{\mathcal{C}}$ for derivability in $\mathcal{C}$. We write $\mathbb{N}$ for the set $\{1, 2, 3, \dots\}$ of natural numbers.

	K $\Box(A \to B) \to (\Box A \to \Box B)$		Nec $\vdash A / \vdash \Box A$	
D $\Box A \to \Diamond A$	T $\Box A \to A$	4 $\Box A \to \Box\Box A$	5 $\Diamond\Box A \to \Box A$	B $A \to \Box\Diamond A$

Fig. 1. Axioms for modal logics

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Delta]\}}{\mathcal{S}\{\Gamma, \Box A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]\}}\ \Box_L \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\ \Rightarrow A]\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, \Box A\}}\ \Box_R$$

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [A \Rightarrow\ ]\}}{\mathcal{S}\{\Gamma, \Box A \Rightarrow \Delta\}}\ \mathsf{d} \qquad \frac{\mathcal{S}\{\Gamma, A \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, \Box A \Rightarrow \Delta\}}\ \mathsf{t} \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\Sigma, \Box A \Rightarrow \Pi]\}}{\mathcal{S}\{\Gamma, \Box A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]\}}\ \mathsf{4}$$

Fig. 2. Nested sequent rules

2.1 Nested Sequents / Tree-Hypersequents

One of the most popular recent extensions of the original sequent framework is that of *nested sequents* or *tree-hypersequents*. Partly, the current interest in this formalism was sparked by [3,20] which contain analytic calculi for a number of modal logics. The main idea of the framework is to replace a sequent with a *tree* of sequents, thus intuitively capturing the tree structure of Kripke models for modal logic. The basic concepts (in slightly adapted notation) are the following.

Definition 1. *The set* NS *of* nested sequents *is given by:*

1. *if* $\Gamma \Rightarrow \Delta$ *is a sequent then* $\Gamma \Rightarrow \Delta \in \mathsf{NS}$
2. *if* $\Gamma \Rightarrow \Delta$ *is a sequent and* $\Sigma_i \Rightarrow \Pi_i \in \mathsf{NS}$ *for* $1 \leq i \leq n$, *then* $\Gamma \Rightarrow \Delta, [\Sigma_1 \Rightarrow \Pi_1], \dots, [\Sigma_n \Rightarrow \Pi_n] \in \mathsf{NS}$.

The interpretation *of a nested sequent is given by*

1. $\iota(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \to \bigvee \Delta$ *if* $\Gamma \Rightarrow \Delta$ *is a sequent*
2. $\iota(\Gamma \Rightarrow \Delta, [\Sigma_1 \Rightarrow \Pi_1], \dots, [\Sigma_n \Rightarrow \Pi_n]) = \bigwedge \Gamma \to \bigvee \Delta \vee \bigvee_{i=1}^{n} \Box(\iota(\Sigma_i \Rightarrow \Pi_i))$ *if* $\Gamma \Rightarrow \Delta$ *is a sequent and* $\Sigma_i \Rightarrow \Pi_i \in \mathsf{NS}$ *for* $i \leq n$.

As usual, empty conjuntions and disjunctions are interpreted as $\top$ resp. $\bot$. Thus the structural connective $[\cdot]$ of nested sequents is interpreted by the logical connective $\Box$. Fig. 2 shows the basic logical rules $\Box_L$ and $\Box_R$ for modal logic K and some rules for extensions [21]. Following [3] we write $\mathcal{S}\{.\}$ to signify that the rules can be applied in a *context*, i.e., at an arbitrary node of the nested sequent. The propositional part of the system consists of the standard sequent rules for each node in the nested sequent. This framework captures all logics of the modal cube in a cut-free and modular way [3,20,21,13].

2.2 2-Sequents

While nested sequents have a tree structure, the basic data structure (modulo notation) in the framework of *2-sequents* [15] is that of an infinite list of sequents which are eventually empty. Intuitively, instead of the whole tree structure of a Kripke model, 2-sequents capture the path from the root to a given state.

$$
\dfrac{\begin{array}{l}(\Gamma_i)_{i<n}\\ \Gamma_n\\ \Gamma_{n+1}, A\\ \mathfrak{G}\end{array} \Rightarrow \mathfrak{H}}{\begin{array}{l}(\Gamma_i)_{i<n}\\ \Gamma_n, \Box A\\ \Gamma_{n+1}\\ \mathfrak{G}\end{array} \Rightarrow \mathfrak{H}}\ \Box\Rightarrow
\qquad
\dfrac{\mathfrak{G} \Rightarrow \begin{array}{l}(\Delta_i)_{i<n}\\ \Delta_n\\ A\\ \varepsilon\end{array}}{\mathfrak{G} \Rightarrow \begin{array}{l}(\Delta_i)_{i<n}\\ \Delta_n, \Box A\\ \varepsilon\end{array}}\ \Rightarrow\Box
\qquad
\dfrac{\mathfrak{G} \Rightarrow \begin{array}{l}(\Delta_i)_{i<n}\\ \Delta_n, A\\ \mathfrak{H}\end{array} \qquad \mathfrak{G} \Rightarrow \begin{array}{l}(\Delta_i)_{i<n}\\ \Delta_n, B\\ \mathfrak{H}\end{array}}{\mathfrak{G} \Rightarrow \begin{array}{l}(\Delta_i)_{i<n}\\ \Delta_n, A \wedge B\\ \mathfrak{H}\end{array}}\ \Rightarrow\wedge
$$

A maximum of the premiss

Fig. 3. The modal 2-sequent rules and the conjunction rule of $\mathsf{C\text{-}2SC}$

Definition 2 ([15]). *A* 2-sequence *is an infinite list* $(\Gamma_i)_{i\in\mathbb{N}}$ *of multisets of formulae with* $\Gamma_k = \emptyset$ *for some* $n \in \mathbb{N}$ *and all* $k \geq n$. *We write* ε *for the list* $(\emptyset)_{i\in\mathbb{N}}$ *and* $\Sigma : (\Gamma_i)_{i\in\mathbb{N}}$ *for the list* $(\Delta_i)_{i\in\mathbb{N}}$ *with* $\Delta_1 = \Sigma$ *and* $\Delta_{i+1} = \Gamma_i$ *for* $i \in \mathbb{N}$. *A* 2-sequent *is a pair* $\mathfrak{G} \Rightarrow \mathfrak{H}$ *of 2-sequences* $\mathfrak{G}$ *and* $\mathfrak{H}$. *Its* interpretation ι *is:*

1. $\iota(\varepsilon \Rightarrow \varepsilon) = \top \rightarrow \bot$*; and*
2. $\iota(\Gamma : \varepsilon \Rightarrow \Delta : \varepsilon) = \bigwedge \Gamma \rightarrow \bigvee \Delta$ *if* $\Gamma \cup \Delta \neq \emptyset$*; and*
3. $\iota(\Gamma : \mathfrak{G} \Rightarrow \Delta : \mathfrak{H}) = \bigwedge \Gamma \rightarrow \bigvee \Delta \vee \Box\iota(\mathfrak{G} \Rightarrow \mathfrak{H})$ *if* $\mathfrak{G} \neq \varepsilon$ *and* $\mathfrak{H} \neq \varepsilon$.

