Cut Elimination for Shallow Modal Logics*

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Abstract. Motivated by the fact that nearly all conditional logics are axiomatised by so-called shallow axioms (axioms with modal nesting depth ≤ 1) we investigate sequent calculi and cut elimination for modal logics of this type. We first provide a generic translation of shallow axioms to (one-sided, unlabelled) sequent rules. The resulting system is complete if we admit pseudo-analytic cut, i.e. cuts on modalised propositional combinations of subformulas, leading to a generic (but sub-optimal) decision procedure. In a next step, we show that, for finite sets of axioms, only a small number of cuts is needed between any two applications of modal rules. More precisely, completeness still holds if we restrict to cuts that form a tree of logarithmic height between any two modal rules. In other words, we obtain a small (PSPACE-computable) representation of an extended rule set for which cut elimination holds. In particular, this entails PSPACE decidability of the underlying logic if contraction is also admissible. This leads to (tight) PSPACE bounds for various conditional logics.

1 Introduction

Cut elimination is without doubt a central theme in proof theory. Not only do cut-free sequent systems provide for reasonably simple syntactical proofs of results like interpolation, they also pave the way for decision procedures via backwards proof search. While there are a variety of methods to construct a cut-free sequent system for specific logics (and at least as many different sequent calculi), the general approach is to come up with a sequent system tailored to the logic at hand, and then show cut elimination for this particular system. While this approach works very well for specific logics, a good deal of ingenuity is required to construct the actual system. Since this method consumes both a lot of time and effort, this raises the question whether there is a generic method to construct cut-free calculi, and in particular, whether we can delegate the task of constructing these systems. Our motivation for investigating this question mainly stems from automated proof search and questions of complexity, where the shape and structure of the rules of a cut-free system are not important, as long as we can recognise rule instances fast enough. Our ultimate aim in this somewhat radical endeavour is to synthesise algorithms that recognise instances of a cut-free sequent system, given an axiomatisation of the logic under consideration.

This paper reports on our first results on this programme in the context of modal logic: we study the question to what extent we can convert a Hilbert-style axiomatisation of a general, not necessarily normal modal logic into a cut-free sequent system such

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that rule instances are decidable in a moderate complexity class. Our point of departure is the class of logics that can be axiomatised by *shallow axioms*, i.e. axioms of modal nesting depth ≤ 1. These logics are known to be decidable by semantic arguments via coalgebraic semantics [10] and the finite model property, and (reassuringly) exclude modal logics that are known to be undecidable [6]. Indeed, one of the questions is to what extent the decidability of these logics can be reflected purely syntactically.

The motivating examples in this endeavour are the systems of conditional logics. While there is a plethora of systems [7], nearly all of these are axiomatised by shallow axioms. Recent activity in this area has led to methods for constructing (labelled) sequent systems for some of these systems [8], and to generic cut elimination proofs for unlabelled systems given by a set of rules [11]. We extend the latter approach by a generic method to construct rules of an unlabelled sequent system from a set of shallow axioms. For the system obtained in this way we show two main results, namely completeness and decidability of the system, where the cut rule is replaced by the pseudo-analytic cut rule (a variant of the analytic cut rule), and full cut elimination for the system extended by a tractable set of rules. The latter result breathes the spirit of our radical approach driven by proof search. The crucial fact is that the extended rule set is generically constructed and has a small (polysize) representation. We also show that admissibility of the contraction rule in the extended system implies a PSPACE decidability result for the corresponding logic. While this still leaves the question whether generic positive results concerning admissibility of contraction hold, we apply our method successfully to various conditional logics.

Related work Criteria for cut elimination are discussed for instance in [12] for a wide class of logics, but not touching upon the automatic construction of rules or calculi that admit cut elimination. Cut elimination for canonical calculi (where each sequent rule only allows the introduction of one logical connective) are discussed in [1]. This approach in general is unsuitable for modal calculi, since these typically introduce more than one connective at a time. Algorithmic aspects of cut elimination are investigated in [2] but with a focus on deciding whether a calculus enjoys this property, in contrast to the main aspect of this paper which aims to construct a calculus that enjoys cut elimination algorithmically. The present paper is a continuation of work reported in [11] that gives criteria and a semi-algorithmic method to obtain calculi admitting cut elimination, and our focus here is to obtain these calculi purely algorithmically.

2 Preliminaries and Notation

Throughout the paper, we consider a *modal similarity type* Λ consisting of modal operators with arities and a denumerable set V of propositional variables. Given Λ, the set of Λ-*formulas* is given by the grammar

$$\mathcal{F}(\Lambda) \ni \phi, \psi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \heartsuit(\phi_1, \dots, \phi_n)$$

where $p \in V$ and $\heartsuit \in \Lambda$ is n-ary. We employ a classical reading of the propositional part of the language and use the standard abbreviations for other propositional connectives. The *modal rank* of a formula is given inductively by $\mathsf{rk}\,(p) = 0$, $\mathsf{rk}\,(\neg\phi) = \mathsf{rk}\,(\phi)$,

$\mathsf{rk}\,(\phi_1 \wedge \phi_2) = \max_{i=1,2} \mathsf{rk}\,(\phi_i)$ and $\mathsf{rk}\,(\heartsuit(\phi_1, \dots, \phi_n)) = 1 + \max_{1 \le i \le n} \mathsf{rk}\,(\phi_i)$. If $\sigma : V \to \mathcal{F}(\Lambda)$ is a substitution, we write $\phi\sigma$ for the result of replacing every occurrence of p in ϕ by $\sigma(p)$ and $[\phi/p]$ is the substitution defined by $[\phi/p](q) = \phi$ if $p = q$ and $[\phi/p](q) = q$ otherwise. We denote the propositional variables that occur in a formula ϕ by $\mathsf{Var}(\phi)$. If $S \subseteq \mathcal{F}(\Lambda)$ is a set of formulas, we write $\Lambda(S)$ for the set $\{\heartsuit(\phi_1, \dots, \phi_n) \mid \heartsuit \in \Lambda\ n\text{-ary}, \phi_1, \dots, \phi_n \in S\}$ of formulas that arise by applying precisely one modality $\heartsuit \in \Lambda$ to formulas in S, and $\neg S$ is the set $\{\neg\phi \mid \phi \in S\}$ of negations of formulas in S. Similarly, $\mathsf{Prop}(S)$ is the set of propositional combinations of formulas in S. A *clause* over S is a finite disjunction $l_1 \vee \dots \vee l_n$ of literals $l_i \in S \cup \neg S$ $(i = 1, \dots, l)$. If $l \in S \cup \neg S$, then $\sim l$ is the *normalised negation* of l, given by $\sim l = \neg l$ if $l \in S$ and $\sim l = l'$ if $l = \neg l' \in \neg S$. Two formulas $\phi, \psi \in \mathcal{F}(\Lambda)$ are *propositionally equivalent* if $\phi \leftrightarrow \psi$ is a substitution instance of a propositional tautology. To make contraction explicit, we take a *Λ-sequent* to be a finite multiset of Λ-formulas. If $S \subseteq \mathcal{F}(\Lambda)$ is a set of formulas, we write $\mathcal{S}(S)$ for the set of sequents containing only elements in S and $\mathcal{S}(\Lambda)$ for $\mathcal{S}(\mathcal{F}(\Lambda))$. The number of elements of $\Gamma \in \mathcal{S}(\Lambda)$ counting multiplicities is written as $\|\Gamma\|$. We employ usual notation and identify a formula $\phi \in \mathcal{F}(\Lambda)$ with the singleton sequent ϕ and write Γ, Δ for the (multiset) union of sequents $\Gamma, \Delta \in \mathcal{S}(\Lambda)$. If Γ is a Λ-sequent, $\mathsf{Supp}\,(\Gamma)$ denotes the *support* of Γ, i.e. the set of Λ-formulas that occur in Γ with positive multiplicity. Substitution extends to sequents pointwise (preserving multiplicity), that is, $\Gamma\sigma = \phi_1\sigma, \dots, \phi_n\sigma$ if $\Gamma = \phi_1, \dots, \phi_n$. A sequent $\Gamma \in \mathcal{S}(\Lambda)$ is *propositionally equivalent* to a formula $\phi \in \mathcal{F}(\Lambda)$ if $\bigvee \Gamma \leftrightarrow \phi$ is a propositional tautology. A set $\{\Gamma_1, \dots, \Gamma_n\}$ of sequents is a *conjunctive normal form* (cnf) of a formula $\phi \in \mathcal{F}(\Lambda)$ if ϕ and $(\bigvee \Gamma_1) \wedge \dots \wedge (\bigvee \Gamma_n)$ are propositionally equivalent. If $\phi, \psi \in \mathcal{F}(\Lambda)$, we use the shorthand $\phi = \psi$ to denote the set of sequents containing $\neg\phi, \psi$ and $\phi, \neg\psi$. This convention is extended to chains of equations $\phi_1 = \dots = \phi_n$ in the obvious way.

3 From Hilbert Systems to Sequent Systems

Our starting point in this paper is a modal logic axiomatised by shallow axioms (axioms with modal rank ≤ 1) in a Hilbert system that we convert to a set of sequent rules, taking special care of propositional formulas occurring in the scope of a modality.

