Constructing Cut Free Sequent Systems With Context Restrictions Based on Classical or Intuitionistic Logic*

Björn Lellmann[1] and Dirk Pattinson[2]

[1] Department of Computing, Imperial College London, UK
[2] Research School of Computer Science, The Australian National University

Abstract. We consider a general format for sequent rules for not necessarily normal modal logics based on classical or intuitionistic propositional logic and provide relatively simple local conditions ensuring cut elimination for such rule sets. The rule format encompasses e.g. rules for the boolean connectives and transitive modal logics such as *S4* or its constructive version. We also adapt the method of constructing suitable rule sets by saturation to the intuitionistic setting and provide a criterium for translating axioms for intuitionistic modal logics into sequent rules. Examples include constructive modal logics and conditional logic $\mathbb{VA}$.

1 Introduction

It can hardly be disputed that cut elimination theorems are at the foundation of both theoretical investigation and practical implementation of automated reasoning techniques: the ensuing subformula property implies not only decidability of many logical systems, but also lies – mostly in the form of tableau methods – at the heart of the vast majority of implementations of various logics. Achieving cut elimination is usually a two stage process. First, a (sound and complete) set of sequent rules needs to be exhibited. Second, cut elimination is established. Both steps are equally laborious: finding the 'right' set of rules requires ingenuity and (syntactic) proofs of cut elimination rely on the judicious analysis of a large number of cases. Given the growth of logical systems of interest in particular in computer science, both generic methods with efficient tools for designing cut-free calculi, and meta-theorems that guarantee cut-elimination, decidability, and complexity bounds are therefore increasingly important.

This paper explores the method of cut elimination by saturation and extends previous work into two important directions. First, we can now also allow for the propositional base logic to be intuitionistic which allows us to treat a range of logics that have attracted interest in computer science [17, 2]. Second, we generalise the approach to logics given by axioms of arbitrary modal rank. This is achieved by considering sequent rules with *context restrictions* where each premiss only propagates context formulae of a specific form. A prime example for this rule format are e.g. the rules of modal logic *S4*, where a premiss e.g. copies only boxed formulae on the left hand side. This extended rule format necessitates an extension of the previous characterisation of cut-free systems to deal with additional cases in the proof of cut elimination. In order to make full use of the extended rule format we investigate a method for translating axioms into

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rules which works uniformly for classical and intuitionistic logics. The rules so constructed are by construction sound and complete (in the presence of cut) and give rise to unlabelled sequent systems that are amenable to saturation under cuts between rules. In case the resulting rules fulfil our criteria for cut elimination and are also tractable they give rise to a generic EXPTIME decision algorithm for the logic. Our main contributions are the following: we formalise the notion of a rule with context restrictions (Definition 3), give a general criterion for cut elimination to obtain for a large class of modal logics extending classical or intuitionistic propositional logic (Theorem 17), and show how to construct sequent systems satisfying these requirements from axioms of a certain form (Section 3.2). We illustrate these techniques by reconstructing known cut-free sequent systems for constructive *S4*, constructive *K* and access control logic *CDD*, and also obtain a new cut-free calculus for Lewis' conditional logic *VA*. The techniques used are easily modified to treat e.g. minimal logic [9] or the $\{\wedge, \vee\}$-fragment of intuitionistic logic as base logics, but since we are not aware of modal logics based on either of these we restrict ourselves to the classical and intuitionistic cases.

Related Work: The method of cut elimination by saturation for extensions of classical logic with non-nested axioms was explored e.g. in [16, 10]. The idea of contraction closed rule sets for first order and modal logics seems to have been formulated for the first time in [15, 14], where also translations of axioms into rules of a labelled sequent system are given. Our rules with context restrictions are weaker versions of the rules with context relations from [3], which also allow the context formulae to change. While context relations are more general than context restrictions, apparently no syntactical criteria for cut elimination in such systems have been established yet. Our translations of axioms for intuitionistic modal logics into rules are motivated by the translations of (non-modal) axioms into structural rules for substructural logic in [6].

2 Preliminaries

Throughout, $\mathcal{V}$ denotes a denumerable set of propositional variables and Λ is a set of connectives with associated arities. We write $\boldsymbol{p}$ for finite sequences of propositional variables. The set of Λ-*formulae* is defined by $\mathcal{F}(\Lambda) \ni A_1, \dots, A_n ::= p \mid \heartsuit(A_1, \dots, A_n)$ for $p \in \mathcal{V}$ and $\heartsuit \in \Lambda$ with arity n. We write $\Lambda(S) = \{\heartsuit(A_1, \dots, A_n) \mid \heartsuit \in \Lambda\ n\text{-ary}, A_1, \dots, A_n \in S\}$ for the set of formulae constructed from S using a single connective in Λ. Uniform substitution of all propositional variables in a formula A using a substitution $\sigma : \mathcal{V} \to \mathcal{F}(\Lambda)$ is denoted by $A\sigma$. The set $\mathcal{S}(F)$ of *(symmetric) sequents* over F consists of tuples of multisets Γ, Δ of formulae in F, written $\Gamma \Rightarrow \Delta$. When dealing with extensions of intuitionistic propositional logic we consider *asymmetric sequents*, in which the right hand side Δ consists of at most one formula. The formulae in Γ occur *negatively* in the sequent, those in Δ *positively*. The multiset union of two multisets Γ and Δ is written Γ, Δ and we identify formulae with singleton multisets. Substitution extends to both multisets of formulae and sequents in the obvious way (perserving multiplicity), e.g. $(A_1, A_2 \Rightarrow B)\sigma = A_1\sigma, A_2\sigma \Rightarrow B\sigma$. We use the systems $\mathsf{G2}cp$ and $\mathsf{G2}ip$ of [18] with axioms $\Gamma, A \Rightarrow \Delta, A$ (where A ranges over the set of formulae) and the intuitionistic left implication rule $\dfrac{\Gamma \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C}$ as basis for all systems that extend classical respectively intuitionistic propositional logic and write G resp. $\mathsf{G}i$ for these sets of rules.

Our structural rules are

$$\frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta, \Pi}\ \mathsf{W},\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathsf{ConL},\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \mathsf{ConR},\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{Cut}\,.$$

3 Generic Cut Elimination And Construction of Cut-free Systems

We start our investigation with the observation that while standard sequent rules for the boolean connectives carry over the whole context to the premisses, in standard sequent systems for many modal logics such as *K* or *S4* [19] either no or only modalised context formulae are propagated from conclusion to premisses. At the same time exactly one layer of modalities is added to the principal formulae. In order to fit these different formats into a unified framework we now generalise the notion of a shallow rule [10] using the notion of *context restrictions*, a weaker form of the context relations in [3].

Definition 1. If F is a set of formulae, a *context restriction* C *over* F (or simply a *restriction*) is given by a tuple of sets of formulae in F, i.e. $C = \langle F_1, F_2 \rangle$ with $F_1, F_2 \subseteq F$. We write $\mathfrak{C}(F)$ for the set of context restrictions over F. For a restriction $C = \langle F_1, F_2 \rangle$ and a sequent $\Gamma \Rightarrow \Delta$ we write $(\Gamma \Rightarrow \Delta) \upharpoonright_C$ or $\Gamma \upharpoonright_{F_1} \Rightarrow \Delta \upharpoonright_{F_2}$ for the sequent consisting of the restriction of Γ (resp. Δ) to substitution instances of formulae A with $A \in F_1$ (resp. $A \in F_2$) on the left (resp. right) hand side. An occurrence of a formula in a sequent $\Gamma \Rightarrow \Delta$ *satisfies* context restriction C if it also occurs in $(\Gamma \Rightarrow \Delta) \upharpoonright_C$, and a sequent $\Gamma \Rightarrow \Delta$ satisfies C if $(\Gamma \Rightarrow \Delta) \upharpoonright_C = \Gamma \Rightarrow \Delta$. Finally, a context restriction C' satisfies C if every sequent which satisfies C' also satisfies C.

Example 2. 1. The *trivial restriction* $C_{id} := \langle \{p\}, \{p\} \rangle$ does not restrict a sequent at all, we always have $(\Gamma \Rightarrow \Delta) \upharpoonright_{C_{id}} = \Gamma \Rightarrow \Delta$.

2. The *empty restriction* $C_\emptyset := \langle \emptyset, \emptyset \rangle$ deletes every formula in a sequent: $(\Gamma \Rightarrow \Delta) \upharpoonright_{C_\emptyset} = \ \Rightarrow$.

3. The restriction $C_{4\Box} := \langle \{\Box p\}, \emptyset \rangle$ deletes the right side of a sequent and restricts the left side to boxed formulae. E.g.: $(A, C \wedge D, \Box(A \vee B) \Rightarrow \Box D, B) \upharpoonright_{C_{4\Box}} = \Box(A \vee B) \Rightarrow$.

