Axioms vs Hypersequent Rules with Context Restrictions: Theory and Applications*

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Abstract. We introduce transformations between hypersequent rules with context restrictions and Hilbert axioms extending classical (and intuitionistic) propositional logic and vice versa. The introduced rules are used to prove uniform cut elimination, decidability and complexity results as well as finite axiomatisations for many modal logics given by simple frame properties. Our work subsumes many logic-tailored results and allows for new results. As a case study we apply our methods to the logic of uniform deontic frames.

1 Introduction

The automatic construction of reasoning systems and decision procedures from specifications for various logics is an important emerging area in the field of automated reasoning. Results in this area provide general decision procedures and complexity results applicable to specific logics in the spirit of Logic Engineering [12], and also yield deeper insights into strengths, weaknesses, and fundamental properties of different types of calculi used for reasoning systems. But also from the perspective of producing such systems for specific logics investigating the connections between specifications and different frameworks is important, since this allows choosing the most efficient framework for the logic at hand.

Here we investigate the connection between specifications given as Hilbert axioms and the framework of *hypersequent calculi* for extensions of classical propositional logic. Taking the specifications as Hilbert axioms yields a very flexible and semantics-independent approach and allows to capture non-normal modal logics (unlike e.g. [9]) Also, while often not complexity-optimal, the hypersequent framework is very flexible and captures several logics for which no sequent or tableaux systems seem to exists. Of course correspondence results and general decision procedures demand general results about hypersequent calculi. This necessitates a clarification of which kind of calculi we consider. To this aim we introduce the format of *hypersequent rules with context restrictions* which is general enough to capture many existing calculi, e.g. for modal logics S5 [1] and S4.3 [8] as well as for modal logics without symmetry given by simple frame properties [9]. We obtain sufficient conditions for (syntactic) cut elimination, decidability, and complexity results for such systems. The results apply e.g. to the

* Supported by FWF START Y544-N23

calculi for extensions of K or $\mathsf{K4}$ from [9]. We also show a correspondence between rules of our format and axioms of a certain form (Def. 5.16). This yields general decidability and complexity results for modal logics axiomatised this way, and as a byproduct finite axiomatisations for modal logics given by certain simple frame properties. As application we construct a new cut-free hypersequent calculus for the non-normal logic LUDF from [14], entailing a new complexity bound. While for space reasons the results in this article are given for logics with unary connectives based on classical logic, they extend to higher arities and intuitionistic logic as base logic similar to [10]. The extension of these investigations to more general frameworks such as tree-hypersequents will be considered in future work.

2 Preliminaries and Notation

In the following we write $\mathbb{N}$ for $\{0, 1, 2, \dots\}$. We take $\mathcal{V}$ to be a countable set of propositional variables. The set of *boolean connectives* is $\Lambda_{\mathsf{B}} := \{\wedge, \vee, \rightarrow\}$. For a set $\Lambda \subseteq \Lambda_{\mathsf{U}} \cup \Lambda_{\mathsf{B}}$ with Λ_{U} a set of unary connectives the set $\mathcal{F}(\Lambda)$ of *formulae over* Λ is defined by $\mathcal{F}(\Lambda) \ni \varphi ::= p \mid \bot \mid \heartsuit\varphi \mid \varphi \circ \varphi$ with $p \in \mathcal{V}, \heartsuit \in \Lambda \cap \Lambda_{\mathsf{U}}$ and $\circ \in \Lambda \cap \Lambda_{\mathsf{B}}$. The connectives $\leftrightarrow$ and $\neg$ are introduced as abbreviations as usual. Connectives in Λ_{U} are called *modalities*. The set $\{\Box\} \cup \Lambda_{\mathsf{B}}$ is denoted $\Lambda_{\Box}$. For $F \subseteq \mathcal{F}(\Lambda)$ we write $\Lambda(F)$ for $\{\heartsuit\varphi : \heartsuit \in \Lambda \setminus \Lambda_{\mathsf{B}} \text{ and } \varphi \in F\} \cup \{\varphi \circ \psi : \circ \in \Lambda \cap \Lambda_{\mathsf{B}} \text{ and } \varphi, \psi \in F\}$. The *modal rank* of a formula φ, denoted $\mathsf{mrk}(\varphi)$, is the maximum nesting depth of modalities in φ, and its *complexity* is the number of symbols occurring in it. Sequences $\varphi_1, \dots, \varphi_n$ of formulae are written $\boldsymbol{\varphi}$, and $|\boldsymbol{\varphi}|$ denotes the length of $\boldsymbol{\varphi}$. Similarly $*\varphi_1, \dots, *\varphi_n$ is written $\boldsymbol{*\varphi}$ for $* \in \Lambda$.

A *multiset* Γ over a set F of formulae is a function $F \to \mathbb{N}$ with finite support, and we write $\varphi \in \Gamma$ for $\Gamma(\varphi) > 0$. The *union* of multisets Γ and Δ is denoted by Γ, Δ and defined by $(\Gamma, \Delta)(\varphi) := \Gamma(\varphi) + \Delta(\varphi)$. We also write $\bigsqcup_{i=1}^{n} \Gamma_n$ for $\Gamma_1, \dots, \Gamma_n$ and φ for the multiset containing only one occurrence of φ. The set $\mathcal{S}(F)$ of *sequents over the set F of formulae* contains all tuples of multisets over F, written as $\Gamma \Rightarrow \Delta$. A *hypersequent over* F is a multiset over $\mathcal{S}(F)$, written as $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ We write H for the hypersequent version of a standard context-sharing sequent calculus for classical logic [11] with the standard external and internal weakening and contraction rules [1], see Table 1. The rules of $\mathcal{R}_{\mathsf{K}}$, $\mathcal{R}_{\mathsf{KT}}$ and $\mathcal{R}_{\mathsf{K4}}$ are given in Table 2.

A Λ*-logic* is a set $\mathcal{L}$ of formulae over Λ closed under modus ponens (if $\varphi \in \mathcal{L}$ and $\varphi \to \psi \in \mathcal{L}$, then $\psi \in \mathcal{L}$) and uniform substitution (if $\varphi \in \mathcal{L}$, then $\varphi\sigma \in \mathcal{L}$ for every substitution $\sigma : \mathcal{V} \to \mathcal{F}(\Lambda)$) and containing classical propositional logic. For a set $\mathcal{A}$ of formulae, $\mathcal{L}_{\mathcal{A}}$ is the smallest Λ-logic containing $\mathcal{A}$. For a Λ-logic $\mathcal{L}$ and $\varphi \in \mathcal{F}(\Lambda)$ we write $\mathcal{L} \oplus \varphi$ for the smallest Λ-logic $\mathcal{L}'$ with $\mathcal{L} \cup \{\varphi\} \subseteq \mathcal{L}'$. We also write $\models_{\mathcal{L}} \varphi$ for $\varphi \in \mathcal{L}$. For the standard notions of modal logic see [4].

3 Hypersequent Rules with Restrictions

The rule format we consider is an abstraction of the rule format found in many calculi for modal logics. One of the main characteristics is that the format of

Table 1. The propositional and structural rules of H

$$\frac{}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta, \varphi}\ \mathcal{A} \qquad \frac{}{\mathcal{G} \mid \Gamma, \bot \Rightarrow \Delta}\ \bot_{\mathsf{L}}$$

$$\frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \vee \psi \Rightarrow \Delta}\ \vee_{\mathsf{L}} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi, \psi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \vee \psi}\ \vee_{\mathsf{R}}$$

$$\frac{\mathcal{G} \mid \Gamma, \varphi, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ \wedge_{\mathsf{L}} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, \psi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ \wedge_{\mathsf{R}}$$

$$\frac{\mathcal{G} \mid \Gamma, \psi \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta}\ \to_{\mathsf{L}} \qquad \frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta, \psi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \to \psi}\ \to_{\mathsf{R}}$$

$$\frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ \mathsf{EW} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{IW} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \quad \mathcal{G} \mid \Sigma, \varphi \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{Cut}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ \mathsf{EC} \qquad \frac{\mathcal{G} \mid \Gamma, \varphi, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}\ \mathsf{ICL} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi, \varphi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi}\ \mathsf{ICR}$$

Table 2. The standard modal rule sets

$$\frac{\mathcal{G} \mid \boldsymbol{\varphi} \Rightarrow \psi}{\mathcal{G} \mid \Box\boldsymbol{\varphi} \Rightarrow \Box\psi}\ \mathsf{K}_n \quad \frac{\mathcal{G} \mid \Gamma, \boldsymbol{\varphi} \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box\boldsymbol{\varphi} \Rightarrow \Delta}\ \mathsf{T}_n \quad \frac{\mathcal{G} \mid \Box\Gamma, \boldsymbol{\varphi} \Rightarrow \psi}{\mathcal{G} \mid \Box\Gamma, \Box\boldsymbol{\varphi} \Rightarrow \Box\psi}\ \mathsf{4}_n \quad (|\boldsymbol{\varphi}| = n)$$

$$\mathcal{R}_{\mathsf{K}} := \{\mathsf{K}_n : n \geq 0\} \quad \mathcal{R}_{\mathsf{KT}} := \mathcal{R}_{\mathsf{K}} \cup \{\mathsf{T}_n : n \geq 1\} \quad \mathcal{R}_{\mathsf{K4}} := \mathcal{R}_{\mathsf{K}} \cup \{\mathsf{4}_n : n \geq 0\}$$

context formulae which are copied into a premiss can be restricted as in the rule $\mathsf{4}_n$ in Table 2. This is captured by the following notion from [11, 10]:

Definition 3.1. *For $F \subseteq \mathcal{F}(\Lambda)$ the set of* context restrictions over F *is* $\mathfrak{C}(F) := \{\langle F_1, F_2\rangle : F_1, F_2 \subseteq F\}$. *For a sequent $\Gamma \Rightarrow \Delta$ and a context restriction $\mathcal{C} = \langle F_1, F_2\rangle$ the* restriction of $\Gamma \Rightarrow \Delta$ according to $\mathcal{C}$ *is the sequent $(\Gamma \Rightarrow \Delta) \restriction_{\mathcal{C}} := \Gamma \restriction_{F_1} \Rightarrow \Delta \restriction_{F_2}$ where for a multiset Σ and $F \subseteq \mathcal{F}(\Lambda)$ the multiset $\Sigma \restriction_F$ contains those formulae from Σ which are substitution instances of formulae in F.*

Example 3.2. 1. The context restriction $\mathcal{C}_\emptyset := \langle \emptyset, \emptyset\rangle$ intuitively deletes the whole context, we have $(\Gamma \Rightarrow \Delta) \restriction_{\mathcal{C}_\emptyset} = \Rightarrow$ for every sequent $\Gamma \Rightarrow \Delta$.

2. The context restriction $\mathcal{C}_{\mathsf{id}} := \langle \{p\}, \{p\}\rangle$ intuitively copies the whole context, we have $(\Gamma \Rightarrow \Delta) \restriction_{\mathcal{C}_{\mathsf{id}}} = \Gamma \Rightarrow \Delta$ for every sequent $\Gamma \Rightarrow \Delta$.
3. The context restriction $\mathcal{C}_\Box := \langle \{\Box p\}, \emptyset\rangle$ copies only the boxed formulae on the left side of the context.

We take the rules to introduce exactly one layer of connectives in the principal formulae, and we assume that every premiss includes a restriction for each component of the principal part.

Definition 3.3. *A* hypersequent rule with context restrictions, *written as*

$$\frac{\{(\Gamma_i \Rightarrow \Delta_i; \boldsymbol{\mathcal{C}_i}) : i \leq m\}}{\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n}$$

is given by a natural number $n > 0$, *a sequence* $\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n$ *called* principal part *with* $\Sigma_i \Rightarrow \Pi_i \in \mathcal{S}(\Lambda(\mathcal{V}))$*and a set of* premisses*, where each premiss* $(\Gamma_i \Rightarrow \Delta_i; \boldsymbol{\mathcal{C}_i})$ *consists of a sequent of variables and a sequence* $\boldsymbol{\mathcal{C}_i} = \langle F_i^1, G_i^1 \rangle, \ldots, \langle F_i^n, G_i^n \rangle$ *of context restrictions subject to the* variable condition*: every variable occurs at most once in the principal part and it occurs in the principal part whenever it occurs in the premisses. An* application *of such a rule is given by a substitution* $\sigma : \mathcal{V} \to \mathcal{F}(\Lambda)$*, a* side hypersequent $\mathcal{G}$ *and a sequence* $\Omega_1 \Rightarrow \Upsilon_1 \mid \cdots \mid \Omega_n \Rightarrow \Upsilon_n$ *of* context sequents*. It is written as*

$$\frac{\{\mathcal{G} \mid \Omega_1 \restriction_{F_i^1}, \ldots, \Omega_n \restriction_{F_i^n}, \Gamma_i\sigma \Rightarrow \Delta_i\sigma, \Upsilon_1 \restriction_{G_i^1}, \ldots, \Upsilon_n \restriction_{G_i^n} : i \leq m\}}{\mathcal{G} \mid \Omega_1, \Sigma_1\sigma \Rightarrow \Pi_1\sigma, \Upsilon_1 \mid \cdots \mid \Omega_n, \Sigma_n\sigma \Rightarrow \Pi_n\sigma, \Upsilon_n} .$$

The notions of a *derivation* and *derivability* for a set $\mathcal{R}$ of hypersequent rules with restrictions are defined in the usual way, and we write $\vdash_{\mathcal{R}} \mathcal{G}$ if $\mathcal{G}$ is derivable in $\mathcal{R}$. A rule is *derivable in* $\mathcal{R}$ if for all its applications the conclusion can be derived from the premisses in $\mathcal{R}$ and *admissible* if whenever the premisses are derivable in $\mathcal{R}$, then so is the conclusion. We stipulate that sets of rules are closed under variable renaming and permutation of the components in the principal part. Rules are written inline using "/" to separate premisses and conclusion.