Definition 1. A *shallow axiom* over a similarity type Λ is a formula $\phi \in \mathcal{F}(\Lambda)$ with $\mathsf{rk}\,(\phi) \le 1$. A *shallow clause* is of the form $c = c_p \vee c_d$ where c_p is a clause over V and c_d is a clause over $\Lambda(\mathsf{Prop}(V))$. A *decomposition* of a shallow clause c is a triple (c_p, c_d, σ) where c_p, c_d are clauses as above with $\mathsf{Var}(c_d) \cap \mathsf{Var}(c_p) = \emptyset$, and $\sigma : V \to \mathsf{Prop}(V)$ is a substitution with $c = c_p \vee c_d\sigma$.

Insisting that a modal logic is axiomatised purely in terms of shallow axioms clearly excludes a large variety of logics (the most basic example is the modal logic K extended with the transitivity axiom $\Box p \to \Box\Box p$). On the other hand, nearly all conditional logics studied in the literature are axiomatised using shallow axioms [7]. Technically, the restriction to (finitely many) shallow axioms implies that all logics under consideration are in fact decidable, a property that fails for logics that are axiomatised by more general classes of axioms [6].

Example 2. 1. Over the similarity type $\Lambda = \{\Box\}$, the axioms defining the modal logic (K), i.e. $\Box p \wedge \Box q \to \Box(p \wedge q)$ and $\Box\top$ as well as the reflexivity axiom $\Box p \to p$ are shallow. Transitivity $\Box\Box p \to \Box p$ fails to be shallow.

2. The syntax of many conditional logics is given by the similarity type $\Lambda = \{>\}$ where $>$ is a binary operator that we write in infix notation. All of the axioms

$$
\begin{array}{rlrl}
(\mathrm{CM}) & (p > (q \wedge r)) \to (p > q) & (\mathrm{CC}) & (p > q) \wedge (p > r) \to (p > (q \wedge r)) \\
(\mathrm{CS}) & (p \wedge q) \to (p > q) & (\mathrm{CA}) & (p > r) \wedge (q > r) \to ((p \vee q) > r) \\
(\mathrm{MP}) & (p > q) \to (p \to q) & (\mathrm{CMon}) & (p > q) \wedge (p > r) \to ((p \wedge q) > r) \\
(\mathrm{ID}) & (p > p) & (\mathrm{CV}) & (p > q) \wedge \neg(p > \neg r) \to ((p \wedge r) > q)
\end{array}
$$

that define e.g. the conditional systems $B = \{\mathrm{CM}, \mathrm{CC}, \mathrm{CA}, \mathrm{CMon}, \mathrm{ID}\}$, $\mathrm{SS} = B \cup \{\mathrm{CS}, \mathrm{MP}\}$, and $\mathrm{V} = B \cup \{\mathrm{CV}\}$ are shallow [7].

We define modal Hilbert systems in the standard way by closing under modus ponens, uniform substitution and the modal congruence rule. This allows us e.g. to derive the necessitation rule $p/\Box p$ for $\Box$ from the axiom $\Box\top$.

Definition 3. Suppose $\mathcal{A} \subseteq \mathcal{F}(\Lambda)$. The predicate $\mathcal{HA} \vdash$ is the least subset of formulas containing $\mathcal{A}$ and all propositional tautologies that is closed under uniform substitution ($\mathcal{HA} \vdash \phi\sigma$ if $\mathcal{HA} \vdash \phi$), modus ponens ($\mathcal{HA} \vdash \psi$ if $\mathcal{HA} \vdash \phi \to \psi$ and $\mathcal{HA} \vdash \phi$) and congruence ($\mathcal{HA} \vdash \heartsuit(\phi_1, \dots, \phi_n) \leftrightarrow \heartsuit(\psi_1, \dots, \psi_n)$ if $\mathcal{HA} \vdash \phi_i \leftrightarrow \psi_i$ for all $i = 1, \dots, n$).

Given a set of shallow axioms, we now construct an equivalent sequent system that extends propositional logic with *shallow rules*. As we are working in a generic setup, it is more convenient to have negation as an explicit logical operator rather than dealing with formulas in negation normal form as the latter would require that the similarity type Λ is closed under formal duals. Consequently our analysis is based on the system G consisting of all rule instances

$$
\frac{}{\Gamma, p, \neg p} \qquad \frac{}{\Gamma, \neg\bot} \qquad \frac{\Gamma, \neg\phi, \neg\psi}{\Gamma, \neg(\phi \wedge \psi)} \qquad \frac{\Gamma, \phi \quad \Gamma, \psi}{\Gamma, \phi \wedge \psi} \qquad \frac{\Gamma, \phi}{\Gamma, \neg\neg\phi}
$$

where $p \in V$ is a propositional variable, $\phi, \psi \in \mathcal{F}(\Lambda)$ are formulas and $\Gamma \in \mathcal{S}(\Lambda)$ is a sequent. Here, Γ is the *context* and a formula that appears in the conclusion but not the context is called *principal*. The system G is complete for classical propositional logic [14]. Extensions of G with weakening, cut, context-sensitive cut and contraction

$$
(\mathsf{W})\frac{\Gamma}{\Gamma, \phi} \qquad (\mathsf{Cut})\frac{\Gamma, \phi \quad \Delta, \neg\phi}{\Gamma, \Delta} \qquad (\mathsf{Cut_{cs}})\frac{\Gamma, \phi \quad \Gamma, \neg\phi}{\Gamma} \qquad (\mathsf{Con})\frac{\Gamma, \phi, \phi}{\Gamma, \phi}
$$

are denoted by suffixing with the respective rule names so that e.g. GWCon is the system G extended with weakening and contraction. We write $\Omega \vdash_{\mathsf{G}} \Delta$ if Δ can be derived in G from premises in Ω and we use the same notation for extensions of G with a subset of $\{\mathsf{W}, \mathsf{Con}, \mathsf{Cut}, \mathsf{Cut_{cs}}\}$. A sequent Γ is a *propositional consequence* of sequents in Ω if $\Omega \vdash_{\mathsf{GCutCon}} \Gamma$, this is also denoted by $\Omega \vdash_{\mathsf{PL}} \Gamma$. Shallow axioms are incorporated into these systems by converting them into sequent rules of a specific form:

Definition 4. A *shallow rule* is given by a triple $R = (\mathsf{Prem}_c(R), \mathsf{Prem}_n(R), \Sigma)$ consisting of a finite set $\mathsf{Prem}_c(R) = \{\Gamma_1, \dots, \Gamma_l\} \subseteq \mathcal{S}(V \cup \neg V)$ of *contextual premises*, a finite set $\mathsf{Prem}_n(R) = \{\Delta_1, \dots, \Delta_m\} \subseteq \mathcal{S}(V \cup \neg V)$ of *non-contextual premises* and a sequent $\Sigma \in \mathcal{S}(\Lambda(V) \cup \neg\Lambda(V))$ of *principal formulas* where all variables that occur in Σ are pairwise distinct. If $\sigma : V \to \mathcal{F}(\Lambda)$ is a substitution and $\Gamma \in \mathcal{S}(\Lambda)$ is a sequent (the *context*), then

$$(R\sigma)\ \frac{\Gamma, \Gamma_1\sigma, \Sigma\sigma \quad \dots \quad \Gamma, \Gamma_l\sigma, \Sigma\sigma \quad \Delta_1\sigma \quad \dots \quad \Delta_m\sigma}{\Gamma, \Sigma\sigma}$$

is an *instance* of R. If no confusion between contextual and non-contextual premises can arise, we write a shallow rule given by the above data in the more suggestive form

$$(R)\ \frac{\Gamma_1, \Gamma, \Sigma \dots \Gamma_l, \Gamma, \Sigma \quad \Delta_1 \dots \Delta_m}{\Gamma, \Sigma}.$$

The *principal formulas* of a shallow rule R (or rule instance $R\sigma$) of the form above are the (substituted) elements of Σ, written as $\mathsf{PF}(R)$ (resp. $\mathsf{PF}(R\sigma)$). We write $\mathsf{Prem}(R)$ (resp. $\mathsf{Prem}(R\sigma)$) for the set of (substituted) premises of R, and $\mathsf{Concl}(R\sigma)$ for the conclusion of $(R\sigma)$. We identify shallow rules modulo injective renaming of variables.

The requirement that the variables in the principal formulas are pairwise distinct poses no restriction, since we may introduce fresh variables and new premises stating equivalences. The separation between contextual and non-contextual premises is important for two reasons: first, when passing from rules to instances, the contextual premises not only copy the context from premise to conclusion, but also the principal formulas. This is important for admissibility of contraction, as it allows to propagate a contraction between principal formulas and context. *Mutatis mutandis*, it is precisely this mechanism that allows to show admissibility of contraction in a sequent calculus for the modal logic T, i.e. K extended with the rule $\Gamma, \neg\phi, \neg\Box\phi / \Gamma, \neg\Box\phi$. Second, contextual premises receive special treatment in proof search, as the premise is a superset of the conclusion.

Example 5. Over the similarity types introduced in Section 2, we can form the following shallow rules, which we present in the suggestive notation of Definition 4.

