A Meta-Calculus for Krom-Horn logic

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1 Introduction

First-order theorem proving can be considered as a semi-decision procedure for the validity problem of first-order logic. Most clausal theorem provers are based on proof by contradiction and so are a semi-decision methods for unsatisfiability. This point of view, though logically justified, led to a predominant concentration on aspects of completeness. Due to the undecidability of the satisfiability problem for clause logic, the investigation of termination and complexity of theorem provers appeared as a hopeless enterprise. But already in the late sixties and in the seventies some few researchers recognized the potential of theorem proving methods as decision procedures [Joy76], [Mas68] for decidable classes of predicate logic. In the nineties these early approaches were revived and extended to a systematic investigation of resolution refinements as decision procedures [FLTZ93]. About at the same time the first papers appeared addressing the potential of theorem provers in the construction of Herbrand models [CZ91]. Indeed various complete theorem provers show a substantially different behavior on satisfiable problems; in particular investigation of aspects of termination and model building do not only lead to better theorem provers but also to a *mathematical* comparison of different refinements. In [FLTZ93] terminating resolution refinements were found for most of the well-known decision classes of first-order logic; moreover the decidable classes were extended by the methods in a natural way. For some of the classes extensions of the ordinary refinements were necessary in order to achieve termination. In particular, in order to decide some classes, it is necessary to add specific limited forms of instantiation which are not based on most general unification. But even with this technique resolution refinements failed to terminate on known decidable first-order classes. The most prominent of these classes are $\forall\exists\forall$-Horn and $\forall\exists\forall$-Krom, the prenex classes with the prefix $\forall\exists\forall$ and a Horn, respectively Krom, matrix. Both classes are not finitely controllable (they do not possess the finite model property) and thus there exists no trivial decision procedure based on parallel processing of finite domain search and resolution refutation. A closer look at the behavior of hyperresolution on AEA-Horn reveals that the minimal Herbrand models cannot be represented by a finite set of atoms (neither can they be compressed into finite models); on the other hand the infinite set of atoms produced by hyperresolution can easily be described in a finite way using a mathematical meta-formalism. This phenomenon suggests to extend the power of theorem provers by integrating methods of schematization into the object logic and into the inference machinery. Powerful schematization methods for terms and for clause logic have been investigated in several publications in the late eighties and in the nineties, e.g. [Sal89], [Sal92], [Herm92], [ChHs90] (the list is incomplete). Most of these publication focused on the unification problem for meta-formalisms, but already [Sal89] suggested the use of meta-inference rules in theorem proving; here the main challenge consists in the *generation* of cycle clauses finitely representing the set of all clause powers. We use the ideas developed in these papers to define a meta-calculus for Krom-Horn logic, which is simple and natural and terminates on the class $\forall\exists\forall$Krom $\cap$ Horn. Note that the class $\forall\exists\forall$Krom $\cap$ Horn is not finitely controllable as well; moreover prefixing an existential quantifier (creating a constant symbol in the skolemized form) makes the class undecidable.The calculus does not only decide the class but also yields a simple finite representation for Herbrand models and extends the semantic expressivity of first-order provers in a genuine and algorithmic way. Technically we reduce, modify and extend the problems in $\forall\exists\forall$Krom $\cap$ Horn. The problem is extended to all Krom-Horn clauses over the signature of a single one-place function symbol (class $\mathcal{KH}_1$) and containing at most two variables. At the same time we restrict $\mathcal{KH}_1$ to two-place predicate symbols. These transformations are justified by the decision method defined in Dreben and Goldfarb's book [DG79] but have to be carried out in detail. Due to the special form of the arguments the term structure of atoms and meta-atoms can be considerably simplified, as indicated in the next section. The following analysis is based on this form. The main points in this text are the following ones: first we show the finite representability of the deductive closure via meta-clauses; in a second step we prove that this finite set of meta-clauses is generated by the meta-term calculus.

2 Basic Properties of $\mathcal{KH}_1$

Definition 2.1 $\mathcal{KH}_1$ is the class of all finite sets of clauses $\mathcal{C}$ fulfilling the following conditions:

1. $\mathcal{C}$ is Krom and Horn,
2. there are no constant symbols in $\mathcal{C}$,
3. at most one unary function symbol occurs in $\mathcal{C}$,
4. all predicate symbols are binary,
5. every clause in $\mathcal{C}$ contains at most two variables.$\sharp$