Example 3.4. 1. The standard hypersequent rules for the propositional connectives. E.g. the rule $\wedge_{\mathsf{L}}$ is the rule $\{(p, q \Rightarrow ; \mathcal{C}_{\mathsf{id}})\}/p \wedge q \Rightarrow$.
2. The standard rules for modal logics from Table 2. E.g. the rule $\mathsf{4}_n$ is the rule $\{(\boldsymbol{p} \Rightarrow q; \mathcal{C}_{\Box})\}/\Box\boldsymbol{p} \Rightarrow \Box q$ with $|\boldsymbol{p}| = n$.
3. The modalised splitting rule for $\mathsf{S5}$ from [1] with applications $\mathcal{G} \mid \Box\Gamma, \Sigma \Rightarrow \Box\Delta, \Pi/\mathcal{G} \mid \Box\Gamma \Rightarrow \Box\Delta \mid \Sigma \Rightarrow \Pi$ is $\{(\Rightarrow ; \langle\{\Box p\}, \{\Box p\}\rangle, \mathcal{C}_{\mathsf{id}})\}/ \Rightarrow \mid \Rightarrow$.

4 Cut Elimination and Applications

We obtain sufficient criteria for cut elimination by generalising the cut elimination proof in [6]. The cut-elimination strategy is to permute a cut into the premisses of the last applied rule on the left until the cut formula is principal in the last applied rule. Then the cut is permuted into the premisses on the right until it is principal here as well, in which case it is reduced to cuts on formulae of smaller complexity. To state the condition used to reduce principal cuts we use the notion of a *cut between rules*, where intuitively a new rule is constructed from two rules by cutting their conclusions on a formula $\heartsuit p$ and eliminating p from the premisses by cutting on p in all possible ways. To make this precise, write $\boldsymbol{\mathcal{C}} \cup \boldsymbol{\mathcal{D}}$ for the *union* of two sequences $\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{D}} \in \mathfrak{C}^n$ of restrictions, defined component-wise: If the i-th components of $\boldsymbol{\mathcal{C}}$ resp. $\boldsymbol{\mathcal{D}}$ are $\langle F_i, G_i \rangle$ resp. $\langle F_i', G_i' \rangle$, then the i-th component of $\boldsymbol{\mathcal{C}} \cup \boldsymbol{\mathcal{D}}$ is $\langle F_i \cup F_i', G_i \cup G_i' \rangle$. In addition, for permuting the cut into the context on the right we need a condition on the context restrictions which ensures that whenever the cut formula satisfies a context restriction, then so does the whole left premiss of the cut.

Definition 4.1. *For sets* $\mathcal{P}_1, \mathcal{P}_2$ *of premisses and rules* $R_1 = \mathcal{P}_1/\Sigma_1 \Rightarrow \Pi_1 \mid \dots \mid \Sigma_{n-1} \Rightarrow \Pi_{n-1} \mid \Sigma_n \Rightarrow \Pi_n, \heartsuit p$ *and* $R_2 = \mathcal{P}_2/\heartsuit p, \Omega_1 \Rightarrow \Theta_1 \mid \Omega_2 \Rightarrow \Theta_2 \mid \dots \mid \Omega_k \Rightarrow \Theta_k$ *the* cut between R_1 and R_2 on $\heartsuit p$ *is the rule* $\mathsf{cut}(R_1, R_2, \heartsuit p)$ *given by*

$$\frac{\begin{array}{c}\{(\Gamma, \Gamma' \Rightarrow \Delta, \Delta'; \boldsymbol{\mathcal{C}} \cup \boldsymbol{\mathcal{D}}) : (\Gamma \Rightarrow \Delta, p; \boldsymbol{\mathcal{C}}), (\Gamma' \Rightarrow \Delta'; \boldsymbol{\mathcal{D}}) \in \mathcal{P}\} \\ \{(\Gamma \Rightarrow \Delta; \boldsymbol{\mathcal{C}}) \in \mathcal{P} : p \notin \Gamma, \Delta\}\end{array}}{\Sigma_1 \Rightarrow \Pi_1 \mid \dots \mid \Sigma_{n-1} \Rightarrow \Pi_{n-1} \mid \Sigma_n, \Omega_1 \Rightarrow \Pi_n, \Theta_1 \mid \Omega_2 \Rightarrow \Theta_2 \mid \dots \Omega_k \Rightarrow \Theta_k}$$

where $\mathcal{P} := \{(\Gamma \Rightarrow \Delta; \boldsymbol{\mathcal{C}}, \mathcal{C}_\emptyset, \overset{(k-1)\text{-times}}{\dots}, \mathcal{C}_\emptyset) : (\Gamma \Rightarrow \Delta; \boldsymbol{\mathcal{C}}) \in \mathcal{P}_1\} \cup \{(\Gamma \Rightarrow \Delta; \mathcal{D}_\emptyset, \overset{(n-1)\text{-times}}{\dots}, \mathcal{D}_\emptyset, \boldsymbol{\mathcal{D}}) : (\Gamma \Rightarrow \Delta; \boldsymbol{\mathcal{C}}) \in \mathcal{P}_2\}$*. A set* $\mathcal{R}$ *of rules is* principal-cut closed *if it is closed under the addition of cuts between rules. It is* mixed-cut permuting *if for all* $R_1, R_2 \in \mathcal{R}$*: if* $\Gamma \Rightarrow \Delta, \heartsuit p$ *is a component of the principal part of* R_1 *and* $(\heartsuit p \Rightarrow\ ) \restriction_{\mathcal{C}} = \heartsuit p \Rightarrow$ *for a restriction* $\mathcal{C}$ *of* R_2*, then* $(\Gamma \Rightarrow \Delta) \restriction_{\mathcal{C}} = \Gamma \Rightarrow \Delta$ *and* $(\Sigma \Rightarrow \Pi) \restriction_{\mathcal{D}} \restriction_{\mathcal{C}} = (\Sigma \Rightarrow \Pi) \restriction_{\mathcal{D}}$ *for every restriction* $\mathcal{D}$ *for this component and sequent* $\Sigma \Rightarrow \Pi$*.*

Example 4.2. 1. The cut between $\mathsf{K}_n = \{(\boldsymbol{p} \Rightarrow q; \mathcal{C}_\emptyset)\}/\Box\boldsymbol{p} \Rightarrow \Box q$ and $\mathsf{K}_{m+1} = \{(q, \boldsymbol{q} \Rightarrow r; \mathcal{C}_\emptyset)\}/\Box q, \Box\boldsymbol{q} \Rightarrow \Box r$ is the rule $\mathsf{cut}(\mathsf{K}_n, \mathsf{K}_{m+1}, \Box q) = \{(\boldsymbol{p}, \boldsymbol{q} \Rightarrow r); \mathcal{C}_\emptyset\}/\Box\boldsymbol{p}, \Box\boldsymbol{q} \Rightarrow \Box r = \mathsf{K}_{n+m}$. Thus the rule set $\mathcal{R}_\mathsf{K}$ is principal-cut closed.
2. The cut between the rule $\mathsf{K4}_n = \{(\boldsymbol{p} \Rightarrow q; \mathcal{C}_\Box)\}/\Box\boldsymbol{p} \Rightarrow \Box q$ and the rule $\mathsf{5} = \{q \Rightarrow\ ; \mathcal{C}_\emptyset, \mathcal{C}_{\mathsf{id}})\}/\Box q \Rightarrow \mid \Rightarrow$ on $\Box q$ is the rule $\mathsf{cut}(\mathsf{K4}_n, \mathsf{5}, \Box q) = \{(\boldsymbol{p} \Rightarrow \mathcal{C}_\Box, \mathcal{C}_{\emptyset,\emptyset})\}/\Box\boldsymbol{p} \Rightarrow \mid \Rightarrow$ which we denote $\mathsf{5}_n$. Its applications have the form $\mathcal{G} \mid \Box\Gamma, \Sigma, \varphi_1, \dots, \varphi_n \Rightarrow \Pi/\mathcal{G} \mid \Box\Gamma, \Box\varphi_1, \dots, \Box\varphi_n \Rightarrow \mid \Sigma \Rightarrow \Pi$. It is straightforward to see that the rule set $\mathcal{R}_{\mathsf{KT4}} \cup \{\mathsf{5}_n : n \geq 0\}$ is principal-cut closed.

For sequent rules introducing only one connective, principal-cut closure is known as *coherence* [2], and it corresponds to Belnap's condition $C8$ [3]. The two properties of Def. 4.1 ensure that we can eliminate topmost instances of a restricted version of multicut, where the cut formula occurs only once in the left premiss (and is principal in the last applied rule there), but several times in several components in the right premiss by induction on the maximal complexity of a cut formula occurring in a derivation. Allowing the cut formula to occur more than once on the right is necessary due to the internal and external contraction rules. The fact that several instances of the cut formula in the right premiss of such a restricted multicut can be principal also is the reason why we take the cuts between rules of a principal-cut closed rule set to be *in* the rule set and not just derivable: we need to be able to replace iterated cuts by a rule from the rule set. To avoid also several instances of the cut formula being principal in the left premiss and to deal with external contraction we introduce a further restriction.

Definition 4.3. *A rule set* $\mathcal{R}$ *is* right-contraction closed *if applications of internal contraction right to the conclusion of a rule are derived by internal contractions followed by one rule from* $\mathcal{R}$*. It is* single-conclusion right *if the principal part of no rule contains* $\Gamma \Rightarrow \Delta, \heartsuit p \mid \Sigma \Rightarrow \Pi, \heartsuit q$ *for* $\heartsuit \in \Lambda$ *and* $p, q \in \mathcal{V}$*.*

Example 4.4. 1. The rule sets $\mathcal{R}_\mathsf{K}, \mathcal{R}_{\mathsf{K4}}$ and H are trivially right-contraction closed since the right sides of the principal formulae contain only one formula.

2. The rule sets H as well as $\mathcal{R}_{\mathsf{K}}$ and $\mathcal{R}_{\mathsf{K4}}$ are trivially single-conclusion right since their principal parts contain only one component.
3. The rules $\mathsf{5}_n$ from Ex. 4.2.2 are single-conclusion right since no component of the principal part introduces a boxed formula on the right hand side.

The cut formula is not principal in more than one component in the left premiss if the rule set is single-conclusion right, and right-contraction closure prevents the cut formula occurring twice in a single component of the principal part:

Lemma 4.5. *Let $\mathcal{R}$ be right-contraction closed and single-conclusion right. Then whenever $\vdash_{\mathcal{R}\mathsf{Cut}} \mathcal{G}$ there is a derivation of $\mathcal{G}$ in which in every application of a rule from $\mathcal{R}$ the right hand sides of the principal part are fully contracted.*

Proof. We show by induction on n: Suppose there is a derivation of $\mathcal{G}$ in $\mathcal{R}\mathsf{Cut}$ with the property (P_n): whenever the principal part of a rule application contain a component $\Gamma \Rightarrow \Delta, \varphi, \varphi$, then φ has complexity at most n. Then there is a derivation of $\mathcal{G}$ in $\mathcal{R}\mathsf{Cut}$ where the principal part of no rule application contains such a component.

So suppose we have a derivation with property (P_{n+1}). Pick a topmost rule application with principal part containing a component $\Gamma \Rightarrow \Delta, \varphi, \varphi$ and φ of complexity $n + 1$. Using right-contraction closure (possibly repeatedly) this is replaced by contractions on the premisses of this application, an application of a rule from $\mathcal{R}$ which does not contain such a component and applications of Weakening. Since φ was part of the principal part, the newly introduced contractions are on formulae of complexity at most n. Continuing in this fashion we replace all problematic rule applications. The resulting derivation has property (P_n) and we are done using the induction hypothesis. □

Finally, we impose a further restriction which ensures that cuts with cut formula contextual on the left can be permuted into the premisses on the left.