1. Over $\Lambda = \{\Box\}$, both $(\mathcal{R}_{\mathrm{K}})\ \frac{\neg p, \neg q, r \quad \neg r, p \quad \neg r, q}{\Gamma, \neg\Box p, \neg\Box q, \Box r}$ and $(\mathcal{R}_{\mathrm{T}})\ \frac{\Gamma, \neg p, \neg\Box p}{\Gamma, \neg\Box p}$ are shallow. Here the premises in $\mathcal{R}_K$ are non-contextual whereas the premise in $\mathcal{R}_T$ is contextual.
2. Over $\Lambda = \{>\}$, both $(\mathcal{R}_{\mathrm{CC}})\ \frac{p_1 = p_2 = p_3 \quad \neg q, \neg r, s \quad \neg s, q \quad \neg s, r}{\Gamma, \neg(p_1 > q), \neg(p_2 > r), (p_3 > s)}$ and $(\mathcal{R}_{\mathrm{CS}})\ \frac{\Gamma, p \quad \Gamma, q}{\Gamma, (p > q)}$ are shallow.

Every set $\mathcal{R}$ of shallow rules induces a sequent calculus by augmenting instances of rules in $\mathcal{R}$ with the modal congruence rule and propositional reasoning.

Definition 6. Suppose $\mathcal{R}$ is a set of shallow rules. The predicate $\mathsf{G}\mathcal{R} \vdash$ is the least set of sequents closed under the propositional rules of G, instances of shallow rules in $\mathcal{R}$, and instances of the modal congruence rules

$$\frac{\neg\phi_1, \psi_1 \quad \neg\psi_1, \phi_1 \quad \dots \quad \neg\phi_n, \psi_n \quad \neg\psi_n, \phi_n}{\Gamma, \neg\heartsuit(\phi_1, \dots, \phi_n), \heartsuit(\psi_1, \dots, \psi_n)}$$

where $\Gamma \in \mathcal{S}(\Lambda)$, $\heartsuit \in \Lambda$ is n-ary and $\phi_1, \dots, \phi_n \in \mathcal{F}(\Lambda)$. Use of additional rules is indicated by suffixing so that e.g. $\mathsf{G}\mathcal{R}\mathsf{WCut}_{\mathsf{cs}} \vdash$ denotes derivability in $\mathsf{G}\mathcal{R}$ extended with weakening and context-sensitive cut.

We employ the usual definitions of a proof (a tree constructed from proof rules), the depth of a proof (the height of this tree) and (depth-preserving) admissibility of proof rules [14]. Often, a statement holds for extensions of $\mathsf{G}\mathcal{R}$ with several principles. We indicate this using square brackets. For example, a statement involving $\mathsf{G}\mathcal{R}[\mathsf{WCon}]$ holds for an extension of $\mathsf{G}\mathcal{R}$ with a (possibly empty) subset of $\{\mathsf{W}, \mathsf{Con}\}$.

Lemma 7 (Admissibility of weakening and inversion). *Suppose $\mathcal{R}$ is a set of shallow rules over a similarity type Λ. Then the weakening rule* (W) *and the rules*

$$\frac{\Gamma, \phi \wedge \psi}{\Gamma, \phi} \qquad \frac{\Gamma, \phi \wedge \psi}{\Gamma, \psi} \qquad \frac{\Gamma, \neg(\phi \wedge \psi)}{\Gamma, \neg\phi, \neg\psi} \qquad \frac{\Gamma, \neg\neg\phi}{\Gamma, \phi}$$

are depth-preserving admissible in $\mathsf{G}\mathcal{R}[\mathsf{Cut}\mathsf{Cut}_{\mathsf{cs}}\mathsf{WCon}]$.

Proof. Standard by induction on the depth of the proof and the fact that the rules in $\mathcal{R}$ introduce only modalised formulas or their negations. □

Our next goal is to convert shallow axioms into shallow rules and confirm that (for now, with help of cut and contraction) this does not change the notion of derivability.

Definition 8. Suppose that c is a shallow clause with decomposition (c_p, c_d, σ) where $\mathsf{Var}(c_d) = \{q_1, \dots, q_n\}$ and $c_p = l_1 \vee \dots \vee l_m$. If furthermore

- the sequents $\Delta_1, \dots, \Delta_k$ are a cnf of $\bigwedge_{i=1}^n (q_i \leftrightarrow \sigma(q_i))$
- the sequent $\Sigma \subseteq \Lambda(V) \cup \neg\Lambda(V)$ is propositionally equivalent to c_d

then the shallow rule

$$\frac{\Gamma, \sim l_1 \quad \dots \quad \Gamma, \sim l_m \quad \Delta_1 \quad \dots \quad \Delta_k}{\Gamma, \Sigma}$$

is called a *rule form* of c. A *rule form* of a shallow axiom ϕ is a set $R = \{r_1, \dots, r_k\}$ of shallow rules where each r_i is a rule form of a shallow clause c_i such that $\bigwedge_{i=1}^n c_i$ and ϕ are propositionally equivalent. Finally, a rule form of a set $\mathcal{A} = \{\phi_1, \dots, \phi_n\}$ of shallow axioms is a set $\mathcal{R} = \mathcal{R}_1 \cup \dots \cup \mathcal{R}_n$ where each $\mathcal{R}_i$ is a rule form of ϕ_i.

In other words, a rule form of a shallow axiom ϕ is constructed by first converting ϕ into conjunctive normal form, obtaining shallow clauses $c_1, \dots, c_n$. For each shallow clause, we obtain a rule by replacing propositional formulas ϕ_i that occur as arguments of modal operators by new variables q_i and then add the clauses of a conjunctive normal form of $q_i \leftrightarrow \phi_i$ to the premises. The operation of adding a context amounts to considering a shallow clause $c = c_p \vee c_m$ as an implication $\neg c_m \to c_p$ that induces a rule $c_p \to \phi / \neg c_m \to \phi$ which is then interpreted as a sequent rule.

Example 9. The rules $\mathcal{R}_K$, $\mathcal{R}_T$, $\mathcal{R}_{\mathrm{CC}}$ and $\mathcal{R}_{\mathrm{CS}}$ presented in Example 5 are rule forms of the homonymous axioms introduced in Example 2.

As a first sanity check, we confirm that the Hilbert calculus given by a set of shallow axioms is equivalent to the sequent calculus given by their rule forms, at least as long as we admit cut and contraction in the latter.

Proposition 10. *Suppose that $\mathcal{A}$ is a set of shallow axioms and $\mathcal{R}$ is a rule form of $\mathcal{A}$.*

1. $\mathsf{G}\mathcal{R}\mathsf{CutCon} \vdash \phi$ *for every* $\phi \in \mathcal{A}$.

2. $\mathcal{HA} \vdash \bigvee \Gamma_0$ *whenever* $\mathcal{HA} \vdash \bigvee \Gamma_i$ *(all* $1 \leq i \leq n$*) and* $\Gamma_1 \dots \Gamma_n / \Gamma_0$ *is an instance of a shallow rule in* $\mathcal{R}$.

Proof. 1. W.l.o.g. every axiom in $\mathcal{A}$ is a shallow clause. Let $c \in \mathcal{A}$ with $c = l_1 \vee \dots \vee l_m \vee \sigma(q_1) \vee \dots \vee \sigma(q_n)$. Taking $l_1, \dots, l_m$ for the context Γ in the notation of Definition 8, the sequents $\Gamma, \sim l_i$ are instances of the propositional axiom rules, and the sequents $\Delta_i[\sigma(q_1) \dots \sigma(q_n)/q_1 \dots q_n]$ are derivable in Prop. Applying the rule form for c and the disjunction rule gives c.

2. If R is a rule form of a shallow clause c, and $\Gamma_1 \dots \Gamma_n / \Gamma_0$ is an instance of R with $\mathcal{HA} \vdash \bigvee \Gamma_i$ $(1 \leq i \leq n)$, then $\bigvee \Gamma_0$ is derived in $\mathcal{HA}$ from the appropriate substitution instance of c using propositional reasoning and congruence. □

In the presence of cut and contraction, equivalence of both systems is then immediate:

Corollary 11. *Suppose that $\mathcal{A}$ is a set of shallow axioms and $\mathcal{R}$ is a rule form of $\mathcal{A}$. Then* $\mathcal{HA} \vdash \bigvee \Gamma$ *whenever* $\mathsf{G}\mathcal{R}\mathsf{CutCon} \vdash \Gamma$*, for all sequents* $\Gamma \in \mathcal{S}(\Lambda)$.

Clearly, our goal is the elimination of both cut and contraction where the latter can (at least in the first instance) be handled on the basis of rule forms.

Definition 12. A set $\mathcal{R}$ of shallow rules is *contraction closed*, if, for every rule instance $(R\sigma)$ with $\mathsf{Concl}\,(R\sigma) = \Gamma, \phi, \phi$ there exists an instance $(S\tau)$ with $\mathsf{Concl}\,(S\tau) = \Gamma, \phi$ such that $\mathsf{Prem}_c\,(R\sigma) \vdash_{\mathsf{GWCon}} \Delta$ for all $\Delta \in \mathsf{Prem}_c\,(S\tau)$, and $\mathsf{Prem}_n\,(R\sigma) \vdash_{\mathsf{GWCon}} \Pi$ for all $\Pi \in \mathsf{Prem}_n\,(S\tau)$.