As a clause set $\mathcal{C} \in \mathcal{KH}_1$ contains at most one one-place function symbol we can change the usual term notation to a more convenient one. E.g. we write $\neg P(x,y) \vee P(x+2,y+1)$ for $\neg P(x,y) \vee P(f(f(x)),f(y))$. Note that the clause $C: \neg P(x,y) \vee P(x+2,y+1)$ can be iteratively resolved with itself, which gives the sequence of clauses $\neg P(x,y) \vee P(x+2n,y+n)$ for all $n \geq 1$. This infinite set of clauses can be described by

$$\neg P(x,y) \vee P(x+2(\alpha+1), y+\alpha+1)$$

where α is a numerical variable ranging over the natural numbers. Objects of this type are henceforth called meta-clauses. For a more general definition of meta-terms and the corresponding semantics see [Sal92]. Due to the specific form of $\mathcal{KH}_1$ further simplications are possible. So we may replace $\neg P(x,y+1) \vee P(x,y)$ by $\neg P(x,y) \vee P(x+1,y)$ under preservation of sat-equivalence (this holds only because there are no constant symbols). We may even use the subtraction symbol and write $\neg P(x,y) \vee P(x,y-1)$ for $\neg P(x,y) \vee P(x+1,y)$. As only the "difference" of the x and y component counts, we may also replace $\neg P(x,y) \vee P(x+1,y+1)$ by $\neg P(x,y) \vee P(x,y)$; in fact the clause $\neg P(x,y) \vee P(x+1,y+1)$ can be deleted like a tautology without affecting completeness. To sum up we use the following transformations:

$$\begin{aligned}
\neg P(x+a,y+b) \vee P(x+c,y+d) &\Rightarrow \neg P(x,y) \vee P(x+c-a,y+d-b),\\
\neg P(x+a,y+b) \vee P(y+c,x+d) &\Rightarrow \neg P(x,y) \vee P(y+c-b,x+d-a),\\
\neg P(x,y) \vee P(x+c,y+d) &\Rightarrow \neg P(x,y) \vee P(x,y+d-c),\\
\neg P(x,y) \vee P(y+c,x+d) &\Rightarrow \neg P(x,y) \vee P(y,x+d-c).
\end{aligned}$$

An informal justification for these transformations can be given via the effect of rule clauses on facts of type $A: P(x+m,x+n)$. Let

$C = \neg P(x, y + 1) \vee P(x, y)$ and $D = \neg P(x, y) \vee P(x + 1, y)$;

then the resolvent of A and C is $P(x + m, x + n - 1)$, that of A and D is $P(x+m+1, x+n)$. These facts "behave" in the same way as the difference of the f-towers is the same. Note that $P(x+m, x-1)$ stands for $P(x+m+1, x)$. An exact description and justification of these transformations will be given in a forthcoming paper. Only rule clauses with two variables can change the difference of the components for facts of the type $P(x + m, y + n)$; this justifies the focus on the following clause types:

Definition 2.2 (clause types)

I. $\neg P(x, y) \vee Q(x, y + r\alpha + p)$

II. $\neg P(x, y) \vee Q(y + r\alpha + p, x)$

III. $\neg P(x, x + m) \vee Q(x, y)$

IV. $\neg P(x, x + m) \vee Q(y, x)$

V. $\neg P(x, y) \vee Q(x, x + m)$

VI. $\neg P(x, y) \vee Q(y, y + m)$

♯

Remark: r, k are number constants (for integers) and α is a variable ranging over natural numbers. ♯

Definition 2.3 (deduction chain) Let $C_1 : \neg A \vee B$ and $C_2 : \neg C \vee D$ be clauses in Krom ∩ Horn and let $\{B, C\}$ be unifiable by m.g.u. σ then the resolvent $D : \neg A\sigma \vee \mathcal{D}\sigma$ is written as $C_1 \circ C_2$ or simply as C_1C_2 (A or D or both may be $\Box$).

If P is a fact and $C_1, \ldots, C_n$ are rules in Krom∩Horn and $D : ((...(PC_1)...)C_n$ is defined then $\gamma : (P, C_1, \ldots, C_n)$ is called a *deduction chain* or simply a *chain*. The (unit-) clause D is called the *product* of γ.♯

Remark: Note that $(CD)E = C(DE)$ and resolution in Krom ∩ Horn is associative, thus "almost" defining a semi-group (the product need not be defined). Therefore $(...(PC_1)...)C_n$ is henceforth written as $PC_1 \cdots C_n$. ♯

Clause types III,IV:

If clauses of this type appear in a chain $(P, C_1 \cdots C_n)$ then the predicate symbol Q can be deleted from all clauses and only the unit clause $Q(x,y)$ remains. If a negative Q-clause exists in the set of clauses $\mathcal{C}$ then $\mathcal{C}$ is unsatisfiable. In particular a clause of this type can only appear finitely many often in a deduction. The elimination of such predicate symbols can be performed like the one-literal-rule in the Davis-Putnam procedure for propositional logic.