Definition 3. A *rule with context restrictions* (or simply a *rule*) is a tuple $(\, \mathcal{P}; \Sigma \Rightarrow \Pi \,)$ where $\mathcal{P} \subseteq \mathcal{S}(\mathcal{V}) \times \mathfrak{C}(\mathcal{F})$ is the set of *premisses* with associated context restrictions, and $\Sigma \Rightarrow \Pi \in \mathcal{S}(\Lambda(\mathcal{V}))$ are the *principal formulae*, such that no variable occurs twice in the principal formulae and every variable occurs in the principal formulae if it occurs in at least one of the premisses. An *instance* of a rule R is given by a substitution $\sigma : \mathcal{V} \to \mathcal{F}$ and a *context* $\Gamma \Rightarrow \Delta \in \mathcal{S}(\mathcal{F})$ and is written as

$$\frac{\{\ \Gamma \upharpoonright_{F_1}, \Theta\sigma \Rightarrow \Delta \upharpoonright_{F_2}, \Upsilon\sigma \mid (\Theta \Rightarrow \Upsilon; \langle F_1, F_2 \rangle) \in \mathcal{P}\ \}}{\Gamma, \Sigma\sigma \Rightarrow \Delta, \Pi\sigma}\,.$$

Whenever we mention a set of rules we assume that it is closed under injective renaming of variables and for all n-ary $\heartsuit \in \Lambda$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ includes the *congruence rules* $(\{(p_i \Rightarrow q_i; C_\emptyset) \mid i \leq n\} \cup \{(q_i \Rightarrow p_i; C_\emptyset) \mid i \leq n\}; \heartsuit\boldsymbol{p} \Rightarrow \heartsuit\boldsymbol{q})$.

Thus if a formula A is in the left component of a restriction associated with a premiss of a rule, then in an instance of this rule the premiss carries over all substitution instances of A from the left hand side of the context, and dually for the right hand side.

$\wedge_\mathsf{R} := (\{(\Rightarrow p; C_{id}), (\Rightarrow q; C_{id})\}; \Rightarrow p \wedge q)$	$\dfrac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$
$R_{K_n} := (\{(p_1, \dots, p_n \Rightarrow q; C_\emptyset)\}; \Box p_1, \dots, \Box p_n \Rightarrow \Box q)$	$\dfrac{A_1, \dots, A_n \Rightarrow B}{\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta, \Box B}$
$R_{T\Box} := (\{(p \Rightarrow ; C_{id})\}; \Box p \Rightarrow)$	$\dfrac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}$
$R_{4\Box} := (\{(\Rightarrow p ; C_{4\Box})\}; \Rightarrow \Box p)$	$\dfrac{\Box\Sigma \Rightarrow A}{\Gamma, \Box\Sigma \Rightarrow \Delta, \Box A}$

Table 1. Some sequent rules as rules with context restrictions and in standard notation [18, 19]

Example 4. 1. The rules of G are rules with context restriction C_{id}.
2. The rules $\mathcal{R}_K := \{R_{K_n} \mid n \geq 0\}$ of modal logic K are rules with restriction $C_\emptyset$.
3. The rules $\mathcal{R}_{S4} := \{R_{T\Box}, R_{4\Box}\}$ of modal logic *S4* are rules with context restrictions C_{id} resp. $C_{4\Box}$.

Table 1 shows some of these rules as rules with context restrictions and in a more traditional notation.

Definition 5. Let $\mathcal{R}$ be a set of rules and $S \subseteq \mathcal{S}(\mathcal{F})$ a set of sequents. We use the standard notion of *derivations* [18] and say that a sequent $\Gamma \Rightarrow \Delta$ is $\mathcal{R}$*-derivable from* S if there is a derivation of $\Gamma \Rightarrow \Delta$ from S using only instances of rules in $\mathcal{R}$. We then write $S \vdash_\mathcal{R} \Gamma \Rightarrow \Delta$. If we consider rules for asymmetric sequents we will indicate this by writing $\vdash^i_\mathcal{R}$, and we write $\vdash^{[i]}_\mathcal{R}$ if a result holds in both settings. Derivability from $\emptyset$ is denoted by $\vdash^{[i]}_\mathcal{R} \Gamma \Rightarrow \Delta$ and derivability in $\mathcal{R}_1 \cup \mathcal{R}_2$ by $\vdash^{[i]}_{\mathcal{R}_1\mathcal{R}_2}$. We write $\mathcal{R}[\mathsf{CutCon}]$ if a statement holds for $\mathcal{R}$ and extensions with Cut and / or Con.

Admissibility of Weakening is shown by a standard induction on the derivations:

Lemma 6. *Forevery set $\mathcal{R}$ of rules and (asymmetric) sequent $\Gamma \Rightarrow \Delta$ we have $\vdash^{[i]}_{\mathcal{R}[\mathsf{CutCon}]}$ $\Gamma \Rightarrow \Delta$ whenever $\vdash^{[i]}_{\mathcal{R}\mathsf{W}[\mathsf{CutCon}]} \Gamma \Rightarrow \Delta$.*

Proof. By induction on the depth of the derivation. A weakening of the conclusion of a rule with a formula A is pushed into all the premisses whose context restriction is satisfied by A. Then these applications of Weakening are eliminated using the induction hypothesis. □

3.1 Cuts Between Rules and Cut Elimination

The main tool in the construction of cut free rule sets is the notion of *cuts between rules* from [10, 11] that we need to adapt to handle context restrictions. Cut between rules is a two-stage process: first we replace a pair of rules by the rule arising from performing a cut between the conclusions. In a second step we modify the premisses so that variables that no longer appear in the conclusion of the cut are eliminated.

Definition 7. If $\mathcal{P} \subseteq \mathcal{S}(\mathcal{V}) \times \mathfrak{C}(\mathcal{F})$ is a set of premisses with context restrictions, then for $p \in \mathcal{V}$ the *p-elimination of* $\mathcal{P}$ is the set

$$\mathcal{P} \ominus p := \{ (\Gamma, \Sigma \Rightarrow \Delta, \Pi;\ C_1 \cup C_2) \mid (\Gamma \Rightarrow \Delta, p;\ C_1) \in \mathcal{P},\ (p, \Sigma \Rightarrow \Pi;\ C_2) \in \mathcal{P} \} \cup \{ (\Gamma \Rightarrow \Delta;\ C) \mid (\Gamma \Rightarrow \Delta;\ C) \in \mathcal{P},\ p \notin \Gamma, \Delta \},$$

where for restrictions $C_1 = \langle F_1, F_2 \rangle$ and $C_2 = \langle G_1, G_2 \rangle$ we write $C_1 \cup C_2$ for $\langle F_1 \cup G_1, F_2 \cup G_2 \rangle$. Iterated elimination of variables $\boldsymbol{p} = p_1, \dots, p_n$ is denoted by $\mathcal{P} \ominus \boldsymbol{p}$. For rules $R = (\mathcal{P}_R; \Gamma \Rightarrow \Delta, \heartsuit \boldsymbol{p})$ and $Q = (\mathcal{P}_Q; \heartsuit \boldsymbol{p}, \Sigma \Rightarrow \Pi)$ the *cut between R and Q on* $\heartsuit \boldsymbol{p}$ is the rule $\mathsf{cut}(R, Q, \heartsuit \boldsymbol{p}) := (\, (\mathcal{P}_R \cup \mathcal{P}_Q) \ominus \boldsymbol{p};\ \Gamma, \Sigma \Rightarrow \Delta, \Pi\,)$. A rule set $\mathcal{R}$ is *principal-cut closed* if it is closed under cuts between rules.

Example 8. 1. The rule sets $\mathsf{G}[i]$ are principal-cut closed, since cuts between rules can be replaced by the *identity rule* $R_{id} := (\{(\Rightarrow ; C_{id})\};\ \Rightarrow)$.
2. The rule set $\mathcal{R}_K$ is principal-cut closed, since $\mathsf{cut}(R_{K_n}, R_{K_m}, \Box q) = R_{K_{n+m-1}} \in \mathcal{R}_K$.
3. The rule set $\mathcal{R}_{S4}$ is principal-cut closed, since $\mathsf{cut}(R_{4\Box}, R_{T\Box}, \Box p) = R_{id}$.