This definition allows us to propagate contraction over the application of modal rules. Combined with an induction on the depth of the derivation this yields:

Proposition 13. *Suppose that $\mathcal{R}$ is contraction closed. Then* $\mathsf{G}\mathcal{R}[\mathsf{Cut_{cs}}] \vdash \Gamma$ *iff* $\mathsf{G}\mathcal{R}\mathsf{Con}[\mathsf{Cut_{cs}}] \vdash \Gamma$ *for all sequents* $\Gamma \in \mathcal{S}(\Lambda)$*. Moreover, the proof in* $\mathsf{G}\mathcal{R}[\mathsf{Cut_{cs}}]$ *has at most the same height, uses the same number of (instances of) shallow rules and the same cut formulas.*

Proof. By double induction on the modal rank of the contraction formula and the depth of the proof. It is standard to propagate contraction over propositional rules using the inversion lemma without increasing the number of modal rules. In an application of an instance of a shallow rule, contractions involving the conclusion only can be eliminated by contraction closure of $\mathcal{R}$ and the fact that formulas which occur in the premises but not in the context have strictly lower modal rank than principal formulas. Contractions between conclusion and context can be propagated as the conclusion explicitly appears in all contextual premises (Definition 4). □

4 Cut-closure And Pseudo-Analytic Cut

We now set out to establish the first main result of this paper, and show that the cut rule can be restricted to *pseudo-analytic* cut, i.e. cuts on formulas that arise by applying a modal operator to a propositional combination of subformulas of the conclusion of the cut rule, which leads to a generic decidability result for logics axiomatised with shallow rules. To achieve this, we first normalise shallow rules so that only variables occurring in the conclusion are allowed in the premise. This way backwards proof search does not introduce new variables. In a second step, we close a normalised rule set under cuts between rule conclusions, and observe that this closure process can be simulated with pseudo-analytic cut. We first analyse the process of eliminating unnecessary variables.

Definition 14. A set $\mathcal{R}$ of shallow rules is *normalised* if in each rule in $\mathcal{R}$ all variables occurring in the premises also occur in the conclusion.

Superfluous variables in the premises of rules are eliminated as follows.

Definition 15 (p-elimination). Let S be a set of sequents and $p \in V$ a propositional variable. The *p-elimination* of S, written S_p is defined by

$$S_p = \{\Gamma \ominus p, \Delta \ominus \neg p \in N \mid \Gamma, p \in S \text{ and } \Delta, \neg p \in S\} \cup \{\Delta \in S \cap N \mid \{p, \neg p\} \cap \Delta = \emptyset\}$$

where $\Gamma \ominus \phi$ denotes the sequent Γ with all occurrences of ϕ removed (in the multiset sense) and $N = \{\Gamma \in \mathcal{S}(\Lambda) \mid \Gamma \cap \neg\Gamma = \emptyset\}$ is the set of non-axiomatic sequents over Λ. If $\boldsymbol{p} = (p_1, \dots, p_n)$ is a finite sequence of variables, we write $S_{\boldsymbol{p}} = (\dots (S_{p_1}) \dots)_{p_n}$.

In other words, S_p contains all results of multicutting elements of S on p that are not trivially derivable. The next lemma shows that S_p is propositionally equivalent to S.

Lemma 16. *Suppose $S \subseteq \mathcal{S}(\Lambda(V))$ is a finite set of sequents over $\Lambda(V)$ and $p \in V$. Then all $\Delta \in S_p$ are derivable from S in* GCutCon *with cuts only on p. Moreover, there exists a formula $\phi = \phi(S, p)$ such that $\Gamma[\phi/p]$ is derivable from S_p in* G *for each $\Gamma \in S$. The formula ϕ can be chosen as a conjunction of disjunctions of sequents in S_p.*

Proof. The first statement follows directly from the definition of $\mathcal{S}_p$. Now let us construct the formula $\phi(S, p)$. It basically states that all the sequents $\Delta \ominus \neg p$ of the above definition of S_p hold. More formally, let Υ be the set of sequents $\Delta \ominus \neg p$, such that $\Delta \in S$ and there is a $\Gamma \in S$ with $\Gamma \ominus p, \Delta \ominus \neg p \in S_p$. Then

$$\phi(S, p) := \bigwedge_{\Delta \in \Upsilon} \bigvee (\Delta \ominus \neg p) .$$

Now let $\Gamma \in S$ and $\Delta \in S$, where $\Gamma \ominus p = \Gamma'$, $\Delta \ominus \neg p = \Delta'$. Then we have $\vdash_{\mathsf{G}} \Delta' \vee \neg\phi(S, p)$, and thus $\vdash_{\mathsf{G}} \Delta[\phi(S, p)/p]$. On the other hand, since Γ', Δ' is in S_p or an axiom, we get $S_p \vdash_{\mathsf{G}} \Gamma', \phi(S, p)$ and therefore also $S_p \vdash_{\mathsf{G}} \Gamma[\phi(S, p)/p]$. □

Rules with unnecessary variables in the premises can therefore be normalised by successively eliminating these variables.

Example 17. If S contains the sequents $p = t$ and $q \wedge r = s$, then S_r consists of $p = t$ and $\neg s, q$. We may therefore replace the rule form of the axiom $(\mathrm{CM})(p > (q \wedge r)) \rightarrow (p > q)$ on the left

$$\frac{p = t \quad \neg q, \neg r, s \quad \neg s, r \quad \neg s, q}{\Gamma, \neg(p > s), (t > q)} \qquad \mathcal{R}_{\mathrm{CM}} \frac{p = t \quad \neg s, q}{\Gamma, \neg(p > s), (t > q)}$$

with its r-eliminated version (shown on the right).

Lemma 16 allows us to replace shallow rules with their normalised version, and we will assume from now on that all shallow rules are normalised. We now construct a cut-closed set $\mathsf{cc}(\mathcal{R})$ from a set $\mathcal{R}$ of shallow rules: we consider two shallow rules together with an application of cut to their conclusions as a rule in its own right, but eliminate all variables that occur in the premises, but not in the conclusion, of (new) rules that arise in this way. This process then takes the following form:

Definition 18. Let $R_1, R_2 \in \mathcal{R}$ be given by $(\Omega_c, \Omega_n, \Gamma)$ and $(\Upsilon_c, \Upsilon_n, \Delta)$, respectively and suppose that σ, τ are renamings such that $\Gamma\sigma = \Gamma', M$ and $\Delta\tau = \Delta', \neg M$. Then $\mathsf{cut}(R_1\sigma, R_2\tau, M)$ is the shallow rule given by $((\Upsilon_c \cup \Upsilon_n \cup \Omega_c \cup \Omega_n)_{\boldsymbol{p}}, (\Upsilon_n \cup \Omega_n)_{\boldsymbol{p}}, \Gamma', \Delta')$ if $M = \heartsuit\boldsymbol{p}$ for $\boldsymbol{p} = (p_1, \ldots, p_n)$.

This definition ensures that the new (non-) contextual premises arise from the old (non-contextual) premises by removing variables that no longer occur in the conclusion.

Example 19. For the rules $(R_{\mathrm{CC}}) = \frac{p_1 = p_2 = p \quad \neg q_1, \neg q_2, q \quad \neg q, q_1 \quad \neg q, q_2}{\Gamma, \neg(p_1 > q_1), \neg(p_2 > q_2), (p > q)}$ and $(R_{\mathrm{CM}}) = \frac{p = r \quad \neg q, s}{\Gamma, \neg(p > q), (r > s)}$ from $\mathcal{R}_{(CM)}$ we obtain the rule

$$(\mathrm{CC_m}) = \mathsf{cut}(R_{\mathrm{CC}}, R_{\mathrm{CM}}, (p > q)) = \frac{p_1 = p_2 = r \quad \neg q_1, \neg q_2, s}{\Gamma, \neg(p_1 > q_1), \neg(p_2 > q_2), (r > s)} .$$

The *cut closure* of a rule set is then constructed by adding more and more (normalised) cuts until the set is saturated. Formally we have:

Definition 20. Let $\mathcal{R}$ be a set of shallow rules. The *cut closure of* $\mathcal{R}$ is the $\subseteq$-minimal set $\mathsf{cc}(\mathcal{R})$ with $\mathcal{R} \subseteq \mathsf{cc}(\mathcal{R})$, such that for every $R_1, R_2 \in \mathsf{cc}(\mathcal{R})$ and renamings σ, τ with $\mathsf{Concl}(R_1\sigma) = \Gamma, M$ and $\mathsf{Concl}(R_2\tau) = \Delta, \neg M$ we have $\mathsf{cut}(R_1\sigma, R_2\tau, M) \in \mathsf{cc}(\mathcal{R})$.

Not surprisingly, cut is admissible over the cut closure of a rule set.

Proposition 21. $\mathsf{G}\mathcal{R}\mathsf{CutCon} \vdash \Gamma$ *iff* $\mathsf{G\,cc}(\mathcal{R})\,\mathsf{Con} \vdash \Gamma$ *for all sequents* Γ.

Proof. For the left to right direction it suffices to show the statement for *multicuts*

$$\frac{\Gamma, \phi^n \quad \Delta, (\neg\phi)^m}{\Gamma, \Delta}$$

(with $n, m \in \mathbb{N}$) instead of cuts. This is done by double induction on the size of the cut formula and the sum of the sizes of the proofs of the premises, where contraction and the rules in $\mathsf{cc}(\mathcal{R})$ cater for multicuts with cut formula principal in the conclusions of rules in $\mathcal{R}$. For the other direction every instance of a rule in $\mathsf{cc}(\mathcal{R}) \setminus \mathcal{R}$ is replaced by the corresponding tree of cuts with instances of rules in $\mathcal{R}$ at the leaves. By Lemma 16 the premises of these rules can be derived from the premises of the rule in $\mathsf{cc}(\mathcal{R})$. $\square$

In general, we may restrict cuts to formulas that arise as $\phi = \phi(p, S)$ in Lemma 16, i.e. conjunctions of disjunctions, which allows the following restriction on the cut rule:

Definition 22. A *pseudo-analytic cut* is a cut $\dfrac{\Gamma, \varphi \quad \Delta, \neg\varphi}{\Gamma, \Delta}$, where $\varphi = \heartsuit(\psi_1, \ldots, \psi_n)$, and for $1 \leq i \leq n$ each ψ_i is a conjunction of disjunctions of formulas occurring possibly negated under a modal operator in Γ, Δ. For a set $\mathcal{S}$ of sequence rules define $\mathcal{S}\mathsf{Cut}_{\mathsf{pa}}$ to be the set $\mathcal{S}$ together with the cut rule restricted to pseudo-analytic cuts.