Masini's original formulation of 2-sequents used lists instead of multisets of formulae, but in presence of the exchange rule the two formulations are clearly equivalent. Obviously a 2-sequent $(\Gamma_i)_{i\in\mathbb{N}} \Rightarrow (\Delta_i)_{i\in\mathbb{N}}$ can also be seen as the infinite list $(\Gamma_i \Rightarrow \Delta_i)_{i\in\mathbb{N}}$ of sequents, where the head is interpreted in the current world, the tail is interpreted under a box and the empty part of the list is dropped.

The *depth* of a 2-sequence $(\Gamma_i)_{i\in\mathbb{N}}$ is defined as $\sharp(\Gamma_i)_{i\in\mathbb{N}} := \min\{i : i \geq 0, \forall k > i : \Gamma_k = \emptyset\}$ and the *depth of a 2-sequent* $\mathfrak{G} \Rightarrow \mathfrak{H}$ is $\sharp(\mathfrak{G} \Rightarrow \mathfrak{H}) := \max\{\sharp\mathfrak{G}, \sharp\mathfrak{H}\}$. The *level* of an occurrence of a formula A in $(\Gamma_i)_{i\in\mathbb{N}} \Rightarrow (\Delta_i)_{i\in\mathbb{N}}$ is the i such that $\Gamma_i \cup \Delta_i$ contains this occurrence. An occurrence of a formula A is *maximal in* $\mathfrak{G} \Rightarrow \mathfrak{H}$ if its level is $\sharp(\mathfrak{G} \Rightarrow \mathfrak{H})$ and it is *the maximum* in $\mathfrak{G} \Rightarrow \mathfrak{H}$ if it is the unique maximal formula in $\mathfrak{G} \Rightarrow \mathfrak{H}$. The 2-sequent calculus $\mathsf{C\text{-}2SC}$ for the logic KD from [15] uses the modal rules in Fig. 3, with 2-sequences written in a top-down way. The propositional rules again are the local versions of the standard sequent rules for classical logic, i.e., they act only on one component $\Gamma_i \Rightarrow \Delta_i$ of the list. In contrast to Masini's original treatment, here we adopt the *context-sharing* versions of the rules, exemplified by the conjunction right rule in Fig. 3. As usual, in presence of the structural rules the two versions are equivalent.

3 Linear Nested Sequents for KD

The basic data structure of 2-sequents might be that of eventually empty infinite lists, but as the empty part is not interpreted, they can be formulated equivalently in terms of finite lists. But a finite list of sequents is essentially a nested sequent where the tree structure is restricted to the linear structure of a single branch.

Definition 3. *The set* LNS *of* linear nested sequents *is given recursively by:*

1. *if* $\Gamma \Rightarrow \Delta$ *is a sequent, then* $\Gamma \Rightarrow \Delta \in \mathsf{LNS}$*;*
2. *if* $\Gamma \Rightarrow \Delta$ *is a sequent and* $\mathcal{G} \in \mathsf{LNS}$*, then* $\Gamma \Rightarrow \Delta /\!\!/ \mathcal{G} \in \mathsf{LNS}$*.*

The modal formula interpretation $\iota_\Box$ *of a linear nested sequent is given by:*

1. *if* $\Gamma \Rightarrow \Delta$ *is a sequent, then* $\iota_\Box(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \to \bigvee \Delta$
2. $\iota_\Box(\Gamma \Rightarrow \Delta /\!\!/ \mathcal{G}) = \bigwedge \Gamma \to \bigvee \Delta \vee \Box\, \iota_\Box(\mathcal{G})$.

The sequents in a linear sequent are its components*. As in the full nested setting, we use the notation* $\mathcal{S}\{\Gamma \Rightarrow \Delta\}$ *for* $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \mathcal{H}$ *where* $\mathcal{G}, \mathcal{H} \in \mathsf{LNS}$ *or empty to denote a* context*. E.g.,* $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta$ *would be the context above with empty* $\mathcal{H}$*.*

The correspondences between 2-sequents and linear nested sequents are given by the following translations. To take care of the fact that the empty part of a 2-sequent is not interpreted while an empty component in a linear nested sequent is always interpreted we include a marker for the end of the linear nested sequent.

Definition 4. *The translations* τ *and* π *from* LNS *to 2-sequents and vice versa are given by:*

τ*.1. if* $\Gamma \Rightarrow \Delta$ *is a sequent, then* $\tau(\Gamma \Rightarrow \Delta) = \Gamma : \varepsilon \Rightarrow (\Delta, \bot) : \varepsilon$
τ*.2. if* $\Gamma \Rightarrow \Delta$ *is a sequent and* $\mathcal{G} \in \mathsf{LNS}$ *with* $\tau(\mathcal{G}) = \mathfrak{G} \Rightarrow \mathfrak{H}$*, then* $\tau(\Gamma \Rightarrow \Delta /\!\!/ \mathcal{G}) = \Gamma : \mathfrak{G} \Rightarrow \Delta : \mathfrak{H}$*.*
π*.1.* $\pi(\Gamma : \varepsilon \Rightarrow \Delta : \varepsilon) = \Gamma \Rightarrow \Delta$
π*.2.* $\pi(\Gamma : \mathfrak{G} \Rightarrow \Delta : \mathfrak{H}) = \Gamma \Rightarrow \Delta /\!\!/ \pi(\mathfrak{G} \Rightarrow \mathfrak{H})$ *for* $\mathfrak{G} \neq \varepsilon$ *and* $\mathfrak{H} \neq \varepsilon$*.*

By induction on the structure of linear nested sequents resp. 2-sequents it is straightforward to see that the results of the translations indeed are 2-sequents resp. linear nested sequents, and that the interpretations of the original structures and their translations are the same (modulo equivalence of $\top \to \bot$ and $\bot$). The rule set $\mathsf{LNS}_{\mathsf{KD}}$ obtained by rewriting the 2-sequent rules for KD in linear nested sequents notation is given in Fig. 4 (not all propositional rules shown). The rule d captures the case of rule $\Box \Rightarrow$ where the formula A is the maximum of the premiss. But these are exactly the linear versions of the standard nested sequent rules for KD from Fig. 2. In order to see that the marker introduced in the translation does not influence derivability, we first obtain the following lemma using Weakening and easy inductions on the depth of the derivations.