Definition 4.6. *A rule is* right-substitutive *if all restrictions occurring in it have the form $\langle \{p\}; \{p\} \rangle$ or $\langle F; \emptyset \rangle$ for some $F \subseteq \mathcal{F}(\Lambda)$.*

Theorem 4.7 (Cut elimination). *Let $\mathcal{R}$ be right-substitutive, single-conclusion right, right-contraction closed, principal-cut closed and mixed-cut permuting. Then for every hypersequent $\mathcal{G}$ we have: $\vdash_{\mathcal{R}\mathsf{Cut}} \mathcal{G}$ iff $\vdash_{\mathcal{R}} \mathcal{G}$.*

Proof (Sketch, see Appendix for the full proof). By double induction on the maximal complexity of a cut formula in a derivation and the number of applications with cut formula of maximal complexity. Topmost cuts of maximal complexity are eliminated using the fact that with right-substitutivity applications of a restricted version of multicut allowing the cut formula to occur several times in several components on the left can be eliminated by permuting them up on the left until exactly one occurrence is principal (by Lem. 4.5 and single-conclusion right), permuting the non-principal cuts into the premisses and using principal-cut closure and mixed-cut closure as above to eliminate the remaining cut with cut formula principal on the left. □

Corollary 4.8. *The hypersequent calculi* $\mathsf{H}, \mathsf{H}\mathcal{R}_{\mathsf{K}}, \mathsf{H}\mathcal{R}_{\mathsf{K4}}, \mathsf{H}\mathcal{R}_{\mathsf{KT}}$ *and* $\mathsf{H}\mathcal{R}_{\mathsf{KT}}\{\mathsf{5}_n : n \in \mathbb{N}\}$ *with rules* $\mathsf{5}_n$ *from Ex. 4.2.2 admit cut elimination.*

Proof. Inspection of the rules together with Ex. 4.2 shows that these rule sets satisfy the conditions of Thm. 4.7. □

Thm. 4.7 together with the next Lemma also provides the basis of the method of *cut elimination by saturation* used in Sec. 6, where cut-free hypersequent calculi are constructed by saturating a rule set under cuts between rules. Of course we still need to check that the remaining conditions of Thm. 4.7 are satisfied.

Lemma 4.9. *Let* R_1, R_2 *be hypersequent rules with context restrictions. Then the rule* $\mathsf{cut}(R_1, R_2, \heartsuit p)$ *is a derivable rule in* $\mathsf{H}R_1R_2\mathsf{Cut}$.

Proof. Suppose we have two rules

$$R_1 = \frac{\{(\Gamma_i \Rightarrow \Delta_i, p; \boldsymbol{\mathcal{C}_i}) \ : \ i \leq m\} \cup \mathcal{P}_1}{\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_{n-1} \Rightarrow \Pi_{n-1} \mid \Sigma_n \Rightarrow \Pi_n, \heartsuit p}$$

$$R_2 = \frac{\{(p, \Omega_j \Rightarrow \Psi_j; \boldsymbol{\mathcal{D}_j}) \ : \ j \leq \ell\} \cup \mathcal{P}_2}{\heartsuit p, \Upsilon_1 \Rightarrow \Xi_1 \mid \Upsilon_2 \Rightarrow \Xi_2 \mid \cdots \mid \Upsilon_k \Rightarrow \Xi_k}$$

where p does not occur in $\mathcal{P}_1, \mathcal{P}_2$. Furthermore, suppose we have an application of the rule $\mathsf{cut}(R_1, R_2, \heartsuit p)$ given by a substitution σ, a side hypersequent $\mathcal{I}$ and $n + k - 1$ contexts $\Theta_r \Rightarrow \Phi_r$. For the sake of presentation we assume that $\sigma = \mathsf{id}$. We write $\boldsymbol{\mathcal{C}_i} \odot \boldsymbol{\mathcal{D}_j}$ for $(\boldsymbol{\mathcal{C}_i}, \overbrace{\mathcal{C}_\emptyset, \ldots, \mathcal{C}_\emptyset}^{k-1 \text{ times}}) \cup (\overbrace{\mathcal{C}_\emptyset, \ldots, \mathcal{C}_\emptyset}^{n-1 \text{ times}}, \boldsymbol{\mathcal{D}_j})$. Thus we have the premisses from $\mathcal{P}_1, \mathcal{P}_2$ not including p (with context) and the premisses

$$\mathcal{I} \mid \boldsymbol{\Theta} \restriction_{\boldsymbol{\mathcal{C}_i} \odot \boldsymbol{\mathcal{D}_j}} \Gamma_i, \Omega_j \Rightarrow \Delta_i, \Psi_j, \boldsymbol{\Phi} \restriction_{\boldsymbol{\mathcal{C}_i} \odot \boldsymbol{\mathcal{D}_j}}$$

for $i \leq n, j \leq \ell$. Now setting

$$\chi := \bigvee_{i \leq n} (\bigwedge \boldsymbol{\Theta} \restriction_{\boldsymbol{\mathcal{C}_i} \odot \mathcal{C}_\emptyset^k} \wedge \bigwedge \Gamma_i \wedge \neg \bigvee \boldsymbol{\Phi} \restriction_{\boldsymbol{\mathcal{C}_i} \odot \mathcal{C}_\emptyset^k} \wedge \neg \bigvee \Delta_i)$$

we can derive the hypersequents

$$\mathcal{I} \mid \boldsymbol{\Theta} \restriction_{\boldsymbol{\mathcal{C}}_i \odot \mathcal{C}_\emptyset^k}, \Gamma_i \Rightarrow \boldsymbol{\Phi} \restriction_{\boldsymbol{\mathcal{C}}_i \odot \mathcal{C}_\emptyset^k}, \Delta_i, \chi$$

from axioms using propositional logic and the hypersequents

$$\mathcal{I} \mid \boldsymbol{\Theta} \restriction_{\mathcal{C}_\emptyset^n \odot \boldsymbol{\mathcal{D}_j}}, \chi, \Omega_j \Rightarrow \Psi_j, \boldsymbol{\Phi} \restriction_{\mathcal{C}_\emptyset^n \odot \boldsymbol{\mathcal{D}_j}}$$

from the premisses of $\mathsf{cut}(R_1, R_2, \heartsuit p)$. Now applications of R_1 and R_2 give the hypersequents

$$\mathcal{I} \mid \Theta_1, \Sigma_1 \Rightarrow \Phi_1, \Pi_1 \mid \cdots \mid \Theta_n, \Sigma_n \Rightarrow \Phi_n, \Pi_n, \heartsuit \chi$$

and

$$\mathcal{I} \mid \Theta_n, \heartsuit \chi, \Upsilon_1 \Rightarrow \Phi_n, \Xi_1 \mid \cdots \mid \Theta_{n+\ell}, \Upsilon_\ell \Rightarrow \Phi_{n+\ell}, \Xi_\ell$$

Finally, an application of Cut together with external and internal contractions gives the desired conclusion. □

4.1 Applications: Decision Procedures and Complexity Bounds

For general decision procedures apart from cut elimination we also need to deal with Contraction. The idea is to show admissibility of internal contraction under a modified notion of rule applications, where some principal formulae are copied into the premiss (as in Kleene's G3-systems). Then under a mild assumption only a bounded number of components per hypersequent are relevant in a rule application, hence using the subformula property of the rules the total number of hypersequents occurring in a derivation is bounded and we obtain decidability.

Definition 4.10. *A* modified application *of a hypersequent rule* $R = \{(\Gamma_i \Rightarrow \Delta_i; \boldsymbol{\mathcal{C}_i}) : i \in \mathcal{P}\}/\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n$ *is given by a side hypersequent* $\mathcal{G}$*, a substitution* $\sigma : \mathcal{V} \to \mathcal{F}$ *and contexts* $\Theta_1 \Rightarrow \Omega_1 \mid \cdots \mid \Theta_n \Rightarrow \Omega_n$ *and written as*

$$\frac{\left\{\mathcal{G} \mid \mathcal{H} \mid \Gamma_i\sigma, \bigsqcup_{j\leq n}(\Sigma_j\sigma, \Theta_j) \restriction_{\mathcal{C}_i^j} \Rightarrow \Delta_i\sigma, \bigsqcup_{j\leq n}(\Pi_j\sigma, \Omega_j) \restriction_{\mathcal{C}_i^j} : \ i \in \mathcal{P}\right\}}{\mathcal{G} \mid \Sigma_1\sigma, \Theta_1 \Rightarrow \Pi_1\sigma, \Omega_1 \mid \cdots \mid \Sigma_n\sigma, \Theta_n \Rightarrow \Pi_n\sigma, \Omega_n} \ R^*$$

with $\mathcal{H} = \Sigma_1\sigma, \Theta_1 \Rightarrow \Pi_1\sigma, \Omega_1 \mid \cdots \mid \Sigma_n\sigma, \Theta_n \Rightarrow \Pi_n\sigma, \Omega_n$*.*

Thus in addition to the context formulae all principal formulae satisfying the corresponding restriction are copied into the premiss, and all components of the principal part are copied to deal with external contraction. If internal contractions can be permuted with rules this yields admissibility of internal contraction.

Definition 4.11. *A rule set* $\mathcal{R}$ *is* contraction closed *if for every rule* $R \in \mathcal{R}$ *with principal part* $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \heartsuit p, \heartsuit q$ *(resp.* $\mathcal{G} \mid \Gamma, \heartsuit p, \heartsuit q \Rightarrow \Delta$*) there is a rule* $R' \in \mathcal{R}$ *with principal part* $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \heartsuit p$ *(resp.* $\Gamma, \heartsuit p \Rightarrow \Delta$*) whose premisses are derivable from those of* R *by renaming* q *to* p *and contractions.*

Lemma 4.12. *For contraction closed* $\mathcal{R}$ *internal contraction is admissible in* $\mathcal{R}^*$*.*

Proof. By simultaneous double induction on the complexity of φ and the depth of the derivation we show: whenever $\vdash_{\mathcal{R}^*} \mathcal{G} \mid \varphi, \varphi, \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \varphi, \varphi, \Gamma_n \Rightarrow \Delta_n$, then $\vdash_{\mathcal{R}^*} \mathcal{G} \mid \varphi, \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \varphi, \Gamma_n \Rightarrow \Delta_n$ and analogously for φ on the right. Contractions between context and principal formulae are dealt with by modified rule applications and the inner induction hypothesis, those between principal formulae using contraction closure and the outer induction hypothesis. □

Definition 4.13. *A rule set* $\mathcal{R}$ *is* tractable *if there is an encoding* $\ulcorner . \urcorner$ *of applications of rules from* $\mathcal{R}$ *of size polynomial in the size of the conclusion such that given a hypersequent* $\mathcal{G}$ *and an encoding* $\ulcorner R \urcorner$ *of a rule application it is decidable in time exponential in the size of* $\mathcal{G}$ *whether* $\mathcal{G}$ *is the conclusion of* R *and it is decidable in time exponential in the size of* $\ulcorner R \urcorner$ *whether* $\mathcal{G}$ *is a premiss of* R*.*

Definition 4.14. *A rule set* $\mathcal{R}$ *is* bounded component *if there is* $n \in \mathbb{N}$ *such that the principal part of every rule in* $\mathcal{R}$ *has at most* n *components.*

Theorem 4.15. *Let $\mathcal{R}$ be a contraction closed, bounded conclusion and tractable set of rules. Then derivability in $\mathcal{R}$ is decidable in double exponential time.*

Proof. Using Weakening and Contraction derivability in $\mathcal{R}$ is equivalent to derivability in $\mathcal{R}^*$. Moreover, Lem. 4.12 allows us to equivalently work with hypersequents build from *set-set* sequents. Since $\mathcal{R}$ is bounded component, for some k at most k components of a hypersequent contain principal formulae of the last applied rule. Thus w.l.o.g. in a derivation every hypersequent contains at most k copies of the same component. Hence in a derivation of a hypersequent with size n at most $(k+1)^{2^{2n}} = 2^{2^{\mathcal{O}(n)}}$ different hypersequents appear. Thus using the fact that derivability in one step from a set of hypersequents is a monotone operator we compute all derivable hypersequents of this set using tractability of $\mathcal{R}$ and the fact that since the size of an encoding of a rule application is polynomial in the size of its conclusion the number of encodings of rules with a given conclusion is only exponential in the size of the conclusion and check whether the given hypersequent is among these in time doubly exponential in n. $\square$

5 Axioms and Rules

To translate axioms into rules with context restrictions and vice versa we need to interpret hypersequents as formulae. We do this in an abstract way by viewing an interpretation as a family of formulae, one for each number of components in a hypersequent, compatible with the structural rules. Formally:

Definition 5.1. *An* interpretation *for a Λ-logic $\mathcal{L}$ is a set $\iota = \{\iota_n(p_1, \ldots, p_n) : n \geq 1\}$ of formulae in $\mathcal{F}(\Lambda)$ which* respects the structural rules*, i.e. for all $n \geq 1$:*

1. *ι respects (external) exchange: $\models_{\mathcal{L}} \iota_n(\boldsymbol{\varphi}, \psi, \chi, \boldsymbol{\xi})$ iff $\models_{\mathcal{L}} \iota_n(\boldsymbol{\varphi}, \chi, \psi, \boldsymbol{\xi})$*
2. *ι respects external Weakening: if $\models_{\mathcal{L}} \iota_n(\boldsymbol{\varphi})$, then $\models_{\mathcal{L}} \iota_{n+1}(\boldsymbol{\varphi}, \psi)$*
3. *ι respects external Contraction: if $\models_{\mathcal{L}} \iota_{n+1}(\boldsymbol{\varphi}, \psi, \psi)$, then $\models_{\mathcal{L}} \iota_n(\boldsymbol{\varphi}, \psi)$*
4. *ι respects* Cut*: if $\models_{\mathcal{L}} \iota_n(\boldsymbol{\varphi}, \psi \to \chi)$ and $\models_{\mathcal{L}} \iota_m(\chi \to \xi, \boldsymbol{\zeta})$, then we have $\models_{\mathcal{L}} \iota_{n+m-1}(\boldsymbol{\varphi}, \psi \to \xi, \boldsymbol{\zeta})$.*

The interpretation is regular *if for all $\varphi \in \mathcal{F}$ we have $\models_{\mathcal{L}} \varphi$ iff $\models_{\mathcal{L}} \iota_1(\varphi)$.*

An interpretation $\iota = \{\iota_n : n \geq 1\}$ for a logic induces a map $\iota : \mathcal{HS} \to \mathcal{F}$ defined by $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \mapsto \iota_n(\bigwedge \Gamma_1 \to \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \to \bigvee \Delta_n)$.