Corollary 23. $\mathsf{G}\mathcal{R}\mathsf{CutCon} \vdash \Gamma$ *iff* $\mathsf{G}\mathcal{R}\mathsf{Cut}_{\mathsf{pa}}\mathsf{Con} \vdash \Gamma$ *for all sequents* Γ*.*

Distributivity allows us to restrict to a single layer of conjunctions and disjunctions.

Lemma 24. *If* $\mathsf{G}\mathcal{R}\mathsf{Cut}_{\mathsf{pa}}\mathsf{Con} \vdash \Gamma$ *for a set* $\mathcal{R}$ *of shallow rules and a sequent* Γ*, then* $\mathsf{G}\mathcal{R}\mathsf{CutCon} \vdash \Gamma$ *with cuts only on modalised conjunctions of disjunctions of possibly negated subformulas of* Γ*.*

Proof. First note that $(\phi \vee \psi) \wedge \chi \equiv_{\mathsf{GWCon}} (\phi \wedge \chi) \vee (\psi \wedge \chi)$ and $(\phi \wedge \psi) \vee \chi \equiv_{\mathsf{GWCon}} (\phi \vee \chi) \wedge (\psi \vee \chi)$. Given a proof $\mathcal{D}$ of Γ in $\mathsf{G}\mathcal{R}\mathsf{Cut}_{\mathsf{pa}}\mathsf{Con}$ we transform it bottom up into a proof with cuts only on modalised conjunctions of disjunctions of (possibly negated) subformulas of Γ by following a branch in $\mathcal{D}$ until we reach the first (pseudo-analytic) cut on a formula not of this format. Then the cut formula τ must be a modalised conjunction of disjunction of formulas ψ, where each ψ is a (possibly negated) subformula of Γ or a conjunction of disjunctions of (possibly negated) subformulas of Γ. The formula τ either was introduced in the context of an axiom, in which case we may replace it with a formula of the right format, or its modality was introduced by a shallow rule. In this case we use the distributivity laws to transform the formulas occurring under the modality in τ in the premises of the shallow rule into conjunctions of disjunctions of (possibly negated) subformulas of Γ. □

As pseudo-analytic cuts suffice, for a conclusion of a cut rule there are only finitely many possible cut formulas. In order to get a generic decidability result, we need to assume that the rule set is tractable in the following sense.

Definition 25 (from [13]). A set $\mathcal{R}$ of shallow rules is PSPACE-*tractable*, if there are multivalued functions f taking sequents to sets of encodings of instances of rules in $\mathcal{R}$, and g, taking encodings of rule instances to sets of sequents, such that for all sequents Γ, Δ and encodings $\ulcorner R\sigma \urcorner$ of a rule instance we have $\ulcorner R\sigma \urcorner \in f(\Gamma) \iff \mathsf{Concl}\,(R\sigma) = \Gamma$ and $\Delta \in g(\ulcorner R\sigma \urcorner) \iff \Delta \in \mathsf{Prem}\,(R\sigma)$, and whose graphs are decidable in space polynomial in the length of the first argument.

We assume that sequents are encoded as lists of formulas. Note that the length of the encoding of a sequent is at least the number of formulas in the sequent.

Theorem 26. *Let* $\mathcal{R}$ *be a* PSPACE-*tractable and contraction closed set of shallow rules. Then the derivability problem for* $\mathsf{G}\mathcal{R}\mathsf{ConCut}$ *is in* 3EXPTIME*.*

Proof. Applying Lemma 24, and using the context-sensitive cut rule and the fact that contraction is admissible in the resulting system we get that if Γ is derivable, then every sequent occurring in the proof is fully contracted and contains only subformulas of Γ

or formulas of the form specified in Lemma 24. Since the set $\mathcal{S}$ of sequents of this form has size $2^{2^{2^{\mathcal{O}(w)}}}$, where w is the number of subformulas of Γ, and since $\mathcal{R}$ is PSPACE-tractable, the subset of $\mathcal{S}$ consisting of the sequents derivable with G$\mathcal{R}$ and the specified cuts can be computed in 3EXPTIME. □

Example 27. This theorem induces a uniform decidability proof (albeit with a suboptimal complexity bound) for all logics axiomatised by finitely many shallow axioms, e.g. for the conditional logics B, SS and V of Example 2.

5 Cut Elimination Using Small Representations

In the previous section, we have constructed the cut-closure of a given set of shallow rules, and we have argued that a sequent calculus using this set enjoys cut elimination. However, the construction of the cut closure does not yield a concrete representation of a cut-closed rule set. The main result of this section establishes that the rules constituting a cut-closed set can always be represented in space polynomial in the rule conclusion. In particular, we demonstrate that instances of cut-closed rule sets can be decided in PSPACE. This entails that the corresponding derivability problem is decidable in polynomial space. Technically, we show that rules of a cut-closed rule set are represented by proof trees whose inner nodes are applications of cut, and we give explicit bounds on the size of these trees, which yield polynomial representability.

Definition 28. A shallow rule $R_1 = (\Omega_c, \Omega_n, \Sigma)$ *subsumes* a shallow rule $R_2 = (\Xi_c, \Xi_n, \Pi)$, if there is a renaming σ with $\Sigma\sigma = \Pi$ such that $\Xi_c \cup \Xi_n \vdash_{\mathsf{PL}} \Delta\sigma$ for every $\Delta \in \Omega_c$, and $\Xi_n \vdash_{\mathsf{PL}} \Upsilon\sigma$ for every $\Upsilon \in \Omega_n$. Two shallow rules are *equivalent* if they mutually subsume each other.

While the pseudo-analytic cut yields decidability, there is room for improvement in complexity by considering polynomial-size representations of $\mathsf{cc}\,(\mathcal{R})$.

Definition 29. Let $\mathcal{R}$ be a set of shallow rules. An $\mathcal{R}$*-cut tree* with conclusion Γ and leafs $(R_i\sigma)$ (where $1 \leq i \leq n$, $R_i \in \mathcal{R}$ and $\sigma_i : V \to V$ is a renaming) is a proof of Γ from the conclusions of the $(R_i\sigma)$ using only cuts on principal formulas of the $R_i\sigma$. The number of nodes in a cut tree is denoted by $\mathsf{size}\,(\mathcal{D})$, its height by $\mathsf{depth}\,(\mathcal{D})$.

In the above definition, we emphasise that *only* applications of cut are allowed in a cut tree, and the cut formulas have to be principal formulas of the rules at the leafs.

Example 30. The following is a KT-cut-tree for the sequent $u, \neg\Box p, \neg\Box q, \neg\Box r$:

$$\dfrac{\dfrac{\neg p, \neg q, s \quad \neg s, p \quad \neg s, q}{\neg\Box p, \neg\Box q, \Box s}\,(\mathcal{R}_{\mathrm{K}}) \qquad \dfrac{\dfrac{\neg s, \neg r, t \quad \neg t, s \quad \neg t, r}{\neg\Box s, \neg\Box r, \Box t}\,(\mathcal{R}_{\mathrm{K}}) \qquad \dfrac{u, \neg t}{u, \neg\Box t}\,(\mathcal{R}_{\mathrm{T}})}{u, \neg\Box s, \neg\Box r}}{u, \neg\Box p, \neg\Box q, \neg\Box r}$$

Clearly, the cuts introduced in a cut tree may introduce new variables that are present in the premises of the $R_i\sigma$, but not in the conclusion Γ. We eliminate these as before.

Definition 31. Let $\mathcal{R}$ be a set of shallow rules, and $\mathcal{D}$ an $\mathcal{R}$-cut-tree. The shallow rule $r(\mathcal{D})$ *represented by* $\mathcal{D}$ is the leaf of $\mathcal{D}$ if $\mathsf{depth}\,(\mathcal{D}) = 0$. If $\mathsf{depth}\,(\mathcal{D}) > 0$, then $\mathcal{D}$ is of the form $\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{\Gamma}$, where Γ arises from the conclusions of $\mathcal{D}_1$ and $\mathcal{D}_2$ by a cut on M. In this case, $r(\mathcal{D}) = \mathsf{cut}(r(\mathcal{D}_1), r(\mathcal{D}_2), M)$ where $r(\mathcal{D}_1)$ and $r(\mathcal{D}_2)$ are the rules represented by $\mathcal{D}_1$ and $\mathcal{D}_2$.

Equivalence of cut trees and cut closure is clear from the definitions:

Lemma 32. *A shallow rule lies in* $\mathsf{cc}\,(\mathcal{R})$ *iff it is represented by an* $\mathcal{R}$*-cut tree.*

Proof. Straightforward from the definitions of cut closure and represented rules. ☐

Application of Lemma 16 shows that cut trees differing only in the order of the cuts represent basically the same rule instance, a fact that we record here for later use.