Since in the presence of the rules for (intuitionistic) propositional logic it is possible to re-construct the cut formula from the premisses of the cut between two rules, saturating a rule set under cuts between rules does not change the set of derivable sequents:

Lemma 9. *If $\mathcal{R}$ is a set of rules and R is a cut between two rules from $\mathcal{R}$, then $\vdash^{[i]}_{\mathsf{G}[i]\mathsf{CutCon}\mathcal{R}R}$ $\Gamma \Rightarrow \Delta$ iff $\vdash^{[i]}_{\mathsf{G}[i]\mathsf{CutCon}\mathcal{R}}$ $\Gamma \Rightarrow \Delta$.*

Proof. For the symmetric case this was shown in [10]. For the asymmetric case consider a unary modality $\heartsuit$ and the cut $\mathsf{cut}(R_1, R_2, \heartsuit p)$ between two rules $R_1 = (\ \mathcal{P}_1; \Gamma \Rightarrow \heartsuit p\ )$ and $R_2 = (\ \mathcal{P}_2; \heartsuit p, \Delta \Rightarrow B\ )$. By definition we have $\mathsf{cut}(R_1, R_2, \heartsuit p) = (\ (\mathcal{P}_1 \cup \mathcal{P}_2) \ominus p; \Gamma, \Delta \Rightarrow B\ )$. Our goal is to replace an arbitrary instance of $\mathsf{cut}(R_1, R_2, \heartsuit p)$ in a derivation by instances of R_1 and R_2 and an application of the cut rule. Suppose the combined premisses of the two rules are

$$\mathcal{P}_1 \cup \mathcal{P}_2 = \{(\Theta_i \Rightarrow p; C^r_i) \mid i \in I\} \cup \{(p, \Upsilon_j \Rightarrow C_j; C^\ell_j) \mid j \in J\} \cup \{(\Xi_k \Rightarrow D_k; C^n_k) \mid k \in K\},$$

where $p \notin \Xi_k, D_k$ for all $k \in K$. Now consider an instance

$$\frac{\{\Sigma \restriction_{C^\ell_i \cup C^r_j}, (\Theta_i, \Upsilon_j)\sigma \Rightarrow C_j\sigma \mid i \in I, j \in J\} \ \cup\ \{\Sigma \restriction_{C^n_k}, \Xi_k\sigma \Rightarrow D_k\sigma \mid k \in K\}}{\Sigma, (\Gamma, \Delta)\sigma \Rightarrow B\sigma}$$

of the rule $\mathsf{cut}(R_1, R_2, p)$ in a derivation. We construct a substitution instance of p by:

$$P := \bigvee_{i \in I} \left(\bigwedge \Sigma \restriction_{C^\ell_i} \wedge \bigwedge \Theta_i\sigma \right).$$

Then for every $i \in I$ we can derive $\Sigma \restriction_{C^\ell_i}, \Theta_i \Rightarrow P$ from axioms using the right conjunction and disjunction rules. Furthermore for every $i \in I$ and $j \in J$ we get $\Sigma \restriction_{C^\ell_j}, \Sigma \restriction_{C^r_i}, \Theta_i\sigma, \Upsilon_j\sigma \Rightarrow C_j\sigma$ from the premisses of the instance using admissibility of Weakening. Thus for every $j \in J$ we have $\Sigma \restriction_{C^\ell_j}, P, \Upsilon_j\sigma \Rightarrow C_j\sigma$ by left conjunction and disjunction. But now we can apply the rules R_1 and R_2 to these premisses, cut the conclusions and contract duplicate context formulae to arrive at the conclusion of the instance of the cut. If the modality has arity greater than 1 we iterate the process. □

Cuts between rules provide us with a means of eliminating cuts on principal formulae of two rules by replacing the cut with an instance of the cut between the two rules and a number of cuts on formulae of lower complexity. Moreover, Lemma 9 guarantees that we may simply add missing cuts to a rule set without jeopardising soundness. While this is enough for axioms without nested modalities [10], in the more general setting with context restrictions we need additional criteria for cuts involving context formulae:

Definition 10. Two restrictions $C_1 = \langle F_1, F_2\rangle, C_2 = \langle G_1, G_2\rangle$ *overlap* if there are formulae $A_1 \in F_2, A_2 \in G_1$ and substitutions σ_1, σ_2 with $A_1\sigma_1 = A_2\sigma_2$. A rule set $\mathcal{R}$ is

1. *context-cut closed* if whenever $R_0, R_1 \in \mathcal{R}$ and there are context restrictions C_0 of R_0 and C_1 of R_1 which overlap, then there is $i \in \{0, 1\}$ such that all context restrictions of R_i which overlap C_{1-i} and the principal formulae of R_i satisfy C_{1-i}.
2. *mixed-cut closed* if whenever $R, Q \in \mathcal{R}$ and a principal formula A of R satisfies a context restriction of Q, then all context restrictions of R and all principal formulae of R except for A satisfy all those context restrictions of Q satisfied by A.

Intuitively, these conditions allow pushing cuts involving context formulae into the premisses of one of the rules and eliminating them by induction on the cut level.

Example 11. 1. The rule sets $\mathsf{G}[i]$ are context- and mixed-cut closed because all the rules involve only the restriction C_{id} or its asymmetric version $C^i_{id} := \langle\{p\}, \emptyset\rangle$. Hence every restriction is satisfied by every principal formula and every other restriction.

2. The rule set $\mathcal{R}_K$ is trivially context- and mixed-cut closed.
3. The rule set $\mathcal{R}_{S4}$ is mixed-cut closed, since the restriction $C_{4\Box}$ satisfies C_{id}. Since the principal formula of $R_{4\Box}$ also satisfies C_{id}, the set is furthermore context-cut closed.

Since in general rules are not invertible and we need to take care of Contraction we will follow Gentzen's original strategy [8] when proving cut elimination and eliminate multicuts $\dfrac{\Gamma \Rightarrow \Delta, A^n \quad A^m, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}$ instead of cuts. Thus we also need to deal with multiple principal occurrences of the same formula. We do this by elevating contraction to the level of derivation rules and considering rule sets closed under this operation.

Definition 12 ([11]). If $\mathcal{P}$ is a set of premisses with restrictions and $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ are n-tuples of variables, then $\mathcal{P}[\boldsymbol{q} \leftarrow \boldsymbol{p}]$ is the result of replacing every occurrence of q_i in a sequent occurring in a premiss in $\mathcal{P}$ by p_i for all $i = 1, \ldots, n$ and contracting duplicate instances of $p_1, \ldots, p_n$. Let $R = (\mathcal{P}; \Gamma, \heartsuit\boldsymbol{p}, \heartsuit\boldsymbol{q} \Rightarrow \Delta)$ be a rule. The *left contraction of* R *on* $\heartsuit\boldsymbol{p}$ *and* $\heartsuit\boldsymbol{q}$ is the rule $\mathsf{ConL}(R, \heartsuit\boldsymbol{p}, \heartsuit\boldsymbol{q}) = (\mathcal{P}[\boldsymbol{q} \leftarrow \boldsymbol{p}]; \Gamma, \heartsuit\boldsymbol{p} \Rightarrow \Delta)$. The *right contraction* $\mathsf{ConR}(R, \heartsuit\boldsymbol{p}, \heartsuit\boldsymbol{q})$ is defined dually. A rule set $\mathcal{R}$ is *contraction closed* if for every rule $R \in \mathcal{R}$ instances of the rules $\mathsf{ConL}(R, \heartsuit\boldsymbol{p}, \heartsuit\boldsymbol{q})$ and $\mathsf{ConR}(R, \heartsuit\boldsymbol{p}, \heartsuit\boldsymbol{q})$ can be simulated by applications of Weakening and Contraction, followed by at most one application of a rule $R' \in \mathcal{R}$ and Weakening. A set of rules is *saturated* if it is contraction, principal-cut, context-cut, and mixed-cut closed.

Example 13. 1. The rules of $G[i]$ and $\mathcal{R}_{S4}$ are trivially contraction closed.

2. $\mathcal{R}_K$ is contraction closed because $\mathsf{ConL}(R_{K_n}, \Box p_{n-1}, \Box p_n) = R_{K_{n-1}} \in \mathcal{R}_K$.

Thus each of $\mathsf{G}[i], \mathcal{R}_K, \mathcal{R}_{S4}$ are saturated.

Theorem 14 (Cut Elimination). *For every saturated set $\mathcal{R}$ of rules and (asymmetric) sequent $\Gamma \Rightarrow \Delta$ we have $\vdash^{[i]}_{\mathcal{R}\mathsf{Con}} \Gamma \Rightarrow \Delta$ whenever $\vdash^{[i]}_{\mathcal{R}\mathsf{ConCut}} \Gamma \Rightarrow \Delta$.*

Proof. As usual, the *rank* of a formula is defined by $\mathsf{rk}(p) = \mathsf{rk}(\bot) = 0$ and $\mathsf{rk}(\heartsuit(A_1, \ldots, A_n)) = 1 + \max\{\mathsf{rk}(A_i) \mid 1 \leq i \leq n\}$. We basically follow Gentzen's original proof in [8] and eliminate the multicut rule

$$\frac{\Gamma \Rightarrow \Delta, A^n \quad A^m, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{mcut}$$

instead of the cut rule. Since Weakening is admissible and we have the Contraction rule, this rule is derivable using the standard cut rule. The main strategy is to eliminate topmost multicuts by a double induction, where the outer induction is on the rank of the cut formula, and the inner induction is on the sum of the sizes of the derivations of the premisses of the instance of the multicut rule (the *cut level*). We write χ for the cut formula. We will actually show a slightly stronger statement, namely that in a derivation in which every formula occurs at most once amongst the principal formulae of each instance of a rule from $\mathcal{R}$, we can eliminate the topmost multicut, and retain the restriction on the multiplicity of principal occurrences of formulae. This is necessary, since the definition of a principal-cut closed rule set only deals with single occurrences of the cut formula amongst the principal formulae. We need a small lemma.