Lemma 5. *1.* $\vdash_{\mathsf{LNS}_{\mathsf{KD}}} \mathcal{S}\{\Gamma \Rightarrow \Delta\}$ *iff* $\vdash_{\mathsf{LNS}_{\mathsf{KD}}} \mathcal{S}\{\Gamma \Rightarrow \Delta, \bot\}$
2. $\vdash_{\mathsf{C\text{-}2SC}} \mathfrak{G} \Rightarrow (\Delta_i)_{i \leq n} : \Delta : \mathfrak{H}$ *iff* $\vdash_{\mathsf{C\text{-}2SC}} \mathfrak{G} \Rightarrow (\Delta_i)_{i \leq n} : (\Delta, \bot) : \mathfrak{H}$ $\Box$

Proposition 6. *If* $\mathcal{G} \in \mathsf{LNS}$ *and* $\mathfrak{G} \Rightarrow \mathfrak{H}$ *is a 2-sequent, then we have:* $\vdash_{\mathsf{LNS}_{\mathsf{KD}}} \mathcal{G}$ *iff* $\vdash_{\mathsf{C\text{-}2SC}} \tau(\mathcal{G})$ *and* $\vdash_{\mathsf{C\text{-}2SC}} \mathfrak{G} \Rightarrow \mathfrak{H}$ *iff* $\vdash_{\mathsf{LNS}_{\mathsf{KD}}} \pi(\mathfrak{G} \Rightarrow \mathfrak{H})$*.*

Proof. The "$\Leftarrow$" directions follow from the "$\Rightarrow$" directions using Lem. 5. The latter are both shown by induction on the depth of the derivations. For the first statement the only non-trivial cases are if the last rule in the derivation of $\mathcal{G}$ was

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, \Sigma \Rightarrow \Pi, \Delta\}}\ \mathsf{W} \qquad \frac{\mathcal{S}\{\Gamma, A, A \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, A \Rightarrow \Delta\}}\ \mathsf{ICL} \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow A, A, \Delta\}}{\mathcal{S}\{\Gamma \Rightarrow A, \Delta\}}\ \mathsf{ICR}$$

$$\frac{}{\mathcal{S}\{\Gamma, A \Rightarrow A, \Delta\}}\ \mathsf{init} \qquad \frac{\mathcal{S}\{\Gamma, A, B \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, A \wedge B \Rightarrow \Delta\}}\ \wedge_L \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow A, \Delta\} \quad \mathcal{S}\{\Gamma \Rightarrow B, \Delta\}}{\mathcal{S}\{\Gamma \Rightarrow A \wedge B, \Delta\}}\ \wedge_R$$

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \ \Rightarrow A}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \Box A}\ \Box_R \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi\}}{\mathcal{S}\{\Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi\}}\ \Box_L \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{d}$$

Fig. 4. The linear nested sequent calculus $\mathsf{LNS}_{\mathsf{KD}}$ for KD

one of $\Box_R$ or d. In these cases after using the induction hypothesis we use Lem. 5 to delete the marker $\bot$, apply the corresponding 2-sequent rule and add a new marker using Lem. 5 . For the second statement the only interesting case is if the last applied rule was $\Box \Rightarrow$. Depending on whether the rule was applied to the maximum of the premiss or not we apply the corresponding rule d or $\Box_L$. $\Box$

Thus by the results in [15] we immediately obtain cut-free completeness of the calculus $\mathsf{LNS}_{\mathsf{KD}}$ (and hence also its full nested version) for modal logic KD. This connection suggests to construct 2-sequent calculi for other modal logics as well by restricting the established nested sequent rules to the linear setting and formulating the calculi using 2-sequents. E.g., since the rule d is not present in the nested calculus for modal logic K, in the 2-sequent setting we would impose the restriction on the rule $\Box \Rightarrow$ that the formula A is not the maximum in the premiss. However, cut-free completeness is not immediate, since the cut elimination proofs for the nested calculi use the tree structure, and hence do not transfer to the linear setting easily. While instead we could adapt Masini's cut elimination proof for $\mathsf{C\text{-}2SC}$, below we use a much more straightforward method. As the fact that the empty part of a 2-sequent is not interpreted is a slight technical disadvantage for logics not containing KD, from now on we work in the linear nested setting.

4 Connections to Sequent Calculi

While Masini's calculus for KD has some philosophical advantages, there is also a well known sequent calculus for this logic. The connection between the two calculi is given by the observation that linear nested sequents, being lists of sequents, have the same data structure as histories in a backwards proof search procedure for a sequent calculus, with the nesting representing the transitions from conclusion to premisses for non-invertible rules. We use this simple idea to give quick completeness proofs for a number of linear nested calculi for modal logics as well as for the linear version of a nested calculus for intuitionistic logic.

4.1 Other Modal Logics

To make the connection to backwards proof search for sequent calculi clearer, we consider modifications of the linear versions of the rules from Fig. 2 according to

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Box A, \Delta /\!\!/ \ \Rightarrow A}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Box A, \Delta}\ \Box_R^k \qquad \frac{\mathcal{S}\{\Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi\}}{\mathcal{S}\{\Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi\}}\ \Box_L^k$$

$$\frac{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/ A \Rightarrow}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{d}^k \qquad \frac{\mathcal{S}\{\Gamma, \Box A, A \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, \Box A \Rightarrow \Delta\}}\ \mathsf{t}^k$$

$$\frac{\mathcal{S}\{\Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma, \Box A \Rightarrow \Pi\}}{\mathcal{S}\{\Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi\}}\ \mathsf{4}^k \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Box A, \Delta /\!\!/ \Sigma \Rightarrow \Box A, \Pi\}}{\mathcal{S}\{\Gamma \Rightarrow \Box A, \Delta /\!\!/ \Sigma \Rightarrow \Pi\}}\ \mathsf{5}^k$$

Fig. 5. Modal linear nested sequent rules in their Kleene'd versions

Kleene's method for the G3-calculi [9], i.e., we copy the principal formula into the premiss. The resulting rules are shown in Fig. 5, with $\mathsf{5}^k$ motivated directly by sequent rules and not normally considered in nested sequents. The calculus $\mathsf{LNS_K}$ contains the accordingly Kleene'd propositional rules, the structural rules $\mathsf{W}, \mathsf{ICL}, \mathsf{ICR}$ (Fig. 4) and the rules $\Box_R^k, \Box_L^k$. For a set $\mathcal{A} \subseteq \{\mathsf{D}, \mathsf{T}, \mathsf{4}, \mathsf{5}\}$ of modal axioms the calculus $\mathsf{LNS_{K+\mathcal{A}}}$ is obtained from $\mathsf{LNS_K}$ by adding the corresponding rules, e.g., the calculus $\mathsf{LNS_{K+\{T,4\}}}$ is $\mathsf{LNS_K}$ with the additional rules t^k and $\mathsf{4}^k$. We only consider cases where $\mathsf{5}$ never occurs without $\mathsf{4}$, and thus also write $\mathsf{45}$ instead of $\mathsf{4}, \mathsf{5}$. Soundness of the calculi without $\mathsf{5}^k$ follows immediately from the corresponding results for the full nested calculi. For calculi with $\mathsf{5}^k$ we use that axiom $\mathsf{5}$ corresponds to the frame property $\forall xyz(xRy \wedge xRz \to yRz)$ of *Euclideanness* [2] to establish the lemma below, and induction on the derivation.