Lemma 33. *Let* $\mathcal{R}$ *be a set of shallow rules. Let* Γ *be a sequent and let* $\mathcal{D}_1, \mathcal{D}_2$ *be* $\mathcal{R}$*-cut-trees with conclusion* Γ *and leafs* $R_1\sigma_1, \dots, R_n\sigma_n$*. Then the rules represented by* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are equivalent.*

Proof. By Lemma 16 we get $\mathsf{Prem}\,(r(\mathcal{D}_1)) \vdash_{PL} \Delta[\phi_1, \dots, \phi_k / p_1, \dots, p_k]$ for every $\Delta \in \mathsf{Prem}\,(R_i\sigma_i)\,,\ (1 \leq i \leq n)$ and suitable ϕ_i. The premises of $r(\mathcal{D}_2)$ now follow by the appropriate applications of cut. ☐

The main difficulty that we have to overcome in order to obtain small representations of cut-closed rules lies in the fact that the number of literals in either premise of an application of cut may both increase and decrease as we move up a cut tree. This non-monotonic behaviour disappears if we only consider cuts involving sequents consisting of at least three elements. This suffices for our purpose, since we can absorb cuts involving smaller sequents into the rule set at very little extra cost.

Definition 34. A shallow rule is *small* if it has at most two principal formulas. A set $\mathcal{R}$ of shallow rules is *2-cut closed* if for every two rules $R_1, R_2 \in \mathcal{R}$ with conclusions Σ_1 and Σ_2, such that R_1 or R_2 is small, and any two renamings $\sigma_1, \sigma_2 : V \to V$ for which $\Sigma_1\sigma_1 = \Gamma, M$ and $\Sigma_2\sigma_2 = \Delta, \neg M$ there exists a rule $R \in \mathcal{R}$ that subsumes $\mathsf{cut}(R_1\sigma_1, R_2\sigma_2, M)$. The *2-cut closure* $2\mathsf{cc}\,(\mathcal{R})$ of a set $\mathcal{R}$ of shallow rules is the $\subseteq$-minimal, 2-cut closed set of shallow rules containing $\mathcal{R}$.

Example 35. The rule set CK containing $(\mathcal{R}_{\mathrm{CM}})$, $(\mathcal{R}_{\mathrm{CC}})$ and $(\mathrm{CC_m})$ is 2-cut closed, but not cut closed.

Passing from a finite set of shallow rules to its 2-cut closure is a preprocessing step that adds finitely many missing rules. Crucially, computing a 2-cut closure is independent of the size of any sequent to which proof search is applied and therefore adds a constant time overhead. The most important ramification of 2-cut closure is the existence of small representations of elements in the cut closure of a given set of shallow rules. We approach this result by means of a sequence of lemmas, the first one establishing that we may always assume that leafs of a cut tree are labelled with 'large' rules.

Lemma 36. *Let* $\mathcal{R}$ *be a 2-cut closed set of shallow rules, and let* $\mathcal{D}$ *be an* $\mathcal{R}$*-cut-tree with conclusion* Γ *and leafs* $R_1, \sigma_1, \dots, R_n\sigma_n$*. Then there exists an* $\mathcal{R}$*-cut-tree* $\mathcal{D}'$ *with conclusion* Γ *and leafs* $R'_1\sigma'_1, \dots, R'_k, \sigma'_k$ *such that*

1. *if* $\|\Gamma\| \leq 2$ *then* $\mathcal{D}'$ *has depth* 0 *(and therefore consists of a single leaf* $R'_1\sigma'$ *only)*
2. *if* $\|\Gamma\| > 2$ *then* $R'_i\sigma'$ *have at least 3 principal formulas each*
3. $\mathsf{size}\,(\mathcal{D}') \leq \mathsf{size}\,(\mathcal{D})$ *and the rules represented by* $\mathcal{D}$ *and* $\mathcal{D}'$ *are equivalent.*

Proof. By induction on the size of the cut-tree. The base case is trivial. Suppose the size of $\mathcal{D}$ is $n+1$, and that the premises of the lowermost cut are Γ_1, M and $\Gamma_2, \neg M$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the induced $\mathcal{R}$-cut-trees with conclusions Γ_1, M respectively $\Gamma_2, \neg M$. Using the induction hypothesis we get $\mathcal{R}$-cut-trees $\mathcal{D}'_1$ and $\mathcal{D}'_2$ with conclusions Γ_1, M and $\Gamma_2, \neg M$, which have the desired properties. If both $r(\mathcal{D}'_1)$ and $r(\mathcal{D}'_2)$ have at least three principal formulas, then we are done. Otherwise, let w.l.o.g. $\|\,\mathsf{PF}\,(r(\mathcal{D}'_1))\,\| \leq 2$. Then $\mathcal{D}'_1$ consists only of a leaf with conclusion Γ_1, M. By permuting the cut on M up to the appropriate leaf in $\mathcal{D}'_2$, and the fact, that $\mathcal{R}$ is 2-cut closed, we get a cut-tree $\mathcal{D}'$ with the desired properties. Lemma 33 together with the definition of 2-cut closed rule sets ensures that the represented rules are equivalent. □

Since cuts between sequents of length at least three increase the length of the sequent, the size of the cut-tree is bounded in terms of the conclusion of the represented rules.

Corollary 37. *Let* $\mathcal{R}$ *be a 2-cut closed set of shallow rules, and let* Γ *be a sequent with* $\|\Gamma\| \geq 3$. *Then every rule in* $\mathsf{cc}\,(\mathcal{R})$ *with conclusion* Γ *is represented by an* $\mathcal{R}$*-cut tree of size* $\leq 2\|\Gamma\| - 5$.

A bound on the depth of a cut tree is obtained from the following adaption of the 2-3-Lemma of [5]. Here for a tree $\mathcal{T}$ and a node x in $\mathcal{T}$ the subtree of $\mathcal{T}$ generated by x is denoted by $\mathcal{T}_x$, and the number of nodes in $\mathcal{T}$ by $|\mathcal{T}|$.

Lemma 38. *Let* $k \in \mathbb{N}$ *and* $\mathcal{T}$ *be a tree, such that* $k{+}1 < |\mathcal{T}|$ *and each node has at most* k *children. Then there is a node* x *in* $\mathcal{T}$*, such that* $\left\lceil \frac{1}{k+2} \cdot |\mathcal{T}| \right\rceil \leq |\mathcal{T}_x| \leq \left\lfloor \frac{k+1}{k+2} \cdot |\mathcal{T}| \right\rfloor$.

Proof. We construct a series $(x_0, x_1, \ldots, x_d)$ of nodes in $\mathcal{T}$, such that x_0 is the root, and x_d is a leaf in the following way. Let x_0 be the root. For $i \geq 0$ and x_i not a leaf let x_{i+1} be a child of x_i, such that $|\mathcal{T}_{x_{i+1}}|$ is maximal. Since x_i has at most k children we have $|\mathcal{T}_{x_i}| \leq k \cdot |\mathcal{T}_{x_{i+1}}| + 1$. Now let

$$i_0 := \min \left\{ i \in [d] \mid |\mathcal{T}_{x_i}| < \frac{1}{k+2} \cdot |\mathcal{T}| \right\}.$$

Then $x_{i_0 - 1}$ is the desired node. Indeed we have

$$|\mathcal{T}_{x_{i_0-1}}| \;\leq\; k \cdot |\mathcal{T}_{x_{i_0}}| + 1 \;<\; k \cdot \frac{1}{k+2}|\mathcal{T}| + 1 \;\leq\; \frac{k}{k+2}|\mathcal{T}| + \frac{1}{k+2}|\mathcal{T}|\;,$$

which yields the upper bound. The lower bound follows by minimality of i_0. □

Lemma 39. *Let* $\mathcal{R}$ *be a 2-cut closed set of shallow rules where the every rule has at most* k *principal formulas, and let* Γ *be a sequent with* $\|\Gamma\| \geq 3$. *Then every instance of a rule in* $\mathsf{cc}\,(\mathcal{R})$ *with conclusion* Γ *can be represented by an* $\mathcal{R}$*-cut-tree of size at most* $2\|\Gamma\| - 5$ *and depth at most* $c_k \cdot \log_2 \|\Gamma\| + k$ *for* $c_k = (\log_2 \frac{k+2}{k+1})^{-1}$.

Proof. By Lemma 37 we get an $\mathcal{R}$-cut-tree $\mathcal{D}$ with conclusion Γ with at most $\|\Gamma\| - 2$ leaves. Suppose that its depth exceeds $c_k \cdot \log_2 \|\Gamma\| + k$. The main idea is to represent the cuts of $\mathcal{D}$ in a (k-ary) graph, whose nodes are the leafs of $\mathcal{D}$, and where two nodes are connected by an edge iff there is a cut between principal formulas of the rules associated with the leafs corresponding to the nodes. Then we may use Lemma 38 to split the graph in two parts, and take the cut corresponding to the split edge to be the lowermost cut in the newly constructed cut-tree. Continuing upwards in this fashion gives a cut-tree with the desired size and depth. □

Crucially, his bound ensures a small size of the cut-tree and the premises of the represented rule. This provides us with a tractable representation of the cut closure of $\mathcal{R}$.

Definition 40. Let $\mathcal{R}$ be a set of shallow rules with at most k principal formulas each. The *rule set generated by* $\mathcal{R}$ is the set $\mathcal{R}^*$ of rules represented by $2\mathsf{cc}\,(\mathcal{R})$-cut-trees with conclusion Γ and depth at most $c_k \cdot \log_2 \|\Gamma\| + k$.