Lemma 15. *Let $\mathcal{R}$ be a contraction closed rule set. If a sequent is derivable in $\mathcal{R}\mathsf{Con}$ (resp. $\mathcal{R}\mathsf{ConCut}$), then there is a derivation of it in $\mathcal{R}\mathsf{Con}$ (resp. $\mathcal{R}\mathsf{ConCut}$), in which every formula occurs at most once amongst the principal formulae of each instance of a rule from $\mathcal{R}$.*

Proof. Since $\mathcal{R}$ is contraction closed, a topmost instance of a rule R in $\mathcal{R}$ with conclusion $\Sigma, A^k \Rightarrow \Pi$, where all k occurrences of A are principal, and premisses $P_1, \ldots, P_n$ can be replaced by an instance of a rule R' in $\mathcal{R}$ with conclusion $\Sigma, A^{k-1} \Rightarrow \Pi$, with the occurrences of A principal, and derivations of its premisses from $\{P_1, \ldots, P_n\}$ using only Contraction and Weakening. This is done successively, until we end up with an instance of a rule Q with the conclusion $\Gamma, A \Rightarrow \Delta$. The instances of the Weakening rule are then eliminated using admissibility of Weakening, which adds occurrences of formulae only to the contexts of rules in $\mathcal{R}$. □ *Lemma*

If the rank of the cut formula is 0, then it is $\bot$ or a propositional variable. Since no rule in $\mathcal{R}$ introduces $\bot$ in its principal formulae and since we have the left introduction rule for $\bot$ a standard induction on the depth of the derivation shows that $\frac{\Gamma \Rightarrow \Delta, \bot}{\Gamma \Rightarrow \Delta}$ is depth-preserving admissible. Thus we get a derivation of the conclusion of the cut using Weakening. So suppose the cut formula is a variable. Since all the rules introduce a connective in the principal formulae, every occurrence of the cut formula in a conclusion of a rule from $\mathcal{R}$ must be in the context. If the cut level is 0, then both of the premisses of the cut are axioms, and the conclusion of the cut is an axiom as well. So suppose the cut level is $n + 1$.

- If exactly one of the premisses is an axiom, then either the cut formula is not principal in the axiom and the conclusion of the cut is an axiom, or the cut formula is principal in the axiom and we get the conclusion of the cut by using Contraction and Weakening from the other premiss.

- If at least one of the premisses of the cut is the conclusion of the Contraction rule, then we may apply the multicut to the premiss of this rule to get the conclusion of the cut. The newly introduced cut has cut rank at most n and is eliminated by an appeal to the (inner) induction hypothesis.
- If both of the premisses of the cut are conclusions of rules R_1, R_2 from $\mathcal{R}$, then since $\mathcal{R}$ is context-cut closed the conclusion of R_1 satisfies every context restriction of R_2 which is satisfied by a negative occurrence of χ, or the conclusion of R_2 satisfies every context restriction of R_1 satisfied by a positive occurrence of χ. Thus after permuting the cuts into every premiss of R_1 (respectively R_2) whose context restriction is satisfied by the negative (positive) occurrence of χ, we may apply R_2 (respectively R_1) to get the conclusion of the cut. The newly introduced cuts have cut rank at most n and thus are eliminated using the (inner) induction hypothesis.

Now suppose the rank of the cut formula is $m + 1$. If the cut level is 0, then again the two premisses of the cut must be axioms, and we get the conclusion of the cut in the standard way. So suppose the cut level is $n + 1$. We have the following possible cases:

- One of the premisses of the cut is an axiom. If one of the premisses of the cut is an axiom in which the cut formula is not principal, then the conclusion of the cut also is an axiom. If one of the premisses of the cut is an axiom in which the cut formula is principal, then we get the conclusion of the cut by weakening the conclusion of the other premiss.
- At least one of the premisses of the cut is the conclusion of the Contraction rule. As above we apply multicut to the premiss of this rule and eliminate this cut using the (inner) induction hypothesis.
- Both of the premisses of the cut are conclusions of rules R_1, R_2 in $\mathcal{R}$.
  - If every occurrence of the cut formula χ in both of the conclusions of R_1 and R_2 is contextual, then as above we use the context-cut closure of $\mathcal{R}$ to push the multicuts into the premisses of one of the two rules and eliminate these cuts using the (inner) induction hypothesis.
  - If every occurrence of the cut formula χ in one of the conclusions of R_1 or R_2 (say in R_1) is contextual, and at least one occurrence of χ in the other conclusion (say of R_2) is principal, we first apply multicut on χ to the conclusion of R_2 and every premiss of R_1 in which χ occurs as part of the context. Since $\mathcal{R}$ is mixed-cut closed the resulting sequents satisfy the context restrictions of the relevant premisses. Thus we may now apply R_1 to these new premisses (and possibly weakened versions of the other premisses) to get the conclusion of the original multicut. The newly introduced cuts again have lower level and are eliminated using the (inner) induction hypothesis.
  - If occurrences of χ are principal in the conclusions of both R_1 and R_2, then we first use contraction closure of $\mathcal{R}$ in the form of Lemma 15 to ensure that in both conclusions there is exactly one principal occurrence of χ. In a second step we eliminate contextual occurrences of the cut formula χ from the premisses by applying multicut on χ to
    1. the premisses of R_1 whose restriction is satisfied by a negative occurrence of χ and the conclusion of R_2; and

2. the premisses of R_2 whose restriction is satisfied by a positive occurrence of χ and the conclusion of R_1.

These cuts now are eliminated using the (inner) induction hypothesis. Since $\mathcal{R}$ is mixed-cut closed the resulting sequents still satisfy the context restrictions of the relevant premisses and we can now apply R_1 to the sequents from 1 and R_2 to those from 2 to get conclusions $\Gamma \Rightarrow \Delta, \chi$ and $\Sigma, \chi \Rightarrow \Pi$, where the only occurrences of χ are principal. Finally, we use the fact that $\mathcal{R}$ is principal-cut closed to replace the cut on χ by cuts on (proper) subformulae of χ between the premisses of R_1 and R_2 and instances of rules from $\mathcal{R}$. The new cuts are (multi-)cuts on formulae of rank at most m, and thus can be eliminated using the (outer) induction hypothesis. Now contraction of the duplicated formulae from steps 1 and 2 yields the conclusion of the original multicut.

□

An example for the very last case might be instructive. Let $\mathcal{R} := \mathcal{R}_{S4} = \{R_{4\Box}, R_{T\Box}\}$ and suppose we have the following situation:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ \Box\Gamma \Rightarrow A\end{matrix}}{\Box\Gamma \Rightarrow \Box A}\,R_{4\Box} \qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ \vdots\\ \Sigma, \Box A, \Box A, A \Rightarrow \Pi\end{matrix}}{\Sigma, \Box A, \Box A, \Box A \Rightarrow \Pi}\,R_{T\Box}}{\Box\Gamma, \Sigma \Rightarrow \Pi}\,\mathsf{mcut}$$

We first use context-cut closure of $\mathcal{R}_{S4}$ to eliminate the contextual occurrences of the cut formula $\Box A$ by permuting the multicut into the premiss of $R_{T\Box}$ as follows.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ \Box\Gamma \Rightarrow A\end{matrix}}{\Box\Gamma \Rightarrow \Box A}\,R_{4\Box} \qquad \begin{matrix}\mathcal{D}_2\\ \vdots\\ \Sigma, \Box A, \Box A, A \Rightarrow \Pi\end{matrix}}{\Box\Gamma, \Sigma, A \Rightarrow \Pi}\,\mathsf{mcut}}{\Box\Gamma, \Sigma, \Box A \Rightarrow \Pi}\,R_{T\Box}$$

This multicut is has smaller cut level than the original multicut and thus is eliminated by the inner induction hypothesis, resulting in a cut free derivation $\mathcal{D}$ for the sequent $\Box\Gamma, \Sigma, A \Rightarrow \Pi$. Applying rule $R_{T\Box}$ to this sequent and cutting on $\Box A$ we get

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ \Box\Gamma \Rightarrow A\end{matrix}}{\Box\Gamma \Rightarrow \Box A}\,R_{4\Box} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Box\Gamma, \Sigma, A \Rightarrow \Pi\end{matrix}}{\Box\Gamma, \Sigma, \Box A \Rightarrow \Pi}\,R_{T\Box}}{\Box\Gamma, \Box\Gamma, \Sigma \Rightarrow \Pi}\,\mathsf{mcut}$$

Using principal-cut closure of $\mathcal{R}_{S4}$ this is replaced by a cut on the premisses of the two rules and possibly applications of rules from the rule set. In this case we do not need to apply any rule from the rule set, and replace it with

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ \Box\Gamma \Rightarrow A\end{matrix} \quad \begin{matrix}\mathcal{D} \\ \vdots \\ \Box\Gamma, \Sigma, A \Rightarrow \Pi\end{matrix}}{\Box\Gamma, \Box\Gamma, \Sigma \Rightarrow \Pi}\ \mathsf{mcut}$$

Since A has lower rank than $\Box A$, the cut on A can now be eliminated using the outer induction hypothesis. Finally, the duplicated formulae in $\Box\Gamma$ are contracted using the contraction rule.