Lemma 7. *The rule $\mathsf{5}^k$ preserves validity in Euclidean frames w.r.t. $\iota_\Box$.*

Proof. If the negation $\bigwedge \Gamma_1 \wedge \neg \bigvee \Delta_1 \wedge \Diamond(\ldots(\bigwedge \Gamma_n \wedge \Diamond \neg A \wedge \neg \bigvee \Delta_n \wedge \Diamond(\bigwedge \Gamma_{n+1} \wedge \neg \bigvee \Delta_{n+1} \vee \neg \iota_\Box(\mathcal{H})))\ldots)$ of the interpretation of the conclusion is satisfied in a Euclidean frame, there are worlds $w_1, \ldots, w_{n+1}$ with $w_i R w_{i+1}$ such that $w_i \Vdash \bigwedge \Gamma_i \wedge \neg \bigvee \Delta_i$ and $w_n \Vdash \Diamond \neg A$. Thus for a w with $w_n R w$ we have $w \Vdash \neg A$. By Euclideanness we also have $w_{n+1} R w$ and hence $w_{n+1} \Vdash \bigwedge \Gamma_{n+1} \wedge \Diamond \neg A \wedge \neg \bigvee \Delta_{n+1}$ and the negation of the interpretation of the premiss is satisfied in w_1. ☐

The completeness proof then simulates the rules of the sequent calculi from Fig. 6 in the rightmost component. E.g., the sequent rule for K is translated into the derivation steps below right (with double lines for multiple rule applications).

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ \mathsf{k} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\mathcal{G} /\!\!/ \Box\Gamma \Rightarrow \Box A /\!\!/ \Gamma \Rightarrow A}{\mathcal{G} /\!\!/ \Box\Gamma \Rightarrow \Box A /\!\!/ \ \Rightarrow A}\ \Box_L^k}{\mathcal{G} /\!\!/ \Box\Gamma \Rightarrow \Box A}\ \Box_R^k \tag{1}$$

Of course this does not take into account the formula interpretation of nested sequents. But as we are only interested in the theorems of the logic this is enough. Thus, intuitively, while linear nested sequents capture branches of the search tree (i.e., histories), full nested sequents also capture its existential choices.

Theorem 8. *For $\mathcal{A} \subseteq \{\mathsf{D}, \mathsf{T}, \mathsf{4}\}$ or $\mathcal{A} \in \{\{\mathsf{4}, \mathsf{5}\}, \{\mathsf{4}, \mathsf{5}, \mathsf{d}\}\}$ the calculus $\mathsf{LNS_{K+\mathcal{A}}}$ is complete for $\mathsf{K} + \mathcal{A}$, i.e., for all formulae B: if $\models_{\mathsf{K+\mathcal{A}}} B$ then $\vdash_{\mathsf{LNS_{K+\mathcal{A}}}} \Rightarrow B$.*

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\ \mathsf{k} \qquad \frac{\Gamma, A \Rightarrow}{\Box\Gamma, \Box A \Rightarrow}\ \mathsf{d} \qquad \frac{\Gamma, \Box A, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\ \mathsf{t} \qquad \frac{\Box\Gamma, \Delta \Rightarrow A}{\Box\Gamma, \Box\Delta \Rightarrow \Box A}\ \mathsf{4}$$

$$\frac{\Gamma, \Box\Sigma \Rightarrow A, \Box\Pi}{\Box\Gamma, \Box\Sigma \Rightarrow \Box A, \Box\Pi}\ \mathsf{45}\ \text{where } \emptyset \neq \Pi \qquad \frac{\Gamma, \Box\Sigma \Rightarrow \Delta, \Box\Pi}{\Box\Gamma, \Box\Sigma \Rightarrow \Box\Delta, \Box\Pi}\ \mathsf{45d}\ \text{where } |\Delta| \leq 1$$

Fig. 6. Standard modal sequent rules

Proof. We translate a sequent derivation $\mathcal{D}$ bottom-up into a linear nested derivation as follows. If we have constructed a derivation tree with $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta$ at a leaf, and the last rule in the subderivation of $\mathcal{D}$ ending in the corresponding sequent $\Gamma \Rightarrow \Delta$ was one of k, d or $\mathsf{4}$, we add some steps above the leaf of the nested sequent derivation, giving a new leaf corresponding to the premiss of the sequent rule. For k the steps are as in (1) above, for $\mathsf{45}$ they are

$$\frac{\Gamma, \Box\Sigma \Rightarrow A, \Box\Pi}{\Box\Gamma, \Box\Sigma \Rightarrow \Box A, \Box\Pi}\ \mathsf{45} \quad \rightsquigarrow \quad \cfrac{\cfrac{\cfrac{\mathcal{G} /\!\!/ \Box\Gamma, \Box\Sigma \Rightarrow \Box A, \Box\Pi /\!\!/ \Gamma, \Box\Sigma \Rightarrow A, \Box\Pi}{\mathcal{G} /\!\!/ \Box\Gamma, \Box\Sigma \Rightarrow \Box A, \Box\Pi /\!\!/ \Box\Sigma \Rightarrow A, \Box\Pi}\ \Box_L^k}{\mathcal{G} /\!\!/ \Box\Gamma, \Box\Sigma \Rightarrow \Box A, \Box\Pi /\!\!/ \ \Rightarrow A}\ \mathsf{4}^k, \mathsf{5}^k}{\mathcal{G} /\!\!/ \Box\Gamma, \Box\Sigma \Rightarrow \Box A, \Box\Pi}\ \Box_R^k$$

The transformations for the sequent rules $\mathsf{d}, \mathsf{4}$, and $\mathsf{45d}$ are similar, those for the propositional rules and t straightforward. Completeness then follows from the result for the standard sequent calculi, see e.g. [27] for references. ☐

The proof above even shows a slightly stronger statement, namely that it is enough to apply the logical rules only to the rightmost sequent.