Theorem 41. *Let $\mathcal{R}$ be a finite set of shallow rules. Then $\mathcal{R}^*$ is* PSPACE*-tractable and cut elimination holds in* $\mathsf{G}\mathcal{R}\mathsf{Con}$*, i.e.* $\mathsf{G}\mathcal{R}\mathsf{CutCon} \vdash \Gamma$ *iff* $\mathsf{G}\mathcal{R}^*\mathsf{Con} \vdash \Gamma$.

Proof. W.l.o.g. $\mathcal{R}$ is 2-cut closed. The equivalence of $\mathsf{G}\mathcal{R}\mathsf{CutCon}$ and $\mathsf{G}\mathcal{R}^*\mathsf{Con}$ follows from Lemma 39 and Proposition 21. We take the encodings of instances of rules in $\mathcal{R}^*$ to be the representing small cut trees, and f to be the function mapping a sequent Γ to the set of $\mathcal{R}$-cut trees for Γ with depth at most $c_k \cdot \log_2 \|\Gamma\| + k$. Since the cut-trees are small enough, the graph of f is in PSPACE. Furthermore, since the cut-trees have depth logarithmic in $\|\Gamma\|$, each of the premises of the represented rules is constructed by cutting at most $\left(2^\ell\right)^{c_k \cdot \log \|\Gamma\| + k} = \|\Gamma\|^{\ell \cdot c_k} \cdot 2^{\ell \cdot k}$ many premises of rule instances at the leaves, where ℓ is the maximal arity of modalities. Thus the graph of the function g, which takes each cut tree to the set of premises of the rule represented by the cut tree, is in PSPACE. □

6 Proof Search in G$\mathcal{R}^*$

In the previous section, we have seen that cut can be eliminated by passing from a given rule set to its cut closure. The polynomial representability of the latter does not yet guarantee that proof search can be accomplished in polynomial space, as instances of shallow rules propagate the conclusion to contextual premises. In this section, we introduce histories (in the spirit of [4]) that avoid infinite branches during proof search.

Definition 42. Let $\mathcal{R}$ be a set of shallow rules. An $\mathcal{R}$-*history* is a multiset h with $\mathsf{Supp}\,(h) \subseteq \{(R, \sigma) \mid R \in \mathcal{R},\ \sigma : \mathsf{Var}(R) \to \mathcal{F}(\Lambda)\}$ consisting of rule/substitution pairs. A *sequent with history* is a pair (h, Γ), written as $h \mid \Gamma$ where h is a $\mathcal{R}$-history. We write $h, (R, \sigma)$ for the (multiset) union of h and $\{(R, \sigma)\}$.

The notion of $\mathcal{R}$-histories extends to equivalence classes of rules modulo injective renamings in the obvious way. Histories are used to prevent shallow rules from being applied repeatedly to the same formulas in the system $G\mathcal{R}_2$ introduced next. The system $G\mathcal{R}_1$ is an intermediate system, which only keeps track of the rules.

Definition 43. Let $\mathcal{R}$ be a set of shallow rules. The system $\mathsf{G}\mathcal{R}_1$ consists of the propositional rules extended with history

$$\frac{}{h \mid \Gamma, p, \neg p} \quad \frac{}{h \mid \Gamma, \neg\bot} \quad \frac{h \mid \Gamma, \neg\phi, \neg\psi}{h \mid \Gamma, \neg(\phi \wedge \psi)} \quad \frac{h \mid \Gamma, \phi \quad h \mid \Gamma, \psi}{h \mid \Gamma, \phi \wedge \psi} \quad \frac{h \mid \Gamma, \phi}{h \mid \Gamma, \neg\neg\phi}$$

and all instances-with-history

$$(R\sigma)\ \frac{h, (R, \sigma) \mid \Gamma, \Gamma_1\sigma, \Sigma\sigma \quad \ldots \quad h, (R, \sigma) \mid \Gamma, \Gamma_n\sigma, \Sigma\sigma \quad \emptyset \mid \Delta_1\sigma \quad \ldots \quad \emptyset \mid \Delta_k\sigma}{h \mid \Gamma, \Sigma\sigma}$$

of shallow rules $R \in \mathcal{R}$ with contextual premises $\Gamma_1, \ldots, \Gamma_n$, non-contextual premises $\Delta_1, \ldots, \Delta_k$ and principal formulas Σ. In $\mathsf{G}\mathcal{R}_2$, instances-with-history above are subject to the side condition $(R, \sigma) \notin h$.

Since propositional rules do not interfere with histories, it is easy to see that admissibility of weakening, contraction and inversion carries over to $G\mathcal{R}_1$.

Lemma 44 (Admissibility of Weakening and inversion). *For every $\varphi, \psi \in \mathcal{F}(\Lambda)$, sequent Σ, and $\mathcal{R}$-history h the rule instances*

$$\frac{h \mid \Sigma, \neg\neg\varphi}{h \mid \Sigma, \varphi} \quad \frac{h \mid \Sigma, \neg(\varphi \wedge \psi)}{h \mid \Sigma, \neg\varphi, \neg\psi} \quad \frac{h \mid \Sigma, (\varphi \wedge \psi)}{h \mid \Sigma, \varphi} \quad \frac{h \mid \Sigma, (\varphi \wedge \psi)}{h \mid \Sigma, \psi} \quad \frac{h \mid \Gamma}{h, (R, \sigma) \mid \Gamma, \Delta}$$

are depth-preserving admissible in $G\mathcal{R}_1$. Moreover, the number of instances of shallow rules in the proof is preserved.

Proof. As for the system $G\mathcal{R}$. □

Lemma 45 (Admissibility of Contraction). *Let $\mathcal{R}$ be a contraction closed set of shallow rules. Then all instances of*

$$\frac{h, (R, \sigma) \mid \Gamma}{h \mid \Gamma} \qquad \frac{h \mid \Gamma, \phi, \phi}{h \mid \Gamma, \phi}$$

are admissible in $\mathsf{G}\mathcal{R}_1$ preserving the number of shallow rules in a proof.

Proof. As before. □

This gives the equivalency of $G\mathcal{R}\mathsf{Con}$ and $G\mathcal{R}_1$.

Lemma 46. *Let $\mathcal{R}$ be a set of shallow rules and Γ a sequent.*

1. $\mathsf{G}\mathcal{R}_1 \vdash \emptyset \mid \Gamma$ *iff there is a history h such that $G\mathcal{R}_1 \vdash h \mid \Gamma$.*
2. *if $\mathcal{R}$ is contraction closed, then* $\mathsf{G}\mathcal{R}\mathsf{Con} \vdash \Gamma$ *iff* $\mathsf{G}\mathcal{R}_1 \vdash \emptyset \mid \Gamma$.

Proof. 1. By induction on the height of the proof, using Lemma 44.

2. Induction on the height of the proof using 1. shows that $\mathsf{G}\mathcal{R} \vdash \Gamma$ iff $\mathsf{G}\mathcal{R}_1 \vdash \emptyset \mid \Gamma$. Admissibility of Contraction in $\mathsf{G}\mathcal{R}$ (Proposition 13) yields the statement. □

In fact, subsequent applications of a shallow rule to the same formulas in a branch of a proof in $G\mathcal{R}_1$ can be eliminated. This gives us equivalency with $G\mathcal{R}_2$.

Lemma 47. *Let $\mathcal{R}$ be a set of shallow rules and Γ a sequent. If $\mathsf{G}\mathcal{R}_1 \vdash h \mid \Gamma$, then $\mathsf{G}\mathcal{R}_2 \vdash h \mid \Gamma$. Moreover, there exists a proof of $h \mid \Gamma$ in $\mathsf{G}\mathcal{R}_2$ where every contextual premise of an application of a shallow rule contains a formula not in the conclusion.*

Proof. By induction on the number of applications of shallow rules in a proof $\mathcal{D}$ of $h \mid \Gamma$ in $G\mathcal{R}_1$. In the inductive step the induction hypothesis is first applied to the premises of the lowermost rule (w.l.o.g. this is an instance of a shallow rule R). If the resulting proof is not a proof in $G\mathcal{R}_2$ or if the additional property does not hold, then R has been applied to the same tuple of formulas twice. Then a proof with the desired properties can be constructed by skipping the upper application of R, using admissibility of contraction, and applying the induction hypothesis again. □

The fact that contextual premises of applications of shallow rules are bigger than the conclusion ensures that the search space in backwards proof search for $G\mathcal{R}_2$ is of depth polynomial in the number of subformulas of the root sequent. Summing up we get:

Theorem 48. *Let $\mathcal{R}$ be a contraction closed set of shallow rules.*

1. *For every sequent Γ we have $G\mathcal{R}\mathsf{Con} \vdash \Gamma$ iff $G\mathcal{R}_2 \vdash \emptyset \mid \Gamma$.*
2. *For* PSPACE*-tractable $\mathcal{R}$, derivability in $G\mathcal{R}_2$ is in* PSPACE.