Clause types V,VI:
Every deduction chain $(P, C_1, \cdots, C_n)$ with C_n of type V or VI yields a product of the form $Q(x, x+m)$ (via clause normalization). Thus the result is the positive literal of an input clause.Clearly clauses of this form can only appear a fixed number of times without being subsumed.

Therefore the remaing part concentrates on clause sets consisting of clauses of type I and II only.

Definition 2.4 (meta-subsets, meta-equality) Let X and Y be meta-clauses or sets of meta-clauses and $S(X), S(Y)$ be the sets of (first-order) clauses described by X and Y. Then we write $X \subseteq_M Y$ if $S(X) \subseteq S(Y)$ and $X =_M Y$ if $S(X) = S(Y)$. $\sharp$

Type I-II clauses are not closed under resolution, although the resolvents CD can always be transformed into a finite set of I-II clauses $\mathcal{D}$ s.t. $CD =_M \mathcal{D}$. We delay the definition of this transformation, because it is not needed for the representation lemmas, and - for the time being - we prefer CD being a single clause instead of a set of clauses. We define the clause types Ia and IIa which are are closed under resolution.

Definition 2.5 (types Ia,IIa) A clause C is of the form Ia if there exists a predicate symbol P, integer constants $k_1, \ldots, k_n$ and q and meta-variables $\alpha_1, \ldots, \alpha_n$ s.t.

$$C = \neg P(x,y) \vee P(x, y + \textstyle\sum_{i=1}^{n} k_i \alpha_i + q).$$

C is of type IIa if

$$C = \neg P(x,y) \vee P(y + \textstyle\sum_{i=1}^{n} k_i \alpha_i + q, x).$$

$\sharp$

Definition 2.6 (inverse) Let C be a clause of the form Ia and

$$C = \neg P(x,y) \vee P(x, y + \textstyle\sum_{i=1}^{n} k_i \alpha_i + q).$$

Then C^{-1} is defined as

$C^{-1} = \neg P(x,y) \vee P(x, y - \sum_{i=1}^{n} k_i\alpha_i - q)$.

$\sharp$

Remark: If C is a clause without meta-variables then $CC^{-1} = C^{-1}C = I_P$ if C is defined over the predicate symbol P and $I_P = \neg P(x,y) \vee P(x,y)$.$\sharp$

3 Representing the Deductive Closure

Lemma 3.1 (closure lemma) *Let C_1, C_2 and D_1, D_2 be clauses over a predicate symbol P s.t. C_1, C_2 are of type Ia and D_1, D_2 are of type IIa. Then the following properties are fulfilled:*

(a) The products C_iC_j, D_iD_j, C_iD_j and D_jC_i for $i,j = 1,2$ are defined.

(b) C_iD_j and D_iC_j are of type IIa ($i,j = 1,2$).

(c) C_iC_j and D_iD_j are of type Ia ($i,j = 1,2$).

Proof:
Let

$$\begin{aligned}
C_1 &= \neg P(x,y) \vee P(x, y + \sum_{i=1}^{K} k_i\alpha_i + p), \\
C_2 &= \neg P(x,y) \vee P(x, y + \sum_{i=1}^{L} l_i\beta_i + q), \\
D_1 &= \neg P(x,y) \vee P(y + \sum_{i=1}^{M} m_i\gamma_i + r, x), \\
D_2 &= \neg P(x,y) \vee P(y + \sum_{i=1}^{N} n_i\delta_i + s, x).
\end{aligned}$$

It is enough to prove (b) and (c).
point (b): w.l.o.g we may investigate C_1D_1 and D_1C_1.

$$C_1D_1 = \neg P(x,y) \vee P(y + \sum_{i=1}^{M} m_i\gamma_i + r + \sum_{i=1}^{K} k_i\alpha_i + p, x),$$

$$\begin{aligned}
D_1C_1 &= \neg P(x,y) \vee P(y + \sum_{i=1}^{M} m_i\gamma_i + r, x + \sum_{i=1}^{K} k_i\alpha_i + p) \\
&= \neg P(x,y) \vee P(y + \sum_{i=1}^{M} m_i\gamma_i + r - \sum_{i=1}^{K} k_i\alpha_i - p, x).
\end{aligned}$$

It is easy to get the form IIa by renaming the summation indices and meta-variables.

point (c):
w.l.o.g. we only compute C_1C_2 and D_1D_2.