Example 5.2.
1. The interpretation $\iota_{\boxplus}$ for normal $\Lambda_\Box$-logics is given by the formulae $\iota_n^{\boxplus}(\varphi_1, \ldots, \varphi_n) = \bigvee_{i=1}^n (\varphi_i \wedge \Box\varphi_i)$. It is an interpretation by normality of $\Box$ and obviously regular.
2. The standard interpretation for normal $\Lambda_\Box$-logics from [1] is $\iota_\Box$ given by $\iota_n^\Box(\varphi_1, \ldots, \varphi_n) = \bigvee_{i=1}^n \Box\varphi_i$. It is regular for a normal logic iff $\Box\varphi/\varphi$ is admissible, in particular if $\Box p \to p$ is an axiom. It is not regular for e.g. KB.

Depending on whether we involve the interpretation we obtain different notions of soundness. Regular interpretations link these and imply soundness of H.

Definition 5.3. *Let $\mathcal{R}$ be a set of rules and ι an interpretation for the logic $\mathcal{L}$. Then $\mathcal{R}$ is* hypersequent soundness preserving *(briefly:* hssp*) for $(\mathcal{L}, \iota)$ if for every application of a rule from $\mathcal{R}$ with premisses $\mathcal{H}_k$ for $k \leq n$ and conclusion $\mathcal{G}$: if $\models_{\mathcal{L}} \iota(\mathcal{H}_k)$ for all $k \leq n$, then $\models_{\mathcal{L}} \iota(\mathcal{G})$. The calculus is* sound *for $\mathcal{L}$, if $\vdash_{\mathsf{H}\mathcal{R}} \Rightarrow \varphi$ implies $\models_{\mathcal{L}} \varphi$, and* complete *for $\mathcal{L}$, if $\models_{\mathcal{L}} \varphi$ implies $\vdash_{\mathsf{H}\mathcal{R}} \Rightarrow \varphi$.*

Proposition 5.4.
1. *If $\mathcal{R}$ is hssp for $(\mathcal{L}, \iota)$ and ι is a regular interpretation for $\mathcal{L}$, then $\mathcal{R}$ is sound for $\mathcal{L}$.*
2. *If ι is a regular interpretation for $\mathcal{L}$, then H is hssp for $(\mathcal{L}, \iota)$.*

Proof.
1. By induction on the depth of a derivation we have: $\vdash_{\mathcal{R}} \mathcal{H}$ implies $\models_{\mathcal{L}} \iota(\mathcal{H})$. Now regularity of ι gives the statement.
2. Using the fact that $\mathcal{L}$ includes all propositional tautologies, all the modalities have congruence and thus $\models_{\mathcal{L}} \iota_n(\boldsymbol{\varphi}, \psi)$ iff $\models_{\mathcal{L}} \iota_n(\boldsymbol{\varphi}, \top \to \psi)$ and the properties of a regular interpretation. ◻

The interpretation $\iota_\Box$ is regular e.g. for normal $\Lambda_\Box$-logics given by a class of Kripke frames closed under the addition of a predecessor to every world:

Definition 5.5. *A class K of frames is* extensible *if whenever for a frame $\mathfrak{F} = (W, R)$ we have $\mathfrak{F} \in \mathsf{K}$ then also $\mathfrak{F}^\bullet \in \mathsf{K}$ where $\mathfrak{F}^\bullet = (W \cup \{x\}, R \cup \{(x, y) : y \in W \cup \{x\}\})$ with $x \notin W$.*

Lemma 5.6. *If $\mathcal{L}$ is a normal $\Lambda_\Box$-logic defined by an extensible class of frames, then $\iota_\Box$ is a regular interpretation for $\mathcal{L}$.*

Proof. By normality $\iota_\Box$ is an interpretation for $\mathcal{L}$. For regularity suppose that $\neg\varphi$ is satisfiable in $\mathfrak{F} \in \mathsf{K}$ with K the extensible class of frames defining $\mathcal{L}$. Then for some world w of $\mathfrak{F}$ and valuation σ we have $\mathfrak{F}, w, \sigma \not\models \varphi$. Thus for the additional world x in $\mathfrak{F}^\bullet$ we have $\mathfrak{F}^\bullet, x, \sigma \not\models \Box\varphi$, and since $\mathfrak{F}^\bullet \in \mathsf{K}$ we have $\not\models_{\mathcal{L}} \Box\varphi$. ◻

From Rules to Axioms. In the construction of axioms from rules we extend the method from [15, 11, 10]. The idea is to show projectivity (Lem. 5.10) of a formula corresponding to the premisses of the rule and use a substitution witnessing this property to inject the information of the premisses into a formula corresponding to the conclusion. For the sake of presentation here we only consider the normal modality $\Box$ and restrict the context restrictions to $\{\mathcal{C}_\emptyset, \mathcal{C}_{\mathsf{id}}, \mathcal{C}_\Box\}$. In general the method also works for monotone or antitone n-ary modalities and arbitrary context restrictions. To show projectivity we need to assume the following for every premiss $(\Gamma \Rightarrow \Delta; \boldsymbol{\mathcal{C}})$:

$$\text{If } \mathcal{C}_{\mathsf{id}} \notin \boldsymbol{\mathcal{C}} \text{ then } \Gamma, \Delta \neq \emptyset \tag{1}$$

For the rest of this section we fix a rule R with this property. In presence of HCut we may assume furthermore w.l.o.g. that the restriction $\mathcal{C}_{\mathsf{id}}$ does not occur in R: If it does occur we simply convert R into a rule of this format by introducing a *dummy modality* $\cdot$ satisfying $\cdot\varphi \leftrightarrow \varphi$ for all formulae and replacing every restriction $\mathcal{C}_{\mathsf{id}}$ by the sequent $\Rightarrow s$ for a fresh variable s in the premisses and by $\Rightarrow \cdot s$ in the corresponding component in the principal part. By Lem. 4.9 the resulting rule is equivalent to the original one modulo $\mathsf{H}\mathcal{R}_{\mathsf{dm}}\mathsf{Cut}$ where $\mathcal{R}_{\mathsf{dm}} = \{(p \Rightarrow ; \mathcal{C}_{\mathsf{id}})/ \cdot p \Rightarrow\ ,\ (\ \Rightarrow p; \mathcal{C}_{\mathsf{id}})/\ \Rightarrow \cdot p\}$ states equivalence of p and $\cdot p$. Together with property (1) this means that $\Gamma, \Delta \neq \emptyset$ for every premiss $(\Gamma \Rightarrow \Delta; \boldsymbol{\mathcal{C}})$. Since the number of context formulae might vary, a rule can not be translated into a formula directly. This is avoided by fixing the number of context formulae. For normal modalities and the limited restrictions considered here this gives:

Definition 5.7. *The* canonical proto rule *for a rule* $R = \{(\Gamma_i \Rightarrow \Delta_i; \mathcal{C}_i^1, \dots, \mathcal{C}_i^n) : i \leq m\}/\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n$ *is given by the context sequents* $\Omega_1 \Rightarrow \mid \cdots \mid \Omega_n \Rightarrow$ *with* $\Omega_j = \Box p_j$ *if* $\mathcal{C}_i^j = \mathcal{C}_\Box$ *for some* i *and empty otherwise, using fresh variables* $\boldsymbol{p}$*. An* application *of the canonical proto rule for* R *given by* $\mathcal{G}$ *and* σ *is the same as the application of* R *given by* $\mathcal{G}, \sigma$ *and the above contexts.*

Example 5.8.
1. The canonical proto rule for $\mathsf{4}_n$ from Tab. 2 is given by the context $\Box p \Rightarrow$ and has applications $\mathcal{G} \mid \Box\chi, \boldsymbol{\varphi} \Rightarrow \psi/\mathcal{G} \mid \Box\chi, \Box\boldsymbol{\varphi} \Rightarrow \Box\psi$.
2. To treat $R_{\mathsf{5}} := (\ \Rightarrow ; \mathcal{C}_\Box, \mathcal{C}_{\mathsf{id}})/\ \Rightarrow \mid \Rightarrow$ we replace $\mathcal{C}_{\mathsf{id}}$ by the dummy modality, giving $(\ \Rightarrow s; \mathcal{C}_\Box, \mathcal{C}_\emptyset)/\ \Rightarrow \mid \Rightarrow \cdot s$. The canonical proto rule for R_{5} is given by the contexts $\Box p \Rightarrow \mid \Rightarrow$ and has applications $\mathcal{G} \mid \Box\varphi \Rightarrow \psi/\mathcal{G} \mid \Box\varphi \Rightarrow \mid \Rightarrow \cdot\psi$.

In the non-normal case or for arbitrary context restrictions we would need to consider a set of proto rules with every possible number of context formulae also on the right hand side, compare [11, 10]. Using the rules for normal modal logics and HCut it is straightforward to see that the canonical proto rule is enough:

Lemma 5.9. *R and its canonical proto rule are interderivable in* $\mathsf{H}\mathcal{R}_{\mathsf{K}}\mathsf{Cut}$.

Proof. Using Cut and the fact that $\bigsqcup_{i \leq n} \Box\varphi_i \Rightarrow \Box \bigwedge_{i \leq n} \varphi_i$ and $\Box \bigwedge_{i \leq n} \varphi_i \Rightarrow \bigwedge_{i \leq n} \Box\varphi_i$ are derivable in $\mathsf{H}\mathcal{R}_{\mathsf{K}}$. ☐

Now suppose we have an interpretation $\iota = \{\iota_n : n \geq 1\}$ and that

$$R = \{(\Gamma_i \Rightarrow \Delta_i; \boldsymbol{\mathcal{C}_i}) : i \leq m\}/\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n$$

with $\mathcal{C}_i^j = \langle F_i^j, G_i^j \rangle$. The canonical proto rule $\hat{R}$ for R is given by the contexts $\Omega_1 \Rightarrow \mid \cdots \mid \Omega_n \Rightarrow$. The formula corresponding to its premisses is

$$\varphi := \bigwedge\nolimits_{i \leq m} \left(\bigwedge (\Omega_1 \restriction_{F_i^1}, \dots, \Omega_n \restriction_{F_i^n}, \Gamma_i) \to \bigvee \Delta_i \right) \ .$$

Now define a substitution θ by $\theta(x) = \varphi \wedge x$ if $x \in \Gamma_i$ for some $i \leq m$ and $\theta(x) = \varphi \to x$ if $x \in \Delta_i$ for some $i \leq m$ and $\theta(x) = x$ otherwise. Since by monotonicity w.l.o.g. no variable occurs both in antecedent and succedent of a premiss, θ is well-defined. Straightforward propositional reasoning gives:

Lemma 5.10. *The substitution θ witnesses projectivity of φ, i.e. the following hold:* $\vdash_{\mathsf{HMonCut}} \Rightarrow \varphi\theta$ *and* $\vdash_{\mathsf{HMonCut}} \varphi \Rightarrow p \leftrightarrow p\theta$ *for every* $p \in \mathcal{V}$. ☐

This gives equivalence of $\hat{R}$ to a *ground hypersequent*, i.e. a set of hypersequents closed under substitution, which we then interpret as an axiom using ι:

Lemma 5.11. *$\hat{R}$ is interderivable over* $\mathsf{HCutMon}$ *with the ground hypersequent* $\mathcal{H}_R := \ \Rightarrow (\bigwedge(\Omega_1, \Sigma_1) \to \bigvee \Pi_1)\,\theta \mid \cdots \mid \Rightarrow (\bigwedge(\Omega_n, \Sigma_n) \to \bigvee \Pi_n)\,\theta$.