Definition 9. *The* end-component *of* $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta \in \mathsf{LNS}$ *is the component* $\Gamma \Rightarrow \Delta$*. For* $\mathsf{LNS}_{\mathcal{L}}$ *one of the calculi above, its* end-variant $\mathsf{LNS}^*_{\mathcal{L}}$ *adds the restriction that the end-component of the conclusion must be active to every rule.*

Corollary 10. *Let* $\mathcal{A} \subseteq \{\mathsf{D}, \mathsf{T}, \mathsf{4}\}$ *or* $\mathcal{A} \in \{\{\mathsf{45}\}, \{\mathsf{d}, \mathsf{45}\}\}$*. Then the end-variant* $\mathsf{LNS}^*_{\mathsf{K}+\mathcal{A}}$ *of the calculus* $\mathsf{LNS}_{\mathsf{K}+\mathcal{A}}$ *is sound and complete for the logic* $\mathsf{K} + \mathcal{A}$. ☐

This might also be shown by permuting rules, as done in [15, Prop. 2] for C-2SC, where derivations in the end-variant are called *leveled*. However, the proof above seems to make the connection to sequent calculi clearer. Of course this result also carries over to the full nested sequent calculi. This method also yields completeness for variants of the calculi formulated using the rules in Fig. 7. For a set $\mathcal{A} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{4}, \mathsf{45}\}$ we write $\dot{\mathcal{A}}$ for the set with the rules $\dot{\mathsf{r}}$ instead of r. The rules $\dot{\mathsf{4}}$ and $\dot{\mathsf{45}}$ differ from the standard nested sequent treatment [3,13], where the structural variant of $\mathsf{4}$ is taken to be rule $\bar{\mathsf{4}}$ of Fig. 7 (which is derivable using $\dot{\mathsf{4}}$).

Proposition 11. *Let* $\mathcal{A} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{4}\}$ *or* $\mathcal{A} \in \{\{\mathsf{45}\}, \{\mathsf{d}, \mathsf{45}\}\}$*. Then the calculus* $\mathsf{LNS}_{\mathsf{K}+\dot{\mathcal{A}}}$ *and its end-variant* $\mathsf{LNS}^*_{\mathsf{K}+\dot{\mathcal{A}}}$ *are (cut-free) complete for* $\mathsf{K} + \mathcal{A}$.

$$\frac{\mathcal{G} /\!\!/ \ \Rightarrow}{\mathcal{G}}\ \dot{\mathsf{d}} \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi\}}{\mathcal{S}\{\Gamma, \Sigma \Rightarrow \Delta, \Pi\}}\ \dot{\mathsf{t}} \qquad \frac{\mathcal{G} /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \ \Rightarrow \ /\!\!/ \mathcal{H}}\ \bar{\mathsf{4}}$$

$$\frac{\mathcal{S}\{\Box\Gamma, \Sigma \Rightarrow \Pi\}}{\mathcal{S}\{\Box\Gamma \Rightarrow \ /\!\!/ \Sigma \Rightarrow \Pi\}}\ \dot{\mathsf{4}} \qquad \frac{\mathcal{S}\{\Box\Gamma, \Sigma \Rightarrow \Box\Delta, \Pi\}}{\mathcal{S}\{\Box\Gamma \Rightarrow \Box\Delta /\!\!/ \Sigma \Rightarrow \Pi\}}\ \dot{\mathsf{45}}$$

Fig. 7. The structural variants of the modal rules

Proof. As above, we simulate a derivation in the corresponding sequent calculus. The rules t and $\mathsf{45}$ are simulated by

$$\dfrac{\dfrac{\dfrac{\mathcal{G} /\!\!/ \Box A \Rightarrow \ /\!\!/ \Gamma, \Box A, A \Rightarrow \Delta}{\mathcal{G} /\!\!/ \Box A \Rightarrow \ /\!\!/ \Gamma, \Box A \Rightarrow \Delta}\ \Box_L^k}{\mathcal{G} /\!\!/ \Gamma, \Box A, \Box A \Rightarrow \Delta}\ \dot{\mathsf{t}}}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{ICL} \qquad \dfrac{\dfrac{\mathcal{G} /\!\!/ \Box\Gamma, \Sigma \Rightarrow \Box\Delta, A}{\mathcal{G} /\!\!/ \Box\Gamma, \Box\Sigma \Rightarrow \Box\Delta, \Box A /\!\!/ \Sigma \Rightarrow A}\ \dot{\mathsf{45}}}{\mathcal{G} /\!\!/ \Box\Gamma, \Box\Sigma \Rightarrow \Box\Delta, \Box A}\ \Box_L^k, \Box_R^k$$

The other rules are similar, e.g., in the case of $\mathsf{45d}$ we replace $\Box_R$ above by $\dot{\mathsf{d}}$. ☐

Hence we obtain modular calculi for logics with axioms from the sets $\{\mathsf{d}, \mathsf{t}, \mathsf{4}\}$ resp. $\{\mathsf{d}, \mathsf{4}, (\mathsf{4} \wedge \mathsf{5})\}$. As the logical rules absorb the structural rules it is not surprising that the latter are admissible. They are made admissible in the structural variants if the rules $\dot{\mathsf{t}}, \dot{\mathsf{4}}$ and $\dot{\mathsf{45}}$ are replaced with the following rules (call the resulting rule sets $\dot{\mathcal{A}}^k$).

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta /\!\!/ \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta\}} \quad \frac{\mathcal{S}\{\Box\Gamma, \Sigma \Rightarrow \Pi\}}{\mathcal{S}\{\Box\Gamma, \Omega \Rightarrow \Theta /\!\!/ \Sigma \Rightarrow \Pi\}} \quad \frac{\mathcal{S}\{\Box\Gamma, \Sigma \Rightarrow \Box\Delta, \Pi\}}{\mathcal{S}\{\Box\Gamma, \Omega \Rightarrow \Box\Delta, \Theta /\!\!/ \Sigma \Rightarrow \Pi\}}$$

Lemma 12. *For* $\mathcal{A} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{4}, \mathsf{45}\}$ *The rules* W *of weakening and* $\mathsf{ICL}, \mathsf{ICR}$ *of contraction are admissible in* $\mathsf{LNS}_{\mathsf{K}+\mathcal{A}}$ *and* $\mathsf{LNS}_{\mathsf{K}+\dot{\mathcal{A}}^k}$ *without these rules.*

Proof. Standard by induction on the depth of the derivation. ☐

4.2 Intuitionistic Logic

The same idea can be used to show completeness for the linear versions of the nested calculi for propositional and (full) first-order intuitionistic logic from [7]. The language is defined as usual using the propositional connectives $\bot, \wedge, \vee, \rightarrow$ and the quantifiers $\forall$ and $\exists$. Following [7] to avoid clashes of variables we make use of a denumerable set $a, b, \ldots$ of special variables called *parameters* which only occur in derivations, but not in their conclusions. *(Intuitionistic) linear nested sequents* then are linear nested sequents built from formulae of this language. In the absence of modalities we reinterpret the nesting in terms of implication.