Proof. 1. Immediate from Lemmas 46 and 47

2. It is easy to see that the rules of $G\mathcal{R}_2$ are PSPACE-tractable as well. For a sequent Γ let $w(\Gamma)$ be the *weight* of Γ, that is the number of subformulas in Γ. Furthermore, let $\mathsf{rk}\,(\Gamma) := \max\{\mathsf{rk}\,(\phi) \mid \phi \in \Gamma\}$. By Lemma 47, for backwards proof search it suffices to consider branches with at most $w(\Gamma)$ many consecutive applications of shallow rules. For $n \in \mathbb{N}$ and a sequent with history $h \mid \Gamma$ define the function f by

$$f(h \mid \Gamma, n) \ := \ (\ \mathsf{rk}\,(\Gamma)\ ,\ n - \|h\|,\ w(\Gamma)\ )\ .$$

Then for every branch $(h^i \mid \Gamma^i)_{i \in I}$ with $h^0 = \emptyset$ and $\Gamma^0 = \Gamma$ in the search tree created by the (modified) backwards proof search algorithm the sequence $(f(h^i \mid \Gamma^i, w(\Gamma)))_{i \in I}$ is strictly decreasing under the lexicographic ordering on $\mathbb{N}^3$. Since the initial value is at most $(w(\Gamma), w(\Gamma), w(\Gamma))$, the search tree has depth at most $w(\Gamma)^3$. □

Together with the results of the previous section this gives the following main theorem:

Theorem 49. *Let $\mathcal{A}$ be a finite set of shallow axioms and $\mathcal{R}$ be a 2-cut-closed rule form of $\mathcal{A}$. If $\mathcal{R}^*$ is contraction closed, then derivability in $\mathcal{HA}$ is in* PSPACE.

Proof. Immediate from Corollary 11 and Theorems 41 and 48. □

Clearly the requirement of $\mathcal{R}^*$ being contraction closed presents a gaping hole in our treatment so far. However, we can establish this property for several examples.

7 Applications: Exemplary Complexity Bounds

Using the machinery of the previous sections, proving PSPACE-bounds for shallow logics boils down to proving admissibility of contraction in the rule set generated by the rules corresponding to the axioms. In Example 35 we have seen that the set

$$CK = \left\{ \begin{array}{l} \dfrac{p_1 = p_2 = p \quad q_1 \wedge q_2 = q}{\Gamma, \neg(p_1 > q_1), \neg(p_2 > q_2), (p > q)}, \dfrac{p = r \quad \neg q, s}{\Gamma, \neg(p > q), (r > s)}, \\ \dfrac{p_1 = p_2 = p \quad \neg q_1, \neg q_2, q}{\Gamma, \neg(p_1 > q_1), \neg(p_2 > q_2), (p > q)} \end{array} \right\}$$

is 2-cut closed. It is clear that it is also contraction closed. This also holds for CK^*:

Lemma 50. *The set CK^* is contraction closed.*

Proof. The rules in CK^* are of the forms

$$\frac{p_1 = \cdots = p_n = p \quad \bigwedge_{i=1}^n q_i = q}{\Gamma, \neg(p_1 > q_1), \ldots, \neg(p_n > q_n), (p > q)} \text{ and } \frac{p_1 = \cdots = p_n = p \quad \bigwedge_{i=1}^n q_i \to q}{\Gamma, \neg(p_1 > q_1), \ldots, \neg(p_n > q_n), (p > q)}$$

for $n \geq 1$. These are easily seen to be contraction closed. □

As another example consider the axiom $CEM = (p > q) \vee (p > \neg q)$ of conditional excluded middle. Turning this into a rule yields

$$(CEM) \, \frac{p_1 = p_2 \quad \neg q_1, \neg q_2 \quad q_1, q_2}{\Gamma, (p_1 > q_1), (p_2 > q_2)} \; .$$

Let $CKCEM := \mathsf{2cc}\,(CK \cup \{CEM\})$. A little computation shows

Lemma 51. *The set $CKCEM^*$ is closed under contraction.*

Proof. Let $CKCEMm$ be the set

$$\left\{ \begin{array}{ll} \dfrac{p_1 = p_2 = p_3 \quad \neg q_1, q_2, q_3}{\Gamma, \neg(p_1 > q_1), (p_2 > q_2), (p_3 > q_3)}, & \dfrac{p_1 = p_2 \quad q_1, q_2}{\Gamma, (p_1 > q_1), (p_2 > q_2)}, \\ \dfrac{p_1 = p_2 = p_3 \quad q_1, q_2, q_3}{\Gamma, (p_1 > q_1), (p_2 > q_2), (p_3 > q_3)}, & \dfrac{q}{\Gamma, (p > q)} \end{array} \right\}$$

As can be seen by an easy induction the rules in $CKCEMm^*$ all have the form

$$\frac{p_1 = \cdots = p_k = p_{k+1} = \cdots = p_{k+m} \quad \neg q_1, \ldots, \neg q_k, q_{k+1}, \ldots, q_{k+m}}{\Gamma, \neg(p_1 > q_1), \ldots, \neg(p_k > q_k), (p_{k+1} > q_{k+1}), \ldots, (p_{k+m} > q_{k+m})}$$

for some natural numbers $k \geq 0, m \geq 1$. Thus $CKCEMm^*$ clearly is contraction closed. Note that the rules in $CKCEMm$ subsume the rules in $CKCEM$. Given a rule $R \in CKCEM^*$ and a renaming σ, replacing the rules at the leaves of the $CKCEM$-cut-tree representing $R\sigma$ with their monotone versions yields a $CKCEMm$-cut-tree, whose represented rule subsumes R. Using contraction closure of $CKCEMm$ and $CKCEMm \subset CKCEM$ we get the desired rule in $CKCEM^*$. □

In order to add more axioms to CK and $CKCEM$ we need to reconcile the definitions of cut and contraction closed rule sets. This can be done by restricting the rule format.

Definition 52. A shallow rule has *complete premises*, if every variable occurring in a principal formula occurs in every premise.

By soundness of the rules of PL and reasoning about propositional valuations it can be seen that for these rules the two definitions are compatible:

Lemma 53. *Let A be a finite set of variables, and let $\Gamma_1, \dots, \Gamma_n, \Gamma$ be sequents over $A \cup \neg A$ with every variable occurring in every sequent. Then $\{\Gamma_1, \dots, \Gamma_n\} \vdash_{\mathsf{PL}} \Gamma$ iff $\{\Gamma_1, \dots, \Gamma_n\} \vdash_{\mathsf{GWCon}} \Gamma$.*

This allows us to add rules with at most one principal formula to a rule set without destroying contraction closure.

Theorem 54. *Let $\mathcal{R}$ be a finite set of shallow rules with complete premises, and let R be a shallow rule with complete premises and one principal formula. If $\mathcal{R}^*$ is contraction closed, then there is a* PSPACE*-tractable set $\mathcal{Q}$ of shallow rules, such that for every sequent Γ we have $\mathsf{G}(\mathcal{R} \cup \{R\})\mathsf{CutCon} \vdash \Gamma$ iff $\mathsf{G}\mathcal{Q} \vdash \Gamma$.*

Proof. The rule set $\mathcal{Q}$ is the rule set generated by the 2-cut closure of $\mathcal{R} \cup \{R\}$, where all the rules have complete premises. The representations again are the small cut-trees. It is easy to see that $\mathcal{Q}$ is PSPACE-tractable, cut closed, and equivalent to $\mathcal{R} \cup \{R\}$. Contraction closure follows from the fact that in a cut-tree the principal formula of a rule with only one principal formula does not occur in the conclusion. □

As a special case, this means that we may add shallow rules with one literal in the conclusion to the sets CK and $CKCEM$, and still retain the PSPACE bound.

Theorem 55. *Let $\mathcal{A} \subseteq \{\mathrm{CEM}, \mathrm{ID}, \mathrm{MP}, \mathrm{CS}\}$. Then the logic $CK + \mathcal{A}$ is in* PSPACE*.*

Proof. The axioms translate into the rules

$$(\mathrm{ID})\ \frac{p = q}{\Gamma, (p > q)}\ , \quad (\mathrm{MP})\ \frac{\Gamma, p \quad \Gamma, \neg q}{\Gamma, \neg(p > q)}\ , \quad (\mathrm{CS})\ \frac{\Gamma, p \quad \Gamma, q}{\Gamma, (p > q)}\ .$$

These and the rules in CK and $CKCEM$ are easily modified to have complete premises by adding missing literals. Applying Theorems 54 and 49 yields the result. □

This reproves the PSPACE upper bounds for these logics found in [9], [8], and [11].

8 Conclusion

In this paper we have reported our first successes in synthetically constructing sequent calculi that admit cut elimination. We have converted shallow modal axioms into sequent rules so that the resulting system together with the cut and contraction rules is sound and complete with respect to the Hilbert-system. It was also shown to stay complete, if the cuts are restricted to pseudo-analytic cuts. This led to a generic decidability result and a 3EXPTIME upper bound for logics axiomatised by a PSPACE-tractable set of shallow axioms. Since in particular all finite sets of axioms are PSPACE-tractable, logics axiomatised by a finite set of shallow axioms are decidable in 3EXPTIME. The method then was extended to generically construct PSPACE-tractable sets of rules from finite sets of shallow axioms in such a way, that the resulting sequent system eliminates the cut rule. If the so constructed rule set is closed under the contraction rule, then the logic axiomatised by the corresponding axioms is decidable in polynomial space.

Our success is clearly partial in that we do not yet know under which conditions closure under contraction can also be obtained. This is the subject of future work, possibly borrowing from the theory of vector addition systems [3] to control the multiplicities of formulas. For now, contraction closure needs to be established by hand, and doing so, we have applied our method to various systems in conditional logics. This led to new proofs of PSPACE upper bounds for these systems.

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