Proof. By Lem. 5.10 we have $\vdash_{\mathsf{HCutMon}} \Rightarrow \varphi\theta$ and thus $\vdash_{\mathsf{HCutMon}} \Rightarrow \varphi\theta\sigma$ for every substitution σ. Now inverting the propositional rules using Cut and an application of $\hat{R}$ give $\mathcal{H}_R\sigma$. For the other direction, Lem. 5.10 implies $\vdash_{\mathsf{HMonCut}} \varphi \Rightarrow \psi \leftrightarrow \psi\theta$ (by induction on the complexity of ψ). Hence $\vdash_{\mathsf{HMonCut}} \varphi\sigma, \chi_i\theta\sigma \Rightarrow \chi_i\sigma$ with $\chi_i = \bigwedge(\Omega_i, \Sigma_i) \to \bigvee \Delta_i$. From the premisses of an application of $\hat{R}$ we obtain $\Gamma \mid \Rightarrow \varphi\sigma$, and cutting these and the ground hypersequent $\mathcal{H}_R\sigma$ followed by invertibility of H and external Contraction yield the conclusion of this application. ◻

Theorem 5.12 (Soundness). *If* $\mathsf{H}\mathcal{R}_{\mathsf{K}}\mathsf{Cut}R$ *is hssp for* $(\mathcal{L}, \iota)$*, then* $\iota(\mathcal{H}_R) \in \mathcal{L}$.

Proof. Since $\mathcal{H}_R$ is derivable in $\mathsf{H}\mathcal{R}_{\mathsf{K}}\mathsf{Cut}R$ by Lem. 5.11 and $\mathsf{H}\mathcal{R}_{\mathsf{K}}\mathsf{Cut}R$ is hssp for $(\mathcal{L}, \iota)$, the former is hssp for $(\mathcal{L}, \iota)$ as well. Thus $\iota(\mathcal{H}_R) \in \mathcal{L}$. ◻

Theorem 5.13 (Completeness). *If for sets $\mathcal{A}$ of axioms and $\mathcal{R}$ of rules* $\mathsf{HCut}\mathcal{R}$ *is complete for $\mathcal{L}_\mathcal{A}$ and the rule $\Rightarrow \varphi_1 \mid \cdots \mid \Rightarrow \varphi_n / \ \Rightarrow \iota_n(\varphi_1, \ldots, \varphi_n)$ is derivable in* $\mathsf{HCut}\mathcal{R}$*, then* $\mathsf{HCut}\mathcal{R}R$ *is complete for* $\mathcal{L}_\mathcal{A} \oplus \iota(\mathcal{H}_R)$.

Proof. By Lem. 5.11 the ground hypersequent $\mathcal{H}_R$ is derivable in $\mathsf{HCut}\mathcal{R}R$, and thus the axiom $\iota(\mathcal{H}_R)$ is derivable in $\mathsf{HCut}\mathcal{R}R$ as well. Simulating modus ponens by Cut we thus obtain completeness of this calculus for $\mathcal{L}_\mathcal{A} \oplus \iota(\mathcal{H}_R)$. ◻

Example 5.14. The premiss of the canonical proto rule for R_5 from Ex. 5.8.2 is turned into $\varphi = \Box p \to s$. Then with θ defined by $\theta(p) = p$ and $\theta(s) = \varphi \to s$ we obtain $\mathcal{H} = \ \Rightarrow \neg\Box p\theta \mid \Rightarrow \cdot s\theta \ = \ \Rightarrow \neg\Box p \mid \Rightarrow \cdot(\varphi \to s)$. Thus R_5 is equivalent under $\iota_\Box$ to the axiom $\iota_\Box(\mathcal{H}) = \Box\neg\Box p \vee \Box \cdot ((\Box p \to s) \to s)$ which modulo propositional reasoning and monotonicity is easily seen to be equivalent (as an axiom) to $\Box\neg\Box p \vee \Box \cdot \Box p$. By idempotency of $\cdot$ this is equivalent to $\Box\neg\Box p \vee \Box\Box p$.

Crucially, Thm. 5.12 also implies that rules stay hssp in extensions of a logic:

Corollary 5.15. *If $\mathcal{L}_1 \subseteq \mathcal{L}_2$, and ι is an interpretation for $\mathcal{L}_1, \mathcal{L}_2$, and* $\mathsf{HCut}\mathcal{R}_\mathsf{K}$ *is hssp for $(\mathcal{L}_1, \iota)$ and $(\mathcal{L}_2, \iota)$, then if R is hssp for $(\mathcal{L}_1, \iota)$ it is also hssp for $(\mathcal{L}_2, \iota)$.*

Proof. Since R and $\mathcal{H}_R$ are interderivable and $\iota(\mathcal{H}_R) \in \mathcal{L}_1 \subseteq \mathcal{L}_2$. ◻

From Axioms to Rules. The translation from axioms to rules proceeds similar to that for sequent rules in [11, 10], but uses the interpretation to peel away one layer of the formula first. The idea is to treat some subformulae of an axiom as *context formulae* and translate the axiom into a proto rule (i.e. a rule with a fixed number of context formulae). To simplify presentation we assume monotonicity of the modalities.

Definition 5.16. *Let $C_\ell, C_r \subseteq \mathcal{F}(\Lambda)$ and $V \subseteq \mathcal{V}$. The class of* translatable clauses *for (C_ℓ, V, C_r) is defined by the following grammar (starting variable S):*

$$
\begin{aligned}
S &::= L \to R \\
L &::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot \\
P_r &::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p \mid \bot \mid \top \\
P_\ell &::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p \mid \bot \mid \top
\end{aligned}
$$

where $\heartsuit \in \Lambda, p \in V$ and $\psi_i \in C_i$ for $i \in \{\ell, r\}$. A formula is hypertranslatable for an interpretation $\iota = \{\iota_n : n \geq 1\}$ *if has the form $\iota_n(\chi_1, \dots, \chi_n)$ with χ_i a translatable clause for (C_ℓ, V, C_r) where no distinct formulae in $C_\ell \cup V \cup C_r$ share a variable, every formula in $C_\ell \cup C_r$ occurs in the χ_i exactly once not in the scope of a modality and at least once in the scope of a modality.*

A little thought shows that hypersequents $\mathcal{G} \mid \Rightarrow \varphi$ (resp. $\mathcal{G} \mid \varphi \Rightarrow$) with φ generated by taking P_r (resp. P_ℓ) as starting variable in the above grammar can be decomposed using invertibility of the propositional rules into sets of hypersequents $\mathcal{G} \mid \Gamma \Rightarrow \Delta$ with $\Gamma \subseteq C_\ell \cup V$ and $\Delta \subseteq C_r \cup V$. The formulae in C_ℓ (resp. C_r) will play the role of context formulae on the left (resp. right). We now fix a logic $\mathcal{L}$, an interpretation $\iota = \{\iota_n : n \geq 1\}$ and a hypertranslatable formula φ for ι and consider the stages of the translation in detail.

Ground hypersequent stage. We have $\varphi = \iota_n(\varphi_1, \dots, \varphi_n)$ where $\varphi_i = \bigwedge \boldsymbol{\psi^i} \wedge \bigwedge \boldsymbol{\chi^i} \to \bigvee \boldsymbol{\xi^i} \vee \bigvee \boldsymbol{\zeta^i}$ with context formulae $\chi^i_j \in C_\ell$, $\zeta^i_j \in C_r$ and formulae ψ^i_j (resp. ξ^i_j) of the form $\heartsuit\delta_j$ with δ_j generated by the above grammar with starting variable P_r (resp. P_ℓ). This is turned into the ground hypersequent $\mathcal{H}_\varphi := \boldsymbol{\psi^1}, \boldsymbol{\chi^1} \Rightarrow \boldsymbol{\xi^1}, \boldsymbol{\zeta^1} \mid \cdots \mid \boldsymbol{\psi^n}, \boldsymbol{\chi^n} \Rightarrow \boldsymbol{\xi^n}, \boldsymbol{\zeta^n}$ which by HCut is hssp for $(\mathcal{L}, \iota)$.

Shaping the conclusion. We replace each $\psi^i_j = \heartsuit\delta^i_j$ with $\heartsuit p^i_j$ where $p^i_j \in \mathcal{V}$ is fresh and add the premiss $p^i_j \Rightarrow \delta^i_j$. Analogously we replace $\xi^i_j = \heartsuit\gamma^i_j$ with $\heartsuit q^i_j$ and add the premiss $\gamma^i_j \Rightarrow q^i_j$. By monotonicity and Cut this is equivalent to $\mathcal{H}_\varphi$.

Resolving propositional logic. Using invertibility of the propositional rules we replace each of these premisses by a number of sequents $\Gamma \Rightarrow \Delta$ with $\Gamma \subseteq C_\ell \cup V$ and $\Delta \subseteq C_r \cup V$. In presence of HCut this gives an equivalent rule.

Cleaning the premisses. To ensure that every variable occurring in the premisses of the rule also occurs in the conclusion we eliminate the variables from V from the premisses by successively cutting the premisses on all variables in V as in Def.4.1 disregarding context restrictions. Reasoning as in Lem. 4.9 the resulting rule is seen to be equivalent to the original rule (compare also [5]).

Introducing context restrictions. The global condition on the context formulae in Def.5.16 guarantees that every formula in $C_\ell \cup C_r$ occurs exactly once in the conclusion and at least once in the premisses. Moreover, it occurs always on the same side of the sequent. Thus we now have a rule with a fixed number of context formulae. Provided the context formulae are *normal* in the sense that formulae in C_ℓ distribute over $\wedge$ and those in C_r over $\vee$ we may replace them with context restrictions by turning a premiss $\chi_1, \dots, \chi_m, \Gamma \Rightarrow \Delta, \zeta_1, \dots, \zeta_k$ with context formulae χ_j and ζ_j occurring in the i_j-th component of the conclusion into the premiss with restriction $(\Gamma \Rightarrow \Delta; \boldsymbol{\mathcal{C}})$ where $\mathcal{C}^i = \langle \{\chi_j : i_j = i\}; \{\zeta_j : i_j = i\}\rangle$ and deleting all context formulae from the conclusion. Call the resulting rule R_φ.

Since all steps in the above construction yield rules interderivable with the original ones using HCut and monotonicity and soundness of these rules is preserved by Cor. 5.15 we immediately obtain soundness and completeness.

Proposition 5.17. *Let ι be a regular interpretation for $\mathcal{L}$ and let $\mathcal{R}$ be hssp and complete for $(\mathcal{L}, \iota)$ with the rule $\Rightarrow p_1 \mid \cdots \mid \Rightarrow p_n / \Rightarrow \iota_n(\boldsymbol{p})$ derivable in $\mathcal{R}$. If φ is hypertranslatable for ι with normal context formulae (C_ℓ, C_r), then $\mathcal{R}R_\varphi$ is hssp and complete for $(\mathcal{L} \oplus \varphi, \iota)$.* $\square$

Example 5.18. Using $\iota_\Box$ the axiom $\Box\neg\Box p \vee \Box\cdot\Box p$ from Ex. 5.14 is converted into the ground hypersequent $\Box p \Rightarrow \mid \Rightarrow \cdot\Box p$. Taking $\Box p$ to be in C_ℓ we introduce a fresh variable q and the corresponding premiss to obtain $\Box p \Rightarrow q / \Box p \Rightarrow \mid \Rightarrow \cdot q$. Using normality of $\Box$ (for $\mathcal{R}_{\mathsf{K}}$) the formula $\Box p$ is now replaced with the context restriction $\langle\{\Box p\}, \emptyset\rangle = \mathcal{C}_\Box$ resulting in the rule $(\Rightarrow q; \mathcal{C}_\Box, \mathcal{C}_\emptyset) / \Rightarrow \mid \Rightarrow \cdot q$.

The translations show that in general a single axiom corresponds to a proto rule, i.e. a rule with a fixed number of context formulae. Thus in general a rule corresponds to an infinite number of (systematically generated) axioms, see [11, 10] for the sequent case. The method also works for non-monotone modalities, where in the second stage we introduce both premisses $p^i_j \Rightarrow \delta^i_j$ and $\delta^i_j \Rightarrow p^i_j$ instead of only one of these. Furthermore, in some cases we still obtain rules with restrictions from axioms with non-normal context formulae, see Sec.6.