While saturated rule sets allow for cut elimination, we are also interested in decision procedures via backwards proof search. For this we also need admissibility of Contraction. While contraction closure of the rule set takes care of contractions of two principal formulae of a rule, for contractions of principal and context formulae we use the standard method of copying the relevant principal formulae into the premisses. This might seem a bit coarse but again is necessary because in general the rules are not invertible.

Definition 16. For a rule $R = (\mathcal{P}; \Sigma \Rightarrow \Pi)$ a *modified instance*

$$\frac{\{\ (\Gamma, \Sigma\sigma) \restriction_{F_1}, \Theta\sigma \Rightarrow (\Delta, \Pi\sigma) \restriction_{F_2}, \Upsilon\sigma \mid (\Theta \Rightarrow \Upsilon;\ \langle F_1, F_2 \rangle) \in \mathcal{P}\ \}}{\Gamma, \Sigma\sigma \Rightarrow \Delta, \Pi\sigma}$$

of R is given by a substitution $\sigma : \mathcal{V} \to \mathcal{F}$ and a context $\Gamma \Rightarrow \Delta \in \mathcal{S}(\mathcal{F})$. We write $\vdash_{\mathcal{R}^*}$ for derivability using modified instances instead of instances of rules in $\mathcal{R}$.

Theorem 17 (Admissibility of Contraction). *For every set $\mathcal{R}$ of rules and (asymmetric) sequent $\Gamma \Rightarrow \Delta$ we have $\vdash^{[i]}_{\mathcal{R}\mathsf{Con}} \Gamma \Rightarrow \Delta$ iff $\vdash^{[i]}_{\mathcal{R}^*\mathsf{Con}} \Gamma \Rightarrow \Delta$ iff $\vdash^{[i]}_{\mathcal{R}^*} \Gamma \Rightarrow \Delta$.*

Proof. Since Weakening is admissible we get the "only if" direction of the first equivalence. The other direction follows by treating the copies of the principal formulae as context formulae, applying the rule, and contracting duplicate formulae.

For the second equivalence the "if" direction is immediate. For the other direction we employ a double induction on the complexity of the contracted formula and on the depth of the derivation. If the Contraction is applied to an axiom or an instance of the left introduction rule for $\bot$ we eliminate it the standard way. So suppose the premiss of the instance of the Contraction rule is the conclusion of a rule in $\mathcal{R}$. If the Contraction is between two context formulae or between a context formula and a principal formula, we permute the instance of Contraction into the premisses of this rule and eliminate it using the inner induction hypothesis. If Contraction is applied to two principal formulae of a rule R we use contraction closure of the rule set to replace the instance of the rule and the Contraction by a number of Contractions on the premisses of that rule and a rule instance Q from the rule set, where we assume wlog that the newly introduced Contractions do not involve context formulae. Since the rules add one layer of connectives in the principal formulae, the newly introduced Contractions must be on formulae of lower complexity and we may eliminate them using the outer induction hypothesis. Finally, instances of Weakening are eliminated using Lemma 6. □

If the rule set is furthermore *tractable* in the sense that given a sequent the rules with this sequent as conclusion have codes of size polynomial in the size of the sequent, which can be recognised in space polynomial in the size of the sequent, and given the code of a rule its premisses can be recognised in space polynomial in the code, we get a generic complexity bound for deciding derivability using modified instances. Due to the more general rule format this bound is slightly higher than the PSPACE bound in [10]. Whether this can be improved in general is subject of ongoing work.

Theorem 18. *For a saturated and tractable set $\mathcal{R}$ of rules, derivability in $\mathcal{R}^*$ is decidable in* EXPTIME.

Proof. The idea is to work on fully contracted sequents and make use of the subformula property of the system to compute all the derivable sequents built from subformulae of the original sequent.

For a multiset Γ of formulae we write $\mathsf{Supp}(\Gamma)$ for the *support* of Γ, that is the multiset of formulae in Γ disregarding their multiplicities. Furthermore, for a sequent $\Gamma \Rightarrow \Delta$ let $\mathsf{Sf}(\Gamma \Rightarrow \Delta)$ denote the set of subformulae of $\Gamma \Rightarrow \Delta$, where as usual we identify different occurrences of the same formula. Since the rule set is saturated and thus Contraction and Weakening are admissible, it is clear that a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathcal{R}^*$ if and only if the sequent $\mathsf{Supp}(\Gamma) \Rightarrow \mathsf{Supp}(\Delta)$ is derivable in $\mathcal{R}^*$. Let $\mathfrak{S}(\Gamma \Rightarrow \Delta)$ denote the set of sequents $\mathsf{Supp}(\Sigma) \Rightarrow \mathsf{Supp}(\Pi)$ with $\Sigma \Rightarrow \Pi \in \mathcal{S}(\mathsf{Sf}(\Gamma \Rightarrow \Delta))$. The following procedure checks derivability in $\mathcal{R}^*$ on input $\Gamma \Rightarrow \Delta$:

- set $\mathcal{D}$ to $\emptyset$.
- repeat until no more new sequents are found:
  for every $\Sigma \Rightarrow \Pi \in \mathfrak{S}(\Gamma \Rightarrow \Delta)$ check whether there is an instance of a rule from $\mathcal{R}^*$ with conclusion $\Sigma \Rightarrow \Pi$ such that for all its premisses $\Theta \Rightarrow \Upsilon$ the sequent $\mathsf{Supp}(\Theta) \Rightarrow \mathsf{Supp}(\Upsilon)$ is in $\mathcal{D}$. If there is, then add $\Sigma \Rightarrow \Pi$ to $\mathcal{D}$.
- If $\mathsf{Supp}(\Gamma) \Rightarrow \mathsf{Supp}(\Delta) \in \mathcal{D}$, then output "derivable", otherwise "not derivable".

Since $\mathcal{D} \subseteq \mathfrak{S}(\Gamma \Rightarrow \Delta)$ and since the number of sequents in $\mathfrak{S}(\Gamma \Rightarrow \Delta)$ is only exponential in $|\mathsf{Sf}(\Gamma \Rightarrow \Delta)| =: s$, the loop in the procedure is executed at most exponentially often (in s). Furthermore, in each execution of the loop the procedure checks at most exponentially many sequents, and since the rule set is tractable, checking each sequent can be done in time exponential in s. Thus the overall runtime of the procedure is exponential in the number of subformulae of the input sequent. □

3.2 Construction of cut-free sequent systems from axioms

With Theorems 14 and 17 we have presented a general *criterion* for a sequent system with context restrictions to admit both cut and contraction. Of course now we need to construct sequent systems satisfying this criterion. As in the case without context restrictions [10] these results suggest constructing saturated rule sets by *saturation*: starting with a set of rules with context restrictions simply add missing cuts and contractions until no more new rules are found. In the presence of context restrictions, however, we then need to check that the resulting rule set is also context-cut closed and mixed-cut closed. The following well-known example shows that this need not be the case.

Example 19. The rule set $\mathcal{R}_{S5} :=\{\ (\{(p \Rightarrow\ ; C_{id})\}; \Box p \Rightarrow\ ),\ (\{(\ \Rightarrow p; C_{5\Box})\};\ \Rightarrow \Box p)\ \}$ with $C_{5\Box} := \langle\{\Box p\}, \{\Box p\}\rangle$ is contraction closed and principal- and context-cut closed. It is not mixed-cut closed, since the occurrence of the principal formula $\Box p$ of $R_{T\Box}$ satisfies the restriction $C_{5\Box}$ of $R_{5\Box}$, but the restriction C_{id} does not.

The method of constructing cut-free rule sets by saturation works reasonably well if we start with a set of sequent rules, but often the modal logics of interest are given in a Hilbert-style system by a set of axioms. Thus the first step in constructing a cut free sequent system from such axioms is to translate the axioms into sequent rules. While this can alway be done if the axioms are *non-nested*, i.e. without nested modalities, and the underlying propositional logic is classical, in the general case we need to be more careful. The notion of cuts between rules will be a useful tool in this step as well.