Definition 13. *The* intuitionistic formula translation ι_{Int} *for* LNS *is given by*

1. *if* $\Gamma \Rightarrow \Delta$ *is a sequent, then* $\iota_{\mathsf{Int}}(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \rightarrow \bigvee \Delta$
2. $\iota_{\mathsf{Int}}(\Gamma \Rightarrow \Delta /\!\!/ \mathcal{G}) = \bigwedge \Gamma \rightarrow (\bigvee \Delta \vee (\iota_{\mathsf{Int}}(\mathcal{G})))$.

$$\frac{\mathcal{S}\{\Gamma, A \to B \Rightarrow A, \Delta\} \quad \mathcal{S}\{\Gamma, A \to B, B \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, A \to B \Rightarrow \Delta\}} \to_L \quad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow A \to B, \Delta /\!\!/ A \Rightarrow B}{\mathcal{G} /\!\!/ \Gamma \Rightarrow A \to B, \Delta} \to_R$$

$$\frac{}{\mathcal{S}\{\Gamma, A \Rightarrow A, \Delta\}} \mathsf{init} \quad \frac{\mathcal{S}\{\Gamma, A \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi\}}{\mathcal{S}\{\Gamma, A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi\}} \mathsf{Lift}$$

$$\frac{\mathcal{G} /\!\!/ \Gamma, \forall x A(x), A(a) \Rightarrow \Delta /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \forall x A(x) \Rightarrow \Delta /\!\!/ \mathcal{H}} \forall_L \quad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \forall x\, A(x), \Delta /\!\!/ \;\Rightarrow A(a)}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \forall x\, A(x), \Delta} \forall_R$$

a does not occur in $\mathcal{H}$ $\quad$ a not in conclusion

$$\frac{\mathcal{S}\{\Gamma, \exists x A(x), A(a) \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, \exists x A(x) \Rightarrow \Delta\}} \exists_L \quad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow A(a), \exists x A(x), \Delta /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \exists x A(x), \Delta /\!\!/ \mathcal{H}} \exists_R$$

a not in conclusion $\quad$ a does not occur in $\mathcal{H}$

Fig. 8. Some representative rules of $\mathsf{LNS_{Int}}$

The calculus $\mathsf{LNS_{Int}}$ contains the linear (and multiset) versions of the rules of the calculus for first-order intuitionistic logic from [7] and the structural rules (Fig. 8). In the linear setting the variable condition on $\forall_L$ and $\exists_R$ is simplified to the parameter a not occurring to the right of the active component. The completeness proof is based on the multi-succedent sequent calculus m-$\mathsf{G3i}$ [26].

Theorem 14. *The calculus* $\mathsf{LNS_{Int}}$ *is complete for first-order intuitionistic logic.*

Proof. We convert a derivation $\mathcal{D}$ in m-$\mathsf{G3i}$ bottom-up into a derivation in $\mathsf{LNS_{Int}}$. To ensure the variable conditions in $\exists_L, \forall_R$ are satisfied we first rename parameters in $\mathcal{D}$ such that no parameter occurs between the end-sequent and an application of $\exists_L$ or $\forall_R$ where the same parameter is eliminated. The $\to_R$ rule converts thus:

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B, \Delta} \quad \rightsquigarrow \quad \cfrac{\cfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow A \to B, \Delta /\!\!/ \Gamma, A \Rightarrow B}{\mathcal{G} /\!\!/ \Gamma \Rightarrow A \to B, \Delta /\!\!/ A \Rightarrow B} \mathsf{Lift}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow A \to B, \Delta} \to_R$$

The other propositional rules are straightforward. For the quantifier rules we also need to verify that the variable condition holds. For $\forall_R$, the conversion is

$$\frac{\Gamma \Rightarrow A(a)}{\Gamma \Rightarrow \forall x A(x), \Delta} \forall_R \quad \rightsquigarrow \quad \cfrac{\cfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \forall x A(x), \Delta /\!\!/ \Gamma \Rightarrow A(a)}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \forall x A(x), \Delta /\!\!/ \;\Rightarrow A(a)} \mathsf{Lift}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \forall x A(x), \Delta} \forall_R$$

Since after the initial renaming the parameter a does not occur below the application of $\forall_R$ on the left, it does not occur in $\mathcal{G}$, and the variable condition for the linear nested $\forall_R$ rule is satisfied. The other quantifier rules are translated directly, where for $\forall_L$ and $\exists_R$ the variable condition is satisfied trivially. □

Again the proof yields completeness of the end-variant $\mathsf{LNS^*_{Int}}$ of the calculus.

Corollary 15. *The calculus* $\mathsf{LNS^*_{Int}}$ *is complete for intuitionistic logic.* □

While soundness follows from soundness of the full nested calculus of [7], there no formula interpretation is considered. However, using Kripke-semantics (see *op. cit.*) it is not hard to check that all the rules preserve soundness under ι_{Int}.

Theorem 16. *The rules of* $\mathsf{LNS}_{\mathsf{Int}}$ *preserve validity in intuitionistic Kripke-frames w.r.t. the formula interpretation* ι_{Int}.

Proof. For the rules $\vee_L, \wedge_R$ and $\rightarrow_R$ this is trivial. For the remaining rules we construct a world falsifying the interpretation of a premiss from a world falsifying the interpretation of the conclusion. E.g., for Lift, suppose that the interpretation $\bigwedge \Gamma_1 \rightarrow \bigvee \Delta_1 \vee (\dots (\bigwedge \Gamma_n \wedge A \rightarrow \bigvee \Delta_n \vee (\bigwedge \Gamma_{n+1} \wedge A \rightarrow \bigvee \Delta_{n+1} \vee \iota_{\mathsf{Int}}(\mathcal{H}))) \dots)$ of its conclusion does not hold in world w in an intuitionistic Kripke-frame. Then there are worlds $w \leq w_1 \leq \dots \leq w_n \leq w_{n+1}$ with $w_i \Vdash \bigwedge \Gamma_i$ and $w_i \nVdash \bigvee \Delta_i$ such that $w_n \Vdash A$ and $w_{n+1} \nVdash \iota_{\mathsf{Int}}(\mathcal{H})$. By monotonicity we have $w_{n+1} \Vdash A$, and thus the formula interpretation of the premiss is falsified in w.

For the quantifier rules $\forall_L$ and $\exists_R$ we use that the domains are expanding. E.g., if the interpretation $\bigwedge \Gamma_1 \rightarrow \bigvee \Delta_1 \vee (\dots (\bigwedge \Gamma_n, \forall x A(x) \rightarrow \bigvee \Delta_n \vee \iota_{\mathsf{Int}}(\mathcal{H})) \dots)$ of the conclusion of $\forall_L$ does not hold at world w in an intuitionistic Kripke-frame, there are worlds $w \leq w_1 \leq \dots \leq w_n \leq w_{n+1}$ with $w_i \Vdash \bigwedge \Gamma_i$ and $w_i \nVdash \Delta_i$ for $i \leq n$ as well as $w_n \Vdash \forall x A(x)$ and $w_{n+1} \nVdash \iota_{\mathsf{Int}}(\mathcal{H})$. Since the domains are expanding, if at a world v with $v \leq w$ the parameter a is interpreted by an element a of the domain of v, then a is in the domain of w_n as well and a is interpreted by a in w_n. Hence $w_n \Vdash A(a)$ and the interpretation of the premiss of $\forall_L$ is falsified at w. If a is not interpreted in a predecessor of w_n we interpret it at w_n arbitrarily. In this case by the variable condition it does not occur in $\iota_{\mathsf{Int}}(\mathcal{H})$, and so this interpretation is legal. Soundness of $\exists_R$ is shown similarly.