6 Case Studies

Logics for simple frame properties. An interesting class of examples are the rules constructed from *simple* frame properties for normal modal logics [9]. A *simple* frame property is a formula $\forall w_1, \dots, \forall w_n \exists u \varphi_S$ in the frame language, with $\varphi_S = \bigvee_{\langle S_R, S_= \rangle \in S} (\bigwedge_{i \in S_R} w_i R u \wedge \bigwedge_{i \in S_=} w_i = u)$ for some non-empty *description* S consisting of a set of tuples $\langle S_R, S_= \rangle$ with $S_R, S_= \subseteq \{1, \dots, n\}$ and $S_R \cup S_= \neq \emptyset$. We identify a simple frame property with its description. In [9] hypersequent rules corresponding to simple frame properties based on $\mathsf{K}, \mathsf{K4}$ and KB are given and cut admissibility is shown via the semantics. Here we consider the rules based on K and $\mathsf{K4}$ (those for KB do not fit our rule format). The set of *hypersequent rules induced by S for $\mathcal{R}_{\mathsf{K}}$* is $\mathcal{R}_S := \{R_{k_1,\dots,k_n} : k_i \geq 0\}$ with

$$R_{k_1,\dots,k_n} := \frac{\left\{ (\bigsqcup_{j \in S_R} p^j_1, \dots, p^j_{k_j} \Rightarrow ; \mathcal{C}^1_{\langle S_R, S_= \rangle}, \dots, \mathcal{C}^n_{\langle S_R, S_= \rangle}) \ : \ \langle S_R, S_= \rangle \in S \right\}}{\Box p^1_1, \dots, \Box p^1_{k_1} \Rightarrow \mid \cdots \mid \Box p^n_1, \dots, \Box p^n_{k_n} \Rightarrow}$$

where $\mathcal{C}^j_{\langle S_R, S_= \rangle} = \mathcal{C}_{\mathsf{id}}$ for $j \in S_=$ and $\mathcal{C}_\emptyset$ otherwise. The set of *hypersequent rules induced by S for $\mathcal{R}_{\mathsf{K4}}$* is the set $\mathcal{R}^4_S := \{R^4_n : n \geq 0\}$ with R^4_n the rule R_n with $\mathcal{C}^j_{\langle S_R, S_= \rangle} = \mathcal{C}_{\mathsf{id}}$ for $j \in S_=$ and $\mathcal{C}_\Box$ for $j \in S_R$ and $\mathcal{C}_\emptyset$ otherwise. Inspection of the rule sets constructed in this way shows that together with $\mathsf{H}\mathcal{R}_{\mathsf{K}}$ (resp. $\mathsf{H}\mathcal{R}_{\mathsf{K4}}$)

they satisfy all conditions given in Thm. 4.7. Thus we obtain a purely syntactic analogue to the semantic cut admissibility proof in [9]:

Corollary 6.1. *If $\mathcal{R}$ is a set of rules induced by simple frame properties for $\mathcal{R}_{\mathsf{K}}$ (resp. $\mathcal{R}_{\mathsf{K4}}$), then $\mathsf{H}\mathcal{R}_{\mathsf{K}}\mathcal{R}$ (resp. $\mathsf{H}\mathcal{R}_{\mathsf{K4}}\mathcal{R}$) has cut elimination.* □

Using the translation from rules to axioms we furthermore obtain finite axiomatisations from the so constructed rules, provided we have a regular interpretation and the rules are hssp for this interpretation. While $\iota_{\boxplus}$ is always regular, the interpretation $\iota_{\Box}$ gives cleaner axioms. Sometimes regularity of $\iota_{\Box}$ can be read of the frame properties directly: if $S_R \neq \emptyset \neq S_=$ for all $(S_R, S_=) \in \theta$ for one property θ, then the logic is reflexive, and if $S_= = \emptyset$ for all $(S_R, S_=) \in \theta$ for every θ, then the logic is extensible (Def. 5.5). Under certain conditions we may also adjust the original soundness proof to our setting:

Proposition 6.2 ([9]). *If S is a simple frame property and $\mathcal{L}_S$ resp. $\mathcal{L}_S^{\mathsf{4}}$ are the logics of the class of frames (resp. transitive frames) with this property, then $\mathcal{R}_S^{\mathsf{4}}$ is hssp for $(\mathcal{L}_S^{\mathsf{4}}, \iota_{\Box})$ and $(\mathcal{L}_S^{\mathsf{4}}, \iota_{\boxplus})$. If $\mathcal{L}_S$ is extensible or if $S_= \neq \emptyset$ for all $(S_R, S_=) \in S$, then $\mathcal{R}_S$ is hssp for $(\mathcal{L}_S, \iota_{\Box})$ and $(\mathcal{L}_S, \iota_{\boxplus})$.*

Proof. We show the statement for $\iota_{\boxplus}$, the case for $\iota_{\Box}$ is similar but easier. We show that if we have a model refuting the interpretation of the conclusion of an application of an induced rule, then there is also a refuting model for the interpretation of one of the premisses. So suppose there is a model $(W, R), w, \sigma$ refuting the interpretation $\iota_{\boxplus}(\mathcal{G} \mid \Gamma_1, \Box\Sigma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, \Box\Sigma_n \Rightarrow \Delta_n)$ of the conclusion of a rule induced by S. Suppose that $\mathcal{G} = \Theta_1 \Rightarrow \Omega_1 \mid \cdots \mid \Theta_m \Rightarrow \Omega_m$. Then w.l.o.g. there are $k \leq m$ and $\ell \leq n$ and worlds $v_1, \dots, v_\ell$ and $w_1, \dots, w_k$ with wRv_i and wRw_j for $i \leq \ell, j \leq k$ such that

- $(W, R), w, \sigma \not\Vdash \bigwedge \Theta_j \to \bigvee \Omega_j$ for $k < j \leq m$
- $(W, R), w, \sigma \not\Vdash \bigwedge \Gamma_i, \bigwedge \Box\Sigma_i \to \bigvee \Delta_i$ for $\ell < i \leq n$
- $(W, R), w_j, \sigma \not\Vdash \bigwedge \Theta_j \to \bigvee \Omega_j$ for $1 \leq j \leq k$
- $(W, R), v_i, \sigma \not\Vdash \bigwedge \Gamma_i \wedge \bigwedge \Box\Sigma_i \to \bigvee \Delta_i$ for $1 \leq i \leq \ell$.

Since the frame (W, R) satisfies $\forall \boldsymbol{v} \exists u \varphi_S$, there is a $\langle S_R, S_= \rangle \in S$ and a world $u \in W$ such that $v_i R u$ for every $i \in S_R, i \leq \ell$ and $v_i = u$ for every $i \in S_=, i \leq \ell$ and wRu (resp. $w = u$) if $S_R \cup \{\ell+1, \dots, n\} \neq \emptyset$ (resp. $S_= \cup \{\ell+1, \dots, n\} \neq \emptyset$). Hence we have

$$(W, R), u, \sigma \not\Vdash \bigwedge_{i \in S_=} \bigwedge \Gamma_i \wedge \bigwedge_{j \in S_R} \bigwedge \Sigma_j \to \bigvee_{i \in S_=} \Delta_i \ .$$

But by construction $\mathcal{H} := \mathcal{G} \mid \bigsqcup_{i \in S_=} \Gamma_i, \bigsqcup_{j \in S_R} \Sigma_j \Rightarrow \bigsqcup_{i \in S_=} \Delta_i$ is a premiss of the (application of the) rule induced by S for $\mathcal{R}_{\mathsf{K}}$. Now if $S_= \neq \emptyset$ for all $\langle S_R, S_= \rangle \in S$, then either $v_i = u$ for some $i \leq n$ and we have wRu and hence $(W, R), w, \sigma \not\Vdash \Box(\bigwedge_{i \in S_=} \bigwedge \Gamma_i \wedge \bigwedge_{j \in S_R} \bigwedge \Sigma_j \to \bigvee_{i \in S_=} \bigvee \Delta_i)$; or $w = u$ and hence $(W, R), w, \sigma \not\Vdash \bigwedge_{i \in S_=} \bigwedge \Gamma_i \wedge \bigwedge_{j \in S_R} \bigwedge \Sigma_j \to \bigvee_{i \in S_=} \bigvee \Delta_i)$. In both cases we have $(W, R), w, \sigma \not\Vdash \iota_{\boxplus}(\mathcal{H})$ and are done. If on the other hand $S_= = \emptyset$ for all $\langle S_R, S_= \rangle \in$

S, then the class of frames defined by $\forall \boldsymbol{v} \exists u \varphi_S$ is extensible and for the new world x in $(W,R)^\bullet$ we have $xR^\bullet u$ and $xR^\bullet w$ as well as $xR^\bullet w_j$ for $j \leq m$. Hence for a valuation $\sigma^\bullet$ with $\sigma^\bullet \upharpoonright_W = \sigma$ we have $(W,R)^\bullet, x, \sigma^\bullet \not\Vdash \iota_{\boxplus}(\mathcal{H})$. Finally, in the transitive case the interpretation of the conclusion has the form $\iota_{\boxplus}(\mathcal{G} \mid \Gamma_1, \Box\Sigma_1, \Box\Pi_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, \Box\Sigma_n, \Box\Pi_n \Rightarrow \Delta_n)$ and constructing u in the same way as above by transitivity we have

$$(W,R), u, \sigma \not\Vdash \bigwedge_{i \in S_=} \bigwedge \Gamma_i \wedge \bigwedge_{j \in S_R} (\bigwedge \Sigma_j \wedge \bigwedge \Box\Pi_j) \to \bigvee_{i \in S_=} \Delta_i \ .$$

But since by transitivity also wRu the model $(W,R), w, \sigma$ refutes the interpretation of the corresponding premiss. □

To obtain the simplest axioms we observe that given $\mathsf{H}\mathcal{R}_{\mathsf{K}}\mathsf{Cut}$ (resp. $\mathsf{H}\mathcal{R}_{\mathsf{K4}}\mathsf{Cut}$) by Lem. 4.9 the set of rules induced by a simple property is equivalent (in both cases!) to a *single* rule $\{(\bigsqcup_{i \in S_R} p_i \Rightarrow ; \boldsymbol{\mathcal{C}_{\langle S_R, S_= \rangle}}) : \langle S_R, S_= \rangle \in S\} / \Box p_1 \Rightarrow | \cdots | \Box p_n$ with $\mathcal{C}^i_{\langle S_R, S_= \rangle} = \mathcal{C}_{\mathsf{id}}$ for $i \in S_=$ and $\mathcal{C}_\emptyset$ otherwise. Translating this rule gives the corresponding axiom. This restricts the shape of the resulting axioms.

Definition 6.3. *A* ι-simple axiom *for an interpretation* $\iota = \{\iota_n : n \geq 1\}$ *is an axiom* $\iota_n(\varphi_1, \ldots, \varphi_n)$ *where* $\mathsf{mrk}\,(\varphi_i) \leq 1$ *and* $\Box$ *occurs only negatively in the* φ_i.

Proposition 6.4. *Let* $\mathcal{L}_S$ *(resp.* $\mathcal{L}_S^4$*) be the logic of the class* F *of frames (resp. transitive frames) satisfying the simple frame property* S*. Then* $\mathcal{L}_S^4$ *is axiomatised over* $\mathsf{K4}$ *by one* $\iota_{\boxplus}$*-simple axiom. The logic* $\mathcal{L}_S$ *is axiomatised by one* $\iota_{\Box}$*-simple axiom if: (*$\mathcal{L}_S$ *is reflexive or* F *is extensible or* $S_= = \emptyset$ *for all* $\langle S_R, S_= \rangle \in S$*) and (*$\mathcal{L}_S$ *is transitive or* F *is extensible or* $S_= \neq \emptyset$ *for all* $\langle S_R, S_= \rangle \in S$*).* □

This extends to finite sets of simple frame properties (if using extensibility to show soundness we need *all* frame classes obtained by successively adding properties to be extensible). While seemingly restrictive, the conditions capture all examples of [9], e.g. directedness, universality, linearity or bounded cardinality.

Example 6.5. The property called *Bounded Acyclic Subgraph* in [9] induces the rule $R_{\mathsf{BAS}} = \{(q_k^k \Rightarrow ; \boldsymbol{\mathcal{C}^i} : k < i \leq n\} / \Box q_1^1 \Rightarrow | \cdots | \Box q_n^n \Rightarrow$ where the i-th component of $\boldsymbol{\mathcal{C}^i}$ is $\mathcal{C}_{\mathsf{id}}$ and all other components are $\mathcal{C}_\emptyset$. Using reflexivity and $\iota_\Box$ the translation of this is $\bigvee_{1 \leq k \leq n} \Box \left(\ell_k \wedge \Box \left(\bigwedge_{1 \leq i < j \leq n} (\ell_j \wedge p_i \to r_j) \wedge p_k \right) \to r_k \right)$, which in particular implies the axiom $\mathsf{BAS}_n = \bigvee_{k=1}^n \Box(\Box p_k \to \bigvee_{m=1}^{k-1} p_m)$. Translating the latter back into a rule (taking $C_r = C_\ell = \emptyset$ and introducing the dummy modality writing $\bigvee_{m=1}^{k-1} \cdot p_m$) again gives the rule R_{BAS}. Thus the logic given by the Bounded Acyclic Subgraph property is axiomatised over KT by BAS_n.