We assume that the underlying propositional logic is classical or intuitionistic. In a first step we extend the method for converting non-nested axioms from [10] to the asymmetric setting using notions from [6]. The main idea is to first treat the modal subformulae in a non-nested axiom like propositional variables, use invertibility of the underlying rule set to break the axiom into a finite number of sequents, and then resolve propositional logic under the modalities by introducing new variables and premisses stating that these variables are equivalent to the original formulae. Finally, these premisses are again broken up using invertibility of the underlying rules. To identify the axioms which can be broken up we loosely follow the idea of the substructural hierarchy from [6] and consider the notions of left resolvable and right resolvable formulae. Intuitively, if a right resolvable formula occurs positively in a sequent, its main (boolean) connective can be broken up. We introduce these notions in a generic form which allows treating classical and intuitionistic logics in the same framework. Also this shows that they are easily adapted to other logics such as minimal or distributive logic.

Definition 20. The sets $\mathcal{F}_r$ of *right resolvable formulae* and $\mathcal{F}_\ell$ of *left resolvable formulae* and their intuitionistic versions ${\mathcal{F}_r}^i$ and ${\mathcal{F}_\ell}^i$ are defined recursively by

1. if $p \in \mathcal{V}$ then $p \in {\mathcal{F}_r}^{[i]}$ and $p \in {\mathcal{F}_\ell}^{[i]}$;
2. $\bot \in {\mathcal{F}_r}^{[i]}$ and $\bot \in {\mathcal{F}_\ell}^{[i]}$;
3. if $A_1, A_2 \in {\mathcal{F}_r}^{[i]}$ then $A_1 \wedge A_2 \in {\mathcal{F}_r}^{[i]}$ and $A_1 \vee A_2 \in \mathcal{F}_r$;
4. if $A_1, A_2 \in {\mathcal{F}_\ell}^{[i]}$ then $A_1 \wedge A_2 \in {\mathcal{F}_\ell}^{[i]}$ and $A_1 \vee A_2 \in {\mathcal{F}_\ell}^{[i]}$;
5. if $A_1 \in {\mathcal{F}_\ell}^{[i]}$ and $A_2 \in {\mathcal{F}_r}^{[i]}$ then $A_1 \to A_2 \in {\mathcal{F}_r}^{[i]}$;
6. if $A_1 \in \mathcal{F}_r$ and $A_2 \in \mathcal{F}_\ell$ then $A_1 \to A_2 \in \mathcal{F}_\ell$

where again we write ${\mathcal{F}_r}^{[i]}$ if a clause applies both to $\mathcal{F}_r$ and ${\mathcal{F}_r}^i$.

Example 21. The formula $p \wedge ((p \vee q) \to r)$ is intuitionistically right resolvable. Both $p \vee q$ and $(p \to q) \to \bot$ are classically right resolvable, but not intuitionistically.

Since the premisses of the right (left) rule for the main connective of a right (left) resolvable formula can be derived from its conclusion in $\mathsf{G}[i]\mathsf{Cut}$, we may decompose an axiom $\overline{\Rightarrow A}$ for $A \in \mathcal{F}_r$ into a number of sequents over $\mathcal{V}$, similar to computing the regular normal form [15] of a formula:

Lemma 22. *Let $\Gamma \subseteq {\mathcal{F}_\ell}^{[i]}$ and $\Delta \subseteq {\mathcal{F}_r}^{[i]}$. Then there are unique sequents $\Gamma_i \Rightarrow \Delta_i \in \mathcal{S}(\mathcal{V})$ such that the axiom $\overline{\Gamma \Rightarrow \Delta}$ is equivalent in $\mathsf{G}[i]\mathsf{CutCon}$ to $\overline{\Gamma_1 \Rightarrow \Delta_1}, \dots, \overline{\Gamma_n \Rightarrow \Delta_n}$.*

Proof. For example if we have the asymmetric sequent $\Gamma, p \wedge q \Rightarrow r \rightarrow s$ we use derivability of the invers $\frac{p \wedge q, \Gamma \Rightarrow B}{p, q, \Gamma \Rightarrow B}$ of the $\wedge$ left-rule and the analogue for the $\rightarrow$ right-rule to derive $\Gamma, p, q, r \Rightarrow s$. The other cases are treated analogously. □

This Lemma is used to break modal axioms into sequents by treating the modalised subformulae as variables. Then the propositional variables are moved into the premisses:

Lemma 23. *For $\Gamma \Rightarrow \Delta \in \mathcal{S}(\mathcal{V})$ and $\Sigma \Rightarrow \Pi \in \mathcal{S}(\Lambda(\mathcal{V}))$ the axiom $\overline{\Gamma, \Sigma \Rightarrow \Delta, \Pi}$ is equivalent in* G[*i*]CutCon *to the rule* $(\{(\Rightarrow p; C_{id}^{[i]}) \mid p \in \Gamma\} \cup \{(p \Rightarrow ; C_{id}) \mid p \in \Delta\}; \Sigma \Rightarrow \Pi)$.

Proof. An instance of the rule is derived from the axiom using cuts on the variables in Γ and Δ and a number of contractions. The axiom is derived using the rule since the sequents $\Gamma \Rightarrow \Delta, p$ with $p \in \Gamma$ and $p, \Gamma \Rightarrow \Delta$ with $p \in \Delta$ are instances of the axiom rule. □

In the case of axioms with modal nesting depth one we now eliminate propositional logic under the modalities by adding new variables for the immediate subformulae and new premisses stating that the variables are equivalent to the original subformulae. Unfortunately the result of this operation is not necessarily a rule in our sense, since the sequents occurring in the premisses do not only include variables. Fortunately, if the modality in question is monotone or antitone (see below), we can cut this "rule" with the monotonicity rule (or its antitone counterpart) to eliminate one of the new premisses. We call the newly introduced premisses the *premisses for the subformula A* and say that the premisses for a subformula can be *resolved* if there are equivalent premisses consisting of sequents over variables. The following Definition and Lemma give criteria on when the premisses can be resolved.

Definition 24. Let $\mathcal{R}$ be a set of rules, $\boldsymbol{p} = p_1, \dots, p_n$ and $\boldsymbol{q} = q_1, \dots, q_n$. For $k \leq n$ a n-ary modality $\heartsuit\boldsymbol{p}$ is *monotone in the k-th argument* if the rule $R_{mon_k} = (\{(p_k \Rightarrow q_k; C_\emptyset)\} \cup \{(p_\ell \Rightarrow q_\ell; C_\emptyset) \mid k \neq \ell \leq n\} \cup \{(q_\ell \Rightarrow p_\ell; C_\emptyset) \mid k \neq \ell \leq n\}; \heartsuit\boldsymbol{p} \Rightarrow \heartsuit\boldsymbol{q})$ is in $\mathcal{R}$. It is *antitone in the k-th argument* if the rule R_{ant_k} with premiss $(q_k \Rightarrow p_k; C_\emptyset)$ instead of $(p_k \Rightarrow q_k; C_\emptyset)$ is in $\mathcal{R}$.

Lemma 25. *Let $\mathcal{R}$ be a rule set. Then for a sequent $\Gamma \Rightarrow \Delta, \heartsuit(\dots, A_k, \dots)$ the premisses for A_k can be resolved if: $A_k \in \mathcal{F}_\ell^{[i]}$ and $\heartsuit$ is monotone in the k-th argument; or $A_k \in \mathcal{F}_r^{[i]}$ and $\heartsuit$ is antitone in the k-th argument; or $A_k \in \mathcal{F}_\ell^{[i]} \cap \mathcal{F}_r^{[i]}$. For a sequent $\heartsuit(\dots, A_k, \dots), \Gamma \Rightarrow \Delta$ we have the analogous result with $\mathcal{F}_\ell^{[i]}$ and $\mathcal{F}_r^{[i]}$ exchanged.*

Proof. We first consider the case of a unary modality $\heartsuit$. Suppose we have the sequent $\Gamma \Rightarrow \Delta, \heartsuit A$. Then in the first step we introduce premisses $A \Rightarrow p_A$ and $p_A \Rightarrow A$ and possibly also premisses for other subformulae of $\Gamma \Rightarrow \Delta$, yielding

$$\frac{\dots \quad A \Rightarrow p_A \quad p_A \Rightarrow A}{\Gamma \Rightarrow \Delta, \heartsuit p_A}\,.$$

If $\heartsuit$ is neither monotone nor antitone we have $A \in \mathcal{F}_\ell^{[i]} \cap \mathcal{F}_r^{[i]}$ by assumption, and thus may use Lemma 22 to resolve the premisses. If $\heartsuit$ is monotone, we have that the rule

$$\frac{p_A \Rightarrow q}{\Sigma, \heartsuit p_A \Rightarrow \heartsuit q, \Pi}\ R_{mon}$$

is in $\mathcal{R}$. Computing the cut between these two rules on $\heartsuit p_A$ yields

$$\frac{\dots \qquad A \Rightarrow q}{\Gamma \Rightarrow \Delta, \heartsuit q} .$$

This rule does not allow to derive new sequents, since using the methods of Lemma 9 we can reconstruct the premisses of the two original rules with p_A substituted by A from the premisses of this rule. Thus we may replace an instance of this rule by the two original rules and an instance of cut. Moreover, since $A \in \mathcal{F}_\ell{}^{[i]}$ we may resolve the premiss $A \Rightarrow q$ again using Lemma 22. The case of antitone $\heartsuit$ is analogue. For n-ary modalities $\heartsuit$ the strategy is the same. □

The previous Lemmata yield general criteria as to which axioms are translatable into rules. For the sake of brevity we only state the result for unary monotone modalities; The generalisations to non-monotone modalities and higher arities are straightforward.