For $\forall_R$ we use that a formula $\forall x A(x)$ is falsified in a world w if the fresh parameter a can be interpreted in a successor of w in a way that $A(a)$ is falsified there. In particular, $\forall x A(x)$ is falsified in w iff the implication $\top \rightarrow A(a)$ for a fresh parameter a is falsified in w. The reasoning for $\exists_L$ is similar but easier. □

Restricting these proofs to the propositional level obviously also shows soundness and completeness of the restrictions $\mathsf{LNS}_{\mathsf{pInt}}$ and $\mathsf{LNS}^*_{\mathsf{pInt}}$ of $\mathsf{LNS}_{\mathsf{Int}}$ resp. $\mathsf{LNS}^*_{\mathsf{Int}}$ to the propositional rules w.r.t. propositional intuitionistic logic .

5 Hypersequents

Another rather successful proof-theoretic framework extending the sequent framework is that of *hypersequent calculi*, introduced independently in [17,22,1] to obtain cut-free calculi for modal logic $\mathsf{S5}$ (and other logics). The fundamental data structure of hypersequent calculi is the same as for LNS: A *hypersequent* is a finite list of sequents, written $\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$. However, the formula interpretation for hypersequents is usually taken as some form of disjunction, in contrast to the nested interpretation of linear nested sequents. E.g., for modal logics the above hypersequent is interpreted as $\bigvee_{i \leq n} \Box(\bigwedge \Gamma_i \rightarrow \bigvee \Delta_i)$, in the intuitionistic setting as $\bigvee_{i \leq n}(\bigwedge \Gamma_i \rightarrow \bigvee \Delta_i)$. This interpretation motivates the

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi\}}{\mathcal{S}\{\Sigma \Rightarrow \Pi /\!\!/ \Gamma \Rightarrow \Delta\}}\ \mathsf{EEX} \qquad \frac{\mathcal{G} /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \mathcal{H}}\ \mathsf{EW} \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta /\!\!/ \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta\}}\ \mathsf{EC}$$

Fig. 9. External structural rules in the linear nested setting

external structural rules which allow to reorder the components, add new components or remove duplicates, mirroring the corresponding properties of disjunction. Disregarding the formula interpretation linear nested sequents thus could be called substructural or non-commutative hypersequents, and hypersequents could be called linear nested sequents with the additional *external* structural rules of *exchange* EEX, *weakening* EW and *contraction* EC shown in Fig. 9.

5.1 Modal Logic S5

We first consider the modal setting. Comparing the external structural rules with the linear nested rules above it can be seen that the rules EW and EC are interderivable (using internal structural rules) with the structural variants $\bar{4}$ and $\dot{\mathsf{t}}$ of the transitivity and reflexivity rules. E.g., EW and EC are derivable via

$$\dfrac{\dfrac{\mathcal{G} /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \ \Rightarrow \ /\!\!/ \mathcal{H}}\ \bar{4}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \mathcal{H}}\ \mathsf{W} \qquad \text{and} \qquad \dfrac{\dfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma, \Gamma \Rightarrow \Delta, \Delta /\!\!/ \mathcal{H}}\ \dot{\mathsf{t}}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \mathcal{H}}\ \mathsf{Con}$$

This might explain why most modal hypersequent calculi in the literature concern extensions of S4. Probably the most-investigated modal logic in the hypersequent framework is modal logic S5 [17,22,1,24,19,11,10]. Before analysing some of these calculi in terms of linear nested sequents we note that the external exchange rule, present in all of them, is sound under the nested interpretation as well.

Lemma 17. *The rule* EEX *preserves* S5*-validity under the interpretation* $\iota_\Box$*.*

Proof. Using transitivity and symmetry of the accessibility relation in S5-models it is straightforward to check that if a world in such a model satisfies the negation $\bigwedge \Gamma_1 \wedge \neg \bigvee \Delta_1 \wedge \Diamond(\dots \Diamond(\wedge \Gamma_n \wedge \neg \bigvee \Delta_n \wedge \Diamond(\bigwedge \Gamma_{n+1} \wedge \neg \bigvee \Delta_{n+1} \wedge \Diamond \iota_\Box(\mathcal{H}))) \dots)$ of the formula translation of the conclusion of EEX, it also satisfies the negation $\bigwedge \Gamma_1 \wedge \neg \bigvee \Delta_1 \wedge \Diamond(\dots \Diamond(\wedge \Gamma_{n+1} \wedge \neg \bigvee \Delta_{n+1} \wedge \Diamond(\bigwedge \Gamma_n \wedge \neg \bigvee \Delta_n \wedge \Diamond \iota_\Box(\mathcal{H}))) \dots)$ of the formula interpretation of the premiss. □

A simple approach to obtaining a linear nested sequent calculus for S5 then would be to extend the calculus $\mathsf{LNS}_{\mathsf{K+45}}$ for modal logic K45 with all the linear nested rules which are sound for S5 and hope to obtain completeness. This amounts to extending $\mathsf{LNS}_{\mathsf{K+45}}$ with t and its structural variant $\dot{t}$ (i.e., external contraction) as well as external exchange EEX (external weakening EW is derivable using $\dot{45}$). But the rule $\dot{45}$ is exactly Avron's *modalised splitting rule MS*, so we obtain (the weak version of) his calculus from [1]. Completeness thus follows from the completeness results for the hypersequent calculus given there. Replacing the rule $\dot{45}$ with the rule $\dot{4}$ yields essentially Kurokawa's system for $\mathsf{S5}$ from [10], apart from the fact that there the standard sequent right rule for $\Box$ from $\mathsf{S4}$ is used. Completeness of this calculus can be seen by showing that the latter rule is derivable, or alternatively by showing that it can derive all the rules from the system $\mathsf{H}\mathcal{R}_{\mathsf{KT}}\{\mathsf{5}_n : n \in \mathbb{N}\}$ from [12, Cor. 4.7].