The logic of uniform deontic frames. The logic LUDF of uniform deontic frames [14] is based on the connectives $\Lambda_\Box \cup \{\mathcal{P}, \mathcal{O}\}$ with $\mathcal{P}$ and $\mathcal{O}$ unary non-normal modalities with intended interpretations "…is permissible" and "…is obligatory" and is axiomatised by the $\mathsf{S5}$-axioms for $\Box$ together with the axioms

Table 3. The additional axioms for LUDF and their translations

(UC) $\mathcal{P}A \wedge \mathcal{P}B \to \mathcal{P}(A \vee B)$	(OiP) $\mathcal{O}A \to \mathcal{P}A$	(Unif-O) $\mathcal{O}A \to \Box\mathcal{O}A$
(W-P) $\mathcal{O}A \to (\mathcal{P}B \to \Box(B \to A))$	(OiC) $\mathcal{O}A \to \neg\Box\neg A$	(Unif-P) $\mathcal{P}A \to \Box\mathcal{P}A$
$\dfrac{\mathcal{G} \mid p \Rightarrow r \quad \mathcal{G} \mid q \Rightarrow r \quad \mathcal{G} \mid r \Rightarrow p, q}{\mathcal{G} \mid \mathcal{P}p, \mathcal{P}q \Rightarrow \mathcal{P}r}$ UC	$\dfrac{\mathcal{G} \mid p, q \Rightarrow}{\mathcal{G} \mid \mathcal{O}p, \Box q \Rightarrow}$ OiC	$\dfrac{\mathcal{G} \mid \mathcal{O}p \Rightarrow q}{\mathcal{G} \mid \mathcal{O}p \Rightarrow \Box q}$ Unif-O
$\dfrac{\mathcal{G} \mid p \Rightarrow r \quad \mathcal{G} \mid \Rightarrow q, r}{\mathcal{G} \mid \mathcal{O}p, \mathcal{P}q \Rightarrow \Box r}$ W-P	$\dfrac{\mathcal{G} \mid p \Rightarrow q \quad \mathcal{G} \mid q \Rightarrow p}{\mathcal{G} \mid \mathcal{O}p \Rightarrow \mathcal{P}q}$ OiP	$\dfrac{\mathcal{G} \mid \mathcal{P}p \Rightarrow q}{\mathcal{G} \mid \mathcal{P}p \Rightarrow \Box q}$ Unif-P

Table 4. The rules in $\mathcal{R}_{\mathsf{LUDF}}$, where $\mathcal{C} := \langle\{\Box p, \mathcal{O}p, \mathcal{P}p\}, \emptyset\rangle$ and writing $\boldsymbol{*p}$ for $*p_1, \ldots, *p_{|\boldsymbol{p}|}$ with $* \in \{\mathcal{O}, \mathcal{P}, \Box\}$.

$$\frac{\{(r \Rightarrow \boldsymbol{p}, \boldsymbol{q}; \mathcal{C}_\emptyset)\} \cup \{(p_i \Rightarrow r; \mathcal{C}_\emptyset), (q_i \Rightarrow r; \mathcal{C}_\emptyset) : p_i \in \boldsymbol{p}, q_i \in \boldsymbol{q}\}}{\boldsymbol{\mathcal{O}p}, \boldsymbol{\mathcal{P}q} \Rightarrow \mathcal{P}r} \ (|\boldsymbol{p}| + |\boldsymbol{q}| \geq 1)$$

$$\frac{(\boldsymbol{p}, \boldsymbol{r} \Rightarrow s; \mathcal{C}), (\boldsymbol{r} \Rightarrow \boldsymbol{q}, s; \mathcal{C})\}}{\boldsymbol{\mathcal{O}p}, \boldsymbol{\mathcal{P}q}, \Box\boldsymbol{r} \Rightarrow \Box s} \qquad \frac{(\boldsymbol{p}, \boldsymbol{r} \Rightarrow \ ; \mathcal{C}_{\mathsf{id}}), (\boldsymbol{r} \Rightarrow \boldsymbol{q}; \mathcal{C}_{\mathsf{id}})\}}{\boldsymbol{\mathcal{O}p}, \boldsymbol{\mathcal{P}q}, \Box\boldsymbol{r} \Rightarrow} \ |\boldsymbol{p}| \geq 1, |\boldsymbol{q}|, |\boldsymbol{r}| \geq 0$$

$$\frac{(\boldsymbol{p}, \boldsymbol{r} \Rightarrow \ ; \mathcal{C}, \mathcal{C}_{\mathsf{id}}), (\boldsymbol{r} \Rightarrow \boldsymbol{q}; \mathcal{C}, \mathcal{C}_{\mathsf{id}})\}}{\boldsymbol{\mathcal{O}p}, \boldsymbol{\mathcal{P}q}, \Box\boldsymbol{r} \Rightarrow \mid \Rightarrow} \ |\boldsymbol{p}| \geq 1, |\boldsymbol{q}|, |\boldsymbol{r}| \geq 0$$

$$\frac{(\boldsymbol{r} \Rightarrow s; \mathcal{C})}{\Box\boldsymbol{r} \Rightarrow \Box s} \ |\boldsymbol{r}| \geq 0 \qquad \frac{(\boldsymbol{r} \Rightarrow \ ; \mathcal{C}_{\mathsf{id}})}{\Box\boldsymbol{r} \Rightarrow} \ |\boldsymbol{r}| \geq 1 \qquad \frac{(\boldsymbol{r} \Rightarrow \ ; \mathcal{C}, \mathcal{C}_{\mathsf{id}})}{\Box\boldsymbol{r} \Rightarrow \mid \Rightarrow} \ |\boldsymbol{r}| \geq 1$$

in Table 3. A hypersequent calculus for the fragment without the axioms (Unif-O) and (Unif-P) based on the calculus for S5 from [13] was given in [7]. We now construct a cut-free calculus for the full logic using the developed methods.

First we convert the axioms into hypersequent rules, building on the calculus for S5 constructed in Ex. 4.2. Since S5 is reflexive, the interpretation $\iota_\Box$ is regular and thus we take it as the underlying interpretation. Under S5 adding an axiom A is equivalent to adding the axiom $\Box A$, hence it suffices to translate the boxed versions of the axioms. Doing this using the methods of Sec. 5 gives the rules in Table 3. Next we saturate the rule set under cuts between rules (Def. 4.1) and (to ensure contraction closure) under contracting principal formulae and the corresponding variables in the premisses (see Appendix A.2 for details). Omitting superfluous premisses this gives the rules in Table 4, where we turned the set of iterated cuts between instances of (Unif-O) and (Unif-P) and 4_n for $n \in \mathbb{N}$ into the rules $(p_1, \ldots, p_n \Rightarrow \ ; \mathcal{C})/\Box p_1, \ldots, \Box p_n \Rightarrow$ with context restriction $\mathcal{C} = \langle\{\Box p, \mathcal{O}p, \mathcal{P}p\}, \emptyset\rangle$. By construction these rules are hssp, and clearly the translations of the axioms are derivable rules using $\mathcal{R}_{\mathsf{LUDF}}$. Finally, it is straightforward to see that $\mathcal{R}_{\mathsf{LUDF}}$ satisfies the conditions for cut elimination and the decision procedure of Sec. 4.1. In particular we obtain an apparently new double exponential complexity bound for LUDF.

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A Appendix: Additional Material

A.1 Proof of Thm. 4.7

Formally, following [6] the proof uses two intermediate Lemmata. To deal with contractions when shifting up the cut on the right we strengthen the induction hypothesis. For m occurrences $\varphi, \dots, \varphi$ resp. $\Gamma, \dots, \Gamma$ resp. $\mathcal{G} \mid \cdots \mid \mathcal{G}$ we write $[\varphi]^m$ resp. $[\Gamma]^m$ resp. $[\mathcal{G}]^m$.

Definition A.1. *The* cut-rank *of a derivation $\mathcal{D}$ is the maximal complexity of the cut formulae occurring in $\mathcal{D}$ and is denoted by $\rho(\mathcal{D})$.*

Lemma A.2 (Shift right). *Let $\mathcal{R}$ be principal-cut closed and mixed-cut permuting. Assume in $\mathcal{R}\mathsf{Cut}$ we have derivations $\mathcal{D}_1$ of $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi$ with last applied rule R and $\mathcal{D}_2$ of $\mathcal{H} \mid \Sigma_1 \Rightarrow [\varphi]^{\lambda_1} \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n, [\varphi]^{\lambda_n} \Rightarrow \Pi_n$ such that φ is principal in the application of R and $\rho(\mathcal{D}_1) < |\varphi| > \rho(\mathcal{D}_2)$. Then there is a derivation $\mathcal{D}$ in $\mathcal{R}\mathsf{Cut}$ of $[\mathcal{G}]^{\sum_{i=1}^n \lambda_i} \mid \mathcal{H} \mid [\Gamma]^{\lambda_1}, \Sigma_1 \Rightarrow [\Delta]^{\lambda_1}, \Pi_1 \mid \cdots \mid [\Gamma]^{\lambda_n}, \Sigma_n \Rightarrow [\Delta]^{\lambda_n}, \Pi_n$ with $\rho(\mathcal{D}) < |\varphi|$.*

Proof. By induction on the depth of $\mathcal{D}_2$. The idea is to permute the (multi-)cut into the premisses of the last applied rule in $\mathcal{D}_2$. If the last applied rule in $\mathcal{D}_2$ was one of $\mathsf{Con}, \mathsf{IW}, \mathsf{EC}, \mathsf{EW}, \mathsf{Cut}, \mathcal{A}$, then the cut is permuted into its premisses or replaced with applications of IW, EW. Otherwise, let Q be the last applied rule in $\mathcal{D}_2$ and for $j \leq \ell, \mu_{i,j} \leq \lambda_i$ let

$$\mathcal{H}_j \mid \Omega_{1,j}, \varphi^{\mu_{1,j}} \Rightarrow \Theta_{1,j} \mid \cdots \mid \Omega_{m_j,j}, \varphi^{\mu_{m_j,j}} \Rightarrow \Theta_{m_j,j}$$

be the premisses of this application. Using the induction hypothesis we have for $j \leq \ell$ derivations $\mathcal{E}_j$ of

$$\mathcal{I}_j := \mathcal{G}^{\sum_{i=1}^{m_j} \mu_{i,j}} \mid \mathcal{H}_j \mid \Gamma^{\mu_{1,j}}, \Omega_{1,j} \Rightarrow \Delta^{\mu_{1,j}}, \Theta_{1,j} \mid \cdots \mid \Gamma^{\mu_{m_j,j}}, \Omega_{m_j,j} \Rightarrow \Delta^{\mu_{m_j,j}}, \Theta_{m_j,j}$$

with $\rho(\mathcal{E}_j) < |\varphi|$. Consider the following derivation:

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ \mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi\end{matrix} \qquad \begin{matrix} \dfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ \mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi\end{matrix} \quad \dfrac{\begin{matrix}\mathcal{E}_1 \\ \vdots \\ \mathcal{I}_1\end{matrix} \quad \dots \quad \begin{matrix}\mathcal{E}_\ell \\ \vdots \\ \mathcal{I}_\ell\end{matrix}}{\mathcal{G}^* \mid \mathcal{H} \mid \Gamma^{\nu_{1,j}}, \varphi^{\lambda'_1}, \Sigma_1 \Rightarrow \Delta^{\nu_{1,j}}, \Pi_1 \mid \cdots \mid \Gamma^{\nu_{n,j}}, \varphi^{\lambda'_n}, \Sigma_n \Rightarrow \Delta^{\nu_{n,j}}, \Pi_n}\,Q}{\vdots}\,\mathsf{cut} \end{matrix}}{\mathcal{G}^{\sum_i \lambda_i} \mid \mathcal{H} \mid \Gamma^{\lambda_1}, \Sigma_1 \Rightarrow \Delta^{\lambda_1}, \Pi_1 \mid \dots \Gamma^{\lambda_n}, \Sigma_n \Rightarrow \Delta^{\lambda_n}, \Pi_n}\,\mathsf{cut}$$

with $\nu_{i,j} := \max_i \mu_{i,j}$. Note that the application of rule Q is possible since the rule set is (right-)mixed-cut permuting (we apply this condition several times for each component). Also note that all the remaining occurrences of the cut formula φ in the conclusion of Q are principal. Because of the latter point we may now apply principal-cut closure of the rule set $\mathcal{R}$ several times and obtain a rule Q' such that

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{E}_1 & & \mathcal{E}_k \\ \vdots & \vdots & \dots & \vdots\end{matrix}}{\mathcal{J}_1 \quad \dots \quad \mathcal{J}_k}\ \mathsf{cut}}{\mathcal{G}^{\sum_i \lambda_i} \mid \mathcal{H} \mid \Gamma^{\lambda_1}, \Sigma_1 \Rightarrow \Delta^{\lambda_1}, \Pi_1 \mid \dots \mid \Gamma^{\lambda_n}, \Sigma_n \Rightarrow \Delta^{\lambda_n}, \Pi_n}\ Q'$$

is a derivation in $\mathsf{H}\mathcal{R}\mathsf{CutConIW}$. Moreover, all the newly introduced cuts are on subformulae of φ and thus this derivation has cut rank $< |\varphi|$. □

Using the previous lemma we may shift cuts up to the left until we reach a principal formula. Provided the rule set is single-component right and right-contraction closed we may then use Lem. A.2 to eliminate this cut.