Definition 26. For $A \in \mathcal{F}_r{}^{[i]}$ and $p \in \mathcal{V}$ we say that p is *positive* (resp. *negative*) in A, if it occurs positively (resp. negatively) in a sequent of the decomposition of the sequent $\Rightarrow A$ according to Lemma 22.

Theorem 27. *Let A be a propositional formula with variables $p_1, \dots, p_n, q_1, \dots, q_m$, and for $i = 1, \dots, m$ let the modality $\heartsuit_i$ be unary monotone and A_i a propositional formula with variables in $p_1, \dots, p_n$ such that: q_i is only positive in A and $A_i \in \mathcal{F}_\ell{}^{[i]}$; or q_i is only negative in A and $A_i \in \mathcal{F}_r{}^{[i]}$; or $A \in \mathcal{F}_\ell{}^{[i]} \cap \mathcal{F}_r{}^{[i]}$. Then there is a rule which is equivalent in* G[i]CutCon *to the axiom* $\overline{\Rightarrow A\sigma}$ *where $\sigma(q_i) = \heartsuit_i(A_i)$ and $\sigma(p_i) = p_i$.*

Proof. First we use Lemma 22 to decompose the formula A into unique sequents $\Gamma_i \Rightarrow \Delta_i \in \mathcal{S}(p_1, \dots, p_n, q_1, \dots, q_m)$ for $i \in I$, such that adding the sequent $\Rightarrow A$ as an axiom is equivalent to adding all the sequents $\Gamma_i \Rightarrow \Delta_i$. If a q_j is only positive in A it never occurs in the Γ_i; if it is only negative, it never occurs in the Δ_i. Thus after substituting the A_j for the q_j we can apply Lemma 25 to resolve the premisses for the A_j. The propositional variables on the top level of the conclusion are moved into the premisses using Lemma 23. Finally, if a variable appears in the premisses but not in the conclusion we eliminate it using variable elimination (Definition 7). The technique used in the proof of Lemma 9 ensures that the resulting rule is equivalent to the original one. □

Remark 28. Since for classical propositional logic all propositional formulae are both right and left resolvable, the previous Theorem yields the translation result for non-nested axioms from [10] as a Corollary.

For axioms with nested modalities we may sometimes use a similar procedure if a modalised formula occurs both on the top level of the axiom and under a modality. The idea is to introduce a fresh variable for this formula and apply the methods above to resolve propositional logic under the modalities, but without moving the top level occurrences of the variable into the premisses with Lemma 23. If now the occurrences of this variable in the premisses and the conclusion are all negative (resp. positive), we may it replace it again with the original formula. Since this formula now occurs both in the premisses and the conclusion this often gives rise to a context restriction. We will illustrate this method using examples in the next section.

4 Applications

Example 29 (Constructive *K*). Constructive modal logic *K* from [4, 13] is based on intuitionistic propositional logic and has rules $Reg_{\Box} = (\{(p \Rightarrow q; C_{\emptyset})\}; \Box p \Rightarrow \Box q)$ and $Reg_{\Diamond} = (\{(p \Rightarrow q; C_{\emptyset})\}; \Diamond p \Rightarrow \Diamond q)$ and axioms $(FS1)$ $\Box\top$, $(FS2)$ $(\Box(p \wedge q) \to (\Box p \wedge \Box q)) \wedge ((\Box p \wedge \Box q) \to \Box(p \wedge q))$ and $(FS6)$ $\Diamond(p \to q) \to (\Box p \to \Diamond q)$. Since the propositional part is intuitionistic we base our treatment on the asymmetric setting. The rules $Reg_{\Box}$ and $Reg_{\Diamond}$ ensure that both modalities $\Box$ and $\Diamond$ are monotone in the sense of Definition 24. Treating modalised subformulae as variables and using Lemma 22 and the fact that $A \to B$ is intuitionistically left resolvable to break up axiom $(FS6)$ first yields the axiom $\overline{\Diamond(p \to q), \Box p \Rightarrow \Diamond q}$. Now introducing a new variable $r_{p\to q}$ and premisses for subformula $p \to q$ yields

$$\frac{r_{p\to q} \Rightarrow p \to q \qquad p \to q \Rightarrow r_{p\to q}}{\Diamond r_{p\to q}, \Box p \Rightarrow \Diamond q} .$$

But now a cut with rule $Reg_{\Diamond}$ on $\Diamond r_{p\to q}$ and resolving the remaining premiss gives the rule $\frac{s, p \Rightarrow q}{\Diamond s, \Box p \Rightarrow \Diamond q} R_{FS6}$. The analogous treatment for axioms $(FS2)$ and $(FS1)$ gives the well-known rules $R_{FS2} := (\{(p, q \Rightarrow r; C_{\emptyset})\}; \Box p, \Box q \Rightarrow \Box r)$ and $R_{FS1} := (\{(\Rightarrow p; C_{\emptyset})\}; \Rightarrow \Box p)$. Now saturating the rule set under cuts yields the rule set

$$\mathcal{R}_{CK} := \left\{ \frac{p_1, \dots, p_n \Rightarrow q}{\Gamma, \Box p_1, \dots, \Box p_n \Rightarrow \Box q} R_{CK_n} \mid n \geq 0 \right\} \cup \left\{ \frac{p_1, \dots, p_n, q \Rightarrow r}{\Gamma, \Box p_1, \dots, \Box p_n, \Diamond q \Rightarrow \Diamond r} \mid n \geq 0 \right\} .$$

Of course this rule set is not new [4]. The point here is that we constructed it in a purely syntactical way from the axioms of the Hilbert-system.

To illustrate the use of Lemma 23 let us add the T-axioms $(T\Box)$ $\Box p \to p$ and $(T\Diamond)$ $p \to \Diamond p$. Again, the axioms are first broken up into $\overline{\Box p \Rightarrow p}$ and $\overline{p \Rightarrow \Diamond p}$. Then they are transformed into equivalent rules $R_{T\Box} = (\{(p \Rightarrow ; C_{id})\}; \Box p \Rightarrow)$ and $R_{T\Diamond} = (\{(\Rightarrow p; C^{i}_{id})\}; \Rightarrow \Diamond p)$. Now saturation under cuts would yield the additional rules $(\{(p_1, \dots, p_n \Rightarrow ; C_{id})\}; \Box p_1, \dots, \Box p_n \Rightarrow)$ and $(\{(p_1, \dots, p_n \Rightarrow r; C^{i}_{id})\}; \Box p_1, \dots, \Box p_n \Rightarrow \Diamond r)$ for $n \geq 0$, but these are simulated by repeated applications of $R_{T\Box}$ and $R_{T\Diamond}$. Thus it is easy to see that the rule sets $\mathcal{R}_{CK}$ and $\mathcal{R}_{CK} \cup \{R_{T\Box}, R_{T\Diamond}\}$ are saturated.

Note that these constructions do not give rise to context restrictions apart from $C_{\emptyset}$ and $C^{[i]}_{id}$. For this we need to consider axioms with nested modalities. Unfortunately, if we are dealing with nested modalities the translation becomes more involved and much less automatic. Nonetheless, as mentioned before in some cases we can still use the method of cutting rules on principal formulae to construct rules with context restrictions. The main idea is to make use of formulae occurring both under a modality and on the top level of the sequent and to construct a context restriction out of this formula.