Dropping the rules $\dot{45}$ resp. $\dot{4}$ and the logical rule t altogether and keeping only the external structural rules $\mathsf{EEX}, \dot{\mathsf{t}}$ and $\bar{4}$ yields essentially Restall's second calculus from [24]. In Restall's calculus external weakening with an empty sequent is not allowed, but clearly in terms of derivability of one-component hypersequents the two systems are equivalent. The external structural rules $\dot{\mathsf{t}}$ and $\bar{4}$ then are exchanged by Poggiolesi in [19] for the logical rule t^k and the (still invertible) un-Kleene'd rule $\Box_R$ (Fig. 4) instead of $\Box_R^k$. Finally, rewriting set-based rules to multisets, the calculus constructed from the frame condition of universality using Lahav's general method [11] is the calculus obtained by adding external exchange and the structural rules absorbing variant of $\bar{4}$ to the direct translation of backwards proof search in a sequent calculus for KT with the rules

$$\frac{\mathcal{G} \,/\!\!/\, \Box\Gamma \Rightarrow \Box A \,/\!\!/\, \Gamma \Rightarrow A}{\mathcal{G} \,/\!\!/\, \Box\Gamma \Rightarrow \Box A} \qquad \frac{\mathcal{S}\{\Gamma, \Box\Sigma \Rightarrow \Delta \,/\!\!/\, \Gamma, \Box\Sigma, \Sigma \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, \Box\Sigma \Rightarrow \Delta\}}$$

and a version of $\Box_L$ which allows to treat multiple formulae at once:

$$\frac{\mathcal{S}\{\Gamma, \Box\Sigma \Rightarrow \Delta \,/\!\!/\, \Omega, \Sigma \Rightarrow \Theta\}}{\mathcal{S}\{\Gamma, \Box\Sigma \Rightarrow \Delta \,/\!\!/\, \Omega \Rightarrow \Theta\}}$$

It is straightforward to check that these rules are equivalent to Restall's rules together with t^k. Again, from the completeness proofs given for the hypersequent calculi we obtain quick completeness proofs for the linear nested sequent calculi.

5.2 Classical Logic

Going the other direction, we can construct a hypersequent calculus from a linear nested sequent calculus by adding the external exchange rule to the calculus for intuitionistic logic from Sec. 4.2. Since this makes excluded middle derivable via

$$\dfrac{\dfrac{\dfrac{A \Rightarrow \bot \,/\!\!/\, A \Rightarrow A, A \to \bot}{A \Rightarrow \bot \,/\!\!/\, \Rightarrow A, A \to \bot}\ \mathsf{Lift}}{\Rightarrow A, A \to \bot \,/\!\!/\, A \Rightarrow \bot}\ \mathsf{EEX}}{\Rightarrow A \vee (A \to \bot)}\ \to_R, \vee_R$$

it should not come as a surprise that this gives a calculus for classical logic. Soundness of the rules is checked by routine methods, while for completeness again we make use of the completeness result for a standard sequent calculus.

Lemma 18. *The rules of* $\mathsf{LNS}_{\mathsf{Int+EEX}}$ *preserve validity of the interpretation of the linear nested sequents in classical logic.* □

Theorem 19. *The calculus* $\mathsf{LNS}_{\mathsf{Int+EEX}}$ *is (cut-free) complete for classical logic.*

Proof. By showing that if a sequent $\Gamma \Rightarrow \Delta$ is derivable in the calculus $\mathsf{G3}$ of [9], then it is derivable in $\mathsf{LNS}_{\mathsf{Int+EEX}}$. For this from a derivation $\mathcal{D}$ in $\mathsf{G3}$ we construct bottom-up a derivation in $\mathsf{LNS}_{\mathsf{Int+EEX}}$ such that every rule application in $\mathcal{D}$ corresponds to a linear subderivation in $\mathcal{D}'$ and every formula in a conclusion of a rule application in $\mathcal{D}$ corresponds to exactly one formula in the conclusion of the corresponding subderivation. The interesting cases are if the last applied rule in $\mathcal{D}$ was $\to_R$ or $\forall_R$. In the former case we perform the following transformation:

$$\dfrac{\Gamma, A \Rightarrow B, A \to B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}\ \to_R \qquad \leadsto \qquad \dfrac{\dfrac{\mathcal{G} /\!\!/ \mathcal{H} /\!\!/ \Sigma \Rightarrow A \to B, \Pi /\!\!/ A \Rightarrow B}{\mathcal{G} /\!\!/ \mathcal{H} /\!\!/ \Sigma \Rightarrow A \to B, \Pi}\ \to_R}{\mathcal{G} /\!\!/ \Sigma \Rightarrow A \to B, \Pi /\!\!/ \mathcal{H}}\ \mathsf{EEX}$$

where the correspondence between formulae extends in the natural way to the premisses of rules resp. subderivations. For $\forall_R$ the transformation is similar, and for the other propositional rules the transformations are the obvious ones.

For the initial sequents we use Lift, distinguishing cases according to where the principal formulae occur in the nested sequent. The most involved case is:

$$\dfrac{}{\Gamma, A \Rightarrow A, \Delta}\ \mathsf{init} \qquad \leadsto \qquad \dfrac{\dfrac{\dfrac{}{\mathcal{S}\{\mathcal{G} /\!\!/ \Omega, A \Rightarrow \Theta /\!\!/ \Sigma, A \Rightarrow A, \Pi\}}\ \mathsf{init}}{\mathcal{S}\{\mathcal{G} /\!\!/ \Omega, A \Rightarrow \Theta /\!\!/ \Sigma \Rightarrow A, \Pi\}}\ \mathsf{Lift}}{\mathcal{S}\{\Sigma \Rightarrow A, \Pi /\!\!/ \mathcal{G} /\!\!/ \Omega, A \Rightarrow \Theta\}}\ \mathsf{EEX}$$

The remaining cases are similar but easier. ☐

The interest of this result lies not so much in the fact that there is (yet another) calculus for classical logic, but in the fact that it is obtained from a calculus for intuitionistic logic just by adding a *structural* rule. In this respect intuitionistic logic could also be seen as a substructural logic obtained by deleting the external exchange rule from the calculus for classical logic. The propositional fragment of the resulting calculus is similar to the hypersequent calculus for classical logic from [4, Rem. 6]. However, since the calculus given there extends a single-conclusion hypersequent calculus for intuitionistic logic, the rules are slightly different, most notably the implication right rule. A similar approach purely on the sequent level was explored in [25,23], where a calculus for intuitionistic logic is obtained from one for classical logic by dropping the internal exchange rule.

6 Conclusion

The presented linear nested sequent calculi show that to capture extensions of K with arbitrary sets of axioms from $\mathsf{d}, \mathsf{t}, \mathsf{4}, (\mathsf{4} \wedge \mathsf{5})$ in a proof-theoretically satisfying way it is sufficient to generalise the sequent framework to lists of sequents instead of trees, thus providing a slightly simpler formalism than that of nested sequents. In particular, in these calculi all connectives have separate left and right rules. Since linear nested sequents are essentially 2-sequents, this might support Masini's idea of the 2-sequent calculus as “a proof theory of modalities” [15]. Furthermore, we obtained linear nested calculi for intuitionistic and classical logic differing only in one structural rule and thus satisfying what has been called Došen's Principle in [27]. These results raise a whole array of open questions for future work, such as: finding a general method for syntactic cut elimination, possibly following [15]; the construction of linear nested calculi for more challenging modal logics such as extensions of K with axiom B or intuitionistic modal logics; more generally, the construction of linear nested rules from axioms to capture e.g. intermediate logics such as Bd_k; or finding limitative results stating that a given logic cannot be captured by structural rules in the linear nested setting.

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