Lemma A.3 (Shift Left). *Let $\mathcal{R}$ be right-substitutive, single-component right and right-contraction closed. Assume in $\mathcal{R}\mathsf{Cut}$ we have derivations $\mathcal{D}_1$ of $\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1, [\varphi]^{\lambda_1} \mid \dots \mid \Gamma_n \Rightarrow \Delta_n, [\varphi]^{\lambda_n}$ and $\mathcal{D}_2$ of $\mathcal{H} \mid \varphi, \Sigma \Rightarrow \Pi$ with $\rho(\mathcal{D}_1) < |\varphi|$ and $\rho(\mathcal{D}_2) < |\varphi|$. Then there is a derivation $\mathcal{D}$ in $\mathcal{R}\mathsf{Cut}$ of $\mathcal{G} \mid \mathcal{H}^{\sum_{i=1}^{n} \lambda_i} \mid \Gamma_1, [\Sigma]^{\lambda_1} \Rightarrow \Delta_1, [\Pi]^{\lambda_1} \mid \dots \mid \Gamma_n, [\Sigma]^{\lambda_n} \Rightarrow \Delta_n, [\Pi]^{\lambda_n}$ with $\rho(\mathcal{D}) < |\varphi|$.*

Proof. By induction on the depth of $\mathcal{D}_1$. We actually show a slightly stronger statement, namely that whenever the principal part of every rule application in $\mathcal{D}_1$ and $\mathcal{D}_2$ are fully contracted on the right, then we can find a derivation $\mathcal{D}$ in which this is again the case. Using Lem. 4.5 we may assume that the original derivations are of this form.

If the last applied rule in $\mathcal{D}_1$ was one of $\mathsf{Con}, \mathsf{EC}, \mathsf{IW}, \mathsf{EW}, \mathsf{Cut}, \mathcal{A}$, then the cut is permuted into the premisses of this rule or replaced by applications of IW, EW. Then we are done using Lem. 4.5.

Otherwise, let Q be the last applied rule in $\mathcal{D}_1$ with premisses

$$\mathcal{G}_j \mid \Omega_{1,j} \Rightarrow \Theta_{1,j}, \varphi^{\mu_{1,j}} \mid \dots \Omega_{m_j,j} \Rightarrow \Theta_{m_j,j}, \varphi^{\mu_{m_j,j}}$$

for $j \leq \ell, \mu_{i,j} \leq \lambda_i$. Using the induction hypothesis again we have for $j \leq \ell$ derivations $\mathcal{E}_j$ of

$$\mathcal{I}_j := \mathcal{G}_j \mid \mathcal{H}^{\sum_{i=1}^{m_j}} \mid \Omega_{1,j}, \Sigma^{\mu_{1,j}} \Rightarrow \Theta_{1,j}, \Pi^{\mu_{1,j}} \mid \Omega_{m_j,j}, \Sigma^{\mu_{m_j,j}} \Rightarrow \Theta_{m_j,j}, \Pi^{\mu_{m_j,j}}$$

with $\rho(\mathcal{E}_j) < |\varphi|$. Thus we have a derivation

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{E}_1 & & \mathcal{E}_\ell \\ \vdots & & \vdots \\ \mathcal{I}'_1 & \dots & \mathcal{I}'_\ell\end{matrix}}{\mathcal{G} \mid \mathcal{H}^* \mid \Gamma_1, \Sigma^{\nu_1} \Rightarrow \Delta_1, \Pi^{\nu_1}, \varphi^{\lambda'_1} \mid \dots \mid \Gamma_n, \Sigma^{\nu_n} \Rightarrow \Delta_n, \Pi^{\nu_n}, \varphi^{\lambda'_n}}\ Q \qquad \begin{matrix}\mathcal{D}_2 \\ \vdots \\ \mathcal{H} \mid \varphi, \Sigma \Rightarrow \Pi\end{matrix}}{\vdots}\ \mathsf{cut} \qquad \begin{matrix}\mathcal{D}_2 \\ \vdots \\ \mathcal{H} \mid \varphi, \Sigma \Rightarrow \Pi\end{matrix}}{\mathcal{G} \mid \mathcal{H}^{\sum \lambda_i} \mid \Gamma_1, \Sigma^{\lambda_1} \Rightarrow \Delta_1, \Pi^{\lambda_1} \mid \Gamma_n, \Sigma^{\lambda_n} \Rightarrow \Delta_n, \Pi^{\lambda_n}}\ \mathsf{cut}$$

where the $\mathcal{I}'_r$ are derived from the $\mathcal{I}_r$ using weakening. Note that we can apply the rule Q since it is right-substitutive, i.e. since the sequent $\Gamma \Rightarrow \Delta$ satisfies every restriction which is satisfied by the sequent $\Rightarrow \varphi$. Since Q is single-component right we furthermore have $\lambda'_i \neq 0$ for at most *one* $i \leq n$, and using right-contraction closure and Lem. 4.5 we have that $\lambda'_i \in \{0, 1\}$. Thus w.l.o.g. we have the situation

$$\dfrac{\dfrac{\mathcal{D}'_1}{\mathcal{G} \mid \mathcal{H}^{\sum \lambda_i - 1} \mid \Gamma_1, \Sigma^{\lambda_1} \Rightarrow \Delta_1, \Pi^{\lambda_1} \mid \cdots \mid \Gamma_n, \Sigma^{\lambda_n - 1} \Rightarrow \Delta_n, \Pi^{\lambda_n - 1}, \varphi}\,Q' \qquad \begin{matrix}\mathcal{D}_2\\ \vdots \\ \Sigma, \varphi \Rightarrow \Pi\end{matrix}}{\mathcal{G} \mid \mathcal{H}^{\sum_i \lambda_i} \mid \Gamma_1, \Sigma^{\lambda_1} \Rightarrow \Delta_1, \Pi^{\lambda_1} \mid \cdots \mid \Gamma_n, \Sigma^{\lambda_n} \Rightarrow \Delta_n, \Pi^{\lambda_n}}\,\mathsf{cut}$$

with $\rho(\mathcal{D}'_1) < |\varphi|$. Now using Lem. A.2 we now obtain a derivation $\mathcal{D}'$ of

$$\mathcal{G} \mid \mathcal{H}^{\sum \lambda_i} \mid \Gamma_1, \Sigma^{\lambda_1} \Rightarrow \Delta_1, \Pi^{\lambda_1} \mid \cdots \mid \Gamma_n, \Sigma^{\lambda_n} \Rightarrow \Delta_n, \Pi^{\lambda_n}$$

with $\rho(\mathcal{D}') < |\varphi|$ and with Lem. 4.5 we turn it into a derivation $\mathcal{D}$ with $\rho(\mathcal{D}) < |\varphi|$ in which the principal parts of all applications of rules are fully contracted on the right. □

The proof of Thm. 4.7 now proceeds by eliminating topmost cuts of maximal rank:

Proof (Thm. 4.7). For a derivation $\mathcal{D}$ let $\#\rho(\mathcal{D})$ be the number of applications of cut on a cut formula φ with $|\varphi| = \rho(\mathcal{D})$. Let $\mathcal{D}$ be a derivation of $\mathcal{G}$ in $\mathsf{H}\mathcal{R}\mathsf{CutConIW}$. The proof is by induction on the tuple $(\rho(\mathcal{D}), \#\rho(\mathcal{D}))$ in the lexicographic ordering. Topmost cuts with maximal rank are eliminated using Lem. A.3, thus reducing $\rho(\mathcal{D})$ or preserving $\rho(\mathcal{D})$ while reducing $\#\rho(\mathcal{D})$. □

A.2 Construction of the rules in $\mathcal{R}_{\mathsf{LUDF}}$

In order to show that $\mathsf{H}\mathcal{R}_{\mathsf{LUDF}}\mathsf{Cut}$ is complete for $\mathcal{L}_{\mathsf{LUDF}}$ we only need to observe that all the rules in Table 3 are derivable rules in $\mathsf{H}\mathcal{R}_{\mathsf{LUDF}}\mathsf{Cut}$. To see that the $\mathcal{R}_{\mathsf{LUDF}}$ is hssp for $(\mathcal{L}_{\mathsf{LUDF}}, \iota_\Box)$ we give the construction of the rules from those in Table 3 by cuts and contractions, where the *(left) contraction* of a rule with principal part $\mathcal{G} \mid \heartsuit p, \heartsuit q, \Sigma \Rightarrow \Pi$ is obtained by uniformly renaming q to p in premisses and conclusion of the rule and applying internal contraction where possible. Lem. 4.9 together with the fact that the contraction of a rule is derivable using Weakening, this rule and contraction then gives the result.

In the first step, iterated cuts and a contraction on the rule UC produce the rules $\{(q \Rightarrow \boldsymbol{p}; \mathcal{C}_\emptyset)\} \cup \{(p_i \in \boldsymbol{p}; \mathcal{C}_\emptyset)\} / \boldsymbol{\mathcal{P}p} \Rightarrow \mathcal{P}q$ with $|\boldsymbol{p}| \geq 1$. Now cuts with OiP give the first rule of $\mathcal{R}_{\mathsf{LUDF}}$ which we call R_1. On the other hand, for arbitrary $n, m \in \mathbb{N}$ cuts between n instances of the proto rule Unif-O as well as m instances of the proto rule Unif-P with the proto rule for the rule $\mathsf{4}_\ell$ given by the context $\Box p_1, \ldots, \Box p_k$ give a proto rule

$$\frac{\mathcal{O}q_1, \ldots, \mathcal{O}q_n, \mathcal{P}r_1, \ldots, \mathcal{P}r_m, \Box p_1, \ldots, \Box p_k, s_1, \ldots, s_\ell \Rightarrow t}{\mathcal{O}q_1, \ldots, \mathcal{O}q_n, \mathcal{P}r_1, \ldots, \mathcal{P}r_m, \Box p_1, \ldots, \Box p_k, \Box s_1, \ldots, \Box s_\ell \Rightarrow \Box t}\ .$$

The set of all these proto rules for $n, m, k \geq 0$ is evidently equivalent to the rule $\mathsf{4}'_\ell = (\boldsymbol{s} \Rightarrow t; \mathcal{C})/\Box \boldsymbol{s} \Rightarrow \Box t$ with context restriction $\mathcal{C} = \langle \{\mathcal{O}p, \mathcal{P}p, \Box p\}, \emptyset, \rangle$ and $|\boldsymbol{s}| = \ell$. In the same way we obtain the rule $(\boldsymbol{r} \Rightarrow\ ; \mathcal{C}, \mathcal{C}_{\mathsf{id}})/\Box \boldsymbol{r} \Rightarrow | \Rightarrow$. Now a cut between R_1 and $\mathrm{W}-\mathrm{P}$ gives the rule $\{(\ \Rightarrow \boldsymbol{p}, \boldsymbol{q}, s; \mathcal{C}_\emptyset), (r \Rightarrow s; \mathcal{C}_\emptyset)\}/\boldsymbol{\mathcal{O}p}, \boldsymbol{\mathcal{P}q}, \mathcal{O}r \Rightarrow \Box s$ and cutting two instances of this rule with $\mathsf{4}'_\ell$ yields

$$\frac{(\boldsymbol{s} \Rightarrow \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{p'}, \boldsymbol{q'}, t; \mathcal{C}) \quad (r, \boldsymbol{s} \Rightarrow \boldsymbol{p'}, \boldsymbol{q'}, t; \mathcal{C}) \quad (w, \boldsymbol{s} \Rightarrow \boldsymbol{p}, \boldsymbol{q}, t; \mathcal{C}) \quad (r, w, \boldsymbol{s} \Rightarrow t; \mathcal{C})}{\boldsymbol{\mathcal{O}p}, \boldsymbol{\mathcal{P}q}, \mathcal{O}r, \boldsymbol{\mathcal{O}p'}, \boldsymbol{\mathcal{P}q'}, \mathcal{O}w, \Box \boldsymbol{s} \Rightarrow \Box t} .$$

Contracting this rule by identifying $\boldsymbol{p}$ and $\boldsymbol{p'}$ as well as $\boldsymbol{q}$ and $\boldsymbol{q'}$ and omitting the two premisses derivable from the first premiss gives the rule $\{(\boldsymbol{s} \Rightarrow \boldsymbol{p}, \boldsymbol{q}, t; \mathcal{C}), (r, w, \boldsymbol{s} \Rightarrow t; \mathcal{C})\}/\boldsymbol{\mathcal{O}p}, \boldsymbol{\mathcal{P}q}, \mathcal{O}r, \mathcal{O}w, \Box \boldsymbol{s} \Rightarrow \Box t$ and iterating this process yields $\{(\boldsymbol{s} \Rightarrow \boldsymbol{p}, \boldsymbol{q}, t; \mathcal{C}), (\boldsymbol{r}, \boldsymbol{s} \Rightarrow t; \mathcal{C})\}/\boldsymbol{\mathcal{O}p}, \boldsymbol{\mathcal{P}q}, \boldsymbol{\mathcal{O}r}, \Box \boldsymbol{s} \Rightarrow \Box t$ with $|\boldsymbol{r}| \geq 1$. Finally, contracting this rule by identifying $\boldsymbol{p}$ and $\boldsymbol{r}$ gives the second rule of $\mathcal{R}_{\mathsf{LUDF}}$. The remaining two rules then are the result of cuts between this rule and the rules T_n resp. $\mathsf{5}_n$.