Example 30 (Constructive *S4*). Constructive modal logic *CS4* from [17, 2] contains the rules $\mathcal{R}_{CK} \cup \{R_{T\Box}, R_{T\Diamond}\}$ and additional axioms $(4\Box)$ $\Box p \to \Box\Box p$ and $(4\Diamond)$ $\Diamond\Diamond p \to \Diamond p$. We make use of the fact that in these we have the same modalised formula occurring on the top level and under a modality as follows: Take axiom $(4\Box)$ and in a first step replace the occurrence of $\Box p$ under the modality by a fresh variable q. The resulting axiom

$\Box p \to \Box q$ is broken up into the sequent $\Box p \Rightarrow \Box q$. Then adding the premisses for q we get $\frac{\Box p \Rightarrow q \quad q \Rightarrow \Box p}{\Box p \Rightarrow \Box q}$ and a cut with the monotonicity rule $Reg_{\Box}$ yields $\frac{\Box p \Rightarrow q}{\Box p \Rightarrow \Box q}$. Now computing principal cuts of a number of instances of this rule with rule R_{CK_n} yields

$$\frac{\Box p_1, \dots, \Box p_m, q_1, \dots, q_k \Rightarrow r}{\Gamma, \Box p_1, \dots, \Box p_m, \Box q_1, \dots, \Box q_k \Rightarrow \Box r} .$$

But since the $\Box p_i$ occur both in conclusion and premiss of the rule, this is exactly the rule $R_{4\Box} = (\{(q_1, \dots, q_k \Rightarrow r; C_{4\Box})\}; \Box q_1, \dots, \Box q_k \Rightarrow \Box r)$. Moreover, the rule $\frac{\Box p \Rightarrow q}{\Box p \Rightarrow \Box q}$ is sound by the methods of the last section, and since $R_{4\Box}$ was constructed from this rule and R_{CK_n} by means of principal cuts, Lemma 9 ensures that it is sound as well. A similar process for axiom ($4\Diamond$) yields the rules $R_{4\Diamond} := (\{(p_1, \dots, p_n, q \Rightarrow C_{4\Diamond})\}; \Box p_1, \dots, \Box p_n, \Diamond q \Rightarrow\ )$ with context restriction $C_{4\Diamond} = \langle \emptyset, \{\Diamond p\} \rangle$. Now adding the missing principal cuts again yields a rule set which is principal-cut closed. It is trivially context-cut closed, and easily checked to be mixed-cut and contraction closed and therefore saturated. Again, the rules are not new, but we constructed them in a purely syntactical way, and their soundness and completeness is guaranteed by construction.

This method can also be applied if the subformula occurs under a modality more than once, or if it is more complex. In the latter case in general this gives rise to more complex context restrictions. The following example shows how sometimes more complex restrictions can be simplified and how context restrictions $C_{id}^{[i]}$ may arise other than as a consequence of Lemma 23.

Example 31 (Access Control Logic *CDD*). Access control logic *CDD* from [1] is based on intuitionistic propositional logic and has indexed normal (and thus monotone) modalities $\bigcirc_k A$ which are interpreted as *principal k says A*. For this example we consider the axiomatisation with the axioms [unit] $p \to \bigcirc_k p$ and [GHO] $\bigcirc_k (p \to \bigcirc_k q) \to (p \to \bigcirc_k q)$ (see [1]). The first axiom straightforwardly translates into the rule $R_{[\mathsf{unit}]} = (\ \{(\ \Rightarrow p; C_{id}^{i})\};\ \Rightarrow \bigcirc_k p\ )$. For the latter axiom we first introduce a variable r for the formula $(p \to \bigcirc_k q)$ under the modality and apply the methods above to obtain

$$\frac{r \Rightarrow p \to \bigcirc_k q}{\Gamma, \bigcirc_k r \Rightarrow p \to \bigcirc_k q} .$$

But now instead of turning this into a rule with context restriction $\langle \emptyset, \{p \to \bigcirc_k q\} \rangle$ we break up the boolean part to arrive at $\frac{r, p \Rightarrow \bigcirc_k q}{\Gamma, \bigcirc_k r, p \Rightarrow \bigcirc_k q}$. Since disjunctions are intuitionistically left resolvable the variable p can be taken to be the context on the left hand side, and this is equivalent to the rule $R_{[\mathsf{GHO}]} = (\ \{(r \Rightarrow\ ; C_{cdd,k})\}; \bigcirc_k r \Rightarrow\ )$ with restriction $C_{cdd,k} := \langle \{p\}, \{\bigcirc_k p\} \rangle$. Since the rules R_{K_n} for $\bigcirc_k$ are simulated by applications of $R_{[\mathsf{unit}]}$ and $R_{[\mathsf{GHO}]}$, this yields the rule set $\mathcal{R}_{CDD} := \{R_{[\mathsf{unit}]}, R_{[\mathsf{GHO}]}\}$ which again is easily seen to be saturated and thus have cut elimination. Of course again this rule set is not new (see e.g. [7, 5]) and there are other ways to construct it, but it nicely illustrates how context restrictions arise.

Example 32 (Conditional Logic with Absoluteness). As a final example let us construct a cut-free set of rules with context restrictions for a logic based on classical propositional logic, namely for Lewis' conditional logic $\mathbb{VA}$ from [12]. The language for this logic contains the binary modality $\preccurlyeq$ called *comparative plausibility operator* with the intuitive reading "A is at least as plausible as B" for $A \preccurlyeq B$. The logic $\mathbb{VA}$ is given as an axiomatic extension of the logic $\mathbb{V}$, where the latter is characterised by non-nested axioms only and does not necessitate the use of rules with context restrictions. For this reason we concentrate on the new axioms and make use of the rules $\mathcal{R}_{\mathbb{V}} = \{R_{n,m} \mid n \geq 1, m \geq 0\}$ from [11] for the logic $\mathbb{V}$, where rule $R_{n,m}$ in our notation is given as $(\ \{(s_k \Rightarrow r_1, \ldots, r_n, q_1, \ldots, q_m; C_\emptyset) \mid k \leq n\} \cup \{(p_k \Rightarrow r_1, \ldots, r_n, q_1, \ldots, q_{k-1}; C_\emptyset) \mid k \leq m\};\ p_1 \preccurlyeq q_1, \ldots, p_m \preccurlyeq q_m \Rightarrow r_1 \preccurlyeq s_1, \ldots, r_n \preccurlyeq s_n\ )$.

For $\mathbb{VA}$ we need to add the two *absoluteness axioms* $(p \preccurlyeq q) \to (\bot \preccurlyeq \neg(p \preccurlyeq q))$ and $\neg(p \preccurlyeq q) \to (\bot \preccurlyeq (p \preccurlyeq q))$. We use the fact that the formula $(p \preccurlyeq q)$ occurs both on the top level and under a modality, and in a first step using monotonicity of $\preccurlyeq$ in the second argument convert the two axioms into $\dfrac{p \Rightarrow \quad q \Rightarrow r \preccurlyeq s}{\Rightarrow p \preccurlyeq q, r \preccurlyeq s}$ and $\dfrac{p \Rightarrow \quad r \preccurlyeq s, q \Rightarrow}{r \preccurlyeq s \Rightarrow p \preccurlyeq q}$. Now computing a principal cut between the first of these rules and a rule $R_{n,m}$ effectively replaces one negative principal formula of $R_{n,m}$ with a positive contextual formula $r \preccurlyeq s$. Repeating this process we get arbitrarily many positive context formulae $r_k \preccurlyeq s_k$ and thus arrive at the context restriction $\langle \emptyset, \{r \preccurlyeq s\}\rangle$. Similarly, principal cuts with the second rule replace negative principal formulae of $R_{n,m}$ with negative contextual formulae $r \preccurlyeq s$, yielding the context restriction $C_{\mathbb{VA}} := \langle\{r \preccurlyeq s\}, \{r \preccurlyeq s\}\rangle$. As usual, since all the cuts involved were cuts on principal formulae, Lemma 9 guarantees soundness of the resulting rule set. Setting $\mathcal{R}_{\mathbb{VA}} = \{R'_{n,m} \mid n \geq 1, m \geq 0\}$ with $R'_{n,m}$ given as $(\ \{(s_k \Rightarrow r_1, \ldots, r_n, q_1, \ldots, q_m; C_{\mathbb{VA}}) \mid k \leq n\} \cup \{(p_k \Rightarrow r_1, \ldots, r_n, q_1, \ldots, q_{k-1}; C_{\mathbb{VA}}) \mid k \leq m\};\ p_1 \preccurlyeq q_1, \ldots, p_m \preccurlyeq q_m \Rightarrow r_1 \preccurlyeq s_1, \ldots, r_n \preccurlyeq s_n\ )$ thus gives a sound and complete rule set for $\mathbb{VA}$, which is easily checked to be saturated and thus cut-free. As far as we are aware this rule set is new. This yields the following Theorem.

Theorem 33. *The rule set* $\mathsf{G}\mathcal{R}_{\mathbb{VA}}\mathsf{ConCut}$ *is sound and complete for* $\mathbb{VA}$. *Moreover, it is saturated and therefore has cut elimination. Since* $\mathcal{R}_{\mathbb{VA}}$ *is tractable, derivability in this system can be checked in* EXPTIME.

5 Conclusion

We presented a generic cut elimination result for symmetric and asymmetric sequent systems consisting of rules with context restrictions which are saturated, i.e. closed under cuts and contractions. This not only extends previous methods to modal axioms of nesting depth greater than one, but also to logics based on intuitionistic logic. Furthermore, we introduced techniques to translate axioms of a Hilbert style system into sequent rules. All the results and techniques are easily adapted to other base logics such as minimal or distributive logic. Examples included the reconstruction of already known sequent systems for constructive modal logics and the construction of an apparently new sequent system for Lewis' conditional logic $\mathbb{VA}$ in the entrenchment language.

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