$$\begin{aligned}
C_1C_2 &= \neg P(x,y) \vee P(x, y + \sum_{i=1}^{K} k_i\alpha_i + p + \sum_{i=1}^{L} l_i\beta_i + q), \\
D_1D_2 &= \neg P(x,y) \vee P(x + \sum_{i=1}^{N} n_i\delta_i + s, y + \sum_{i=1}^{M} m_i\gamma_i + r) \\
&= \neg P(x,y) \vee P(x, y + \sum_{i=1}^{M} m_i\gamma_i + r - \sum_{i=1}^{N} n_i\delta_i - s).
\end{aligned}$$

Again it is easy to transform the products into the form Ia. $\diamond$

Lemma 3.2 (commutation lemma) *Let C, D be clauses over a predicate symbol P and $\pi_P = \neg P(x,y) \vee P(y,x)$. Then*

(a) If C and D are of type Ia then $CD = DC$.

(b) If C is of type IIa then there exists a clause C' of type Ia s.t. $C = C'\pi_P$.

(c) If C is of type Ia then $\pi_P C = C^{-1}\pi_P$

Proof:
point (a): Let

$$\begin{aligned}
C &= \neg P(x,y) \vee P(x, y + \sum_{i=1}^{K} k_i\alpha_i + p), \\
D &= \neg P(x,y) \vee P(x, y + \sum_{i=1}^{L} l_i\beta_i + q).
\end{aligned}$$

Then

$$\begin{aligned}
CD &= \neg P(x,y) \vee P(x, y + \sum_{i=1}^{K} k_i\alpha_i + p + \sum_{i=1}^{L} l_i\beta_i + q), \\
&= DC
\end{aligned}$$

as "+" is commutative.

point (b):

$$\begin{aligned}
C &= \neg P(x,y) \vee P(y + \sum_{i=1}^{K} k_i\alpha_i + p, x), \\
C' &= \neg P(x,y) \vee P(x, y + \sum_{i=1}^{K} k_i\alpha_i + p), \\
C'\pi_P &= C.
\end{aligned}$$

point (c):

$$\begin{aligned}
C &= \neg P(x,y) \vee P(x, y + \sum_{i=1}^{K} k_i\alpha_i + p), \\
C\pi_P &= \neg P(x,y) \vee P(y + \sum_{i=1}^{K} k_i\alpha_i + p, x), \\
\pi_P C &= \neg P(x,y) \vee P(y, x + \sum_{i=1}^{K} k_i\alpha_i + p), \\
&= \neg P(x,y) \vee P(y - \sum_{i=1}^{K} k_i\alpha_i - p, x), \\
&= C^{-1}\pi_P.
\end{aligned}$$

$\diamond$

Definition 3.1 (deductive closure) Let $\mathcal{C}$ be a set of clauses in Krom $\cap$ Horn over a set of two-place predicate symbols $P_1, \ldots, P_n$. Then let

$R(\mathcal{C}) = \mathcal{C} \cup \{CD | C, D \in \mathcal{C}\}$,

$R^{i+1}(\mathcal{C}) = R(R^i(\mathcal{C}))$ for $i \geq 0$.

The deductive closure is defined as

$R^*(\mathcal{C}) = \bigcup_{i=1}^{n} \{\neg P_i(x,y) \vee P_i(x,y)\} \cup \bigcup_{i \in \mathbb{N}} R^i(\mathcal{C})$. $\sharp$

Notation: $\neg P(x,y) \vee P(x,y)$ is denoted by I_P or shortly by I if no confusion may arise. Similarly we may write π instead of π_P.

Remark: Type Ia-IIa clauses over a predicate symbol P define a monoid with neutral element I. The clauses of type Ia form a commutative submonoid. The clauses of type Ia without meta-variables even define an Abelian group. $\sharp$

Lemma 3.3 (first representation lemma) *Let $\mathcal{C}$ be a set of type Ia-IIa clauses over a predicate symbol P containing at least one clause of type IIa, i.e. $\mathcal{C}$ is of the form*

$$\mathcal{C} = \{C_1, \ldots, C_n\} \cup \{D_1\pi, \ldots, D_m\pi\}$$

for C_i, D_j of type Ia and $m > 0$. Moreover let $\mathcal{D} = \{D_1, \ldots D_m\}$ and

$$\hat{\mathcal{C}} = \{C_1, \ldots, C_n, C_1^{-1}, \ldots, C_n^{-1}\} \cup \bigcup_{i,j=1}^{m}\{D_i D_j^{-1}\}.$$

Then

$$R^*(\mathcal{C}) =_M R^*(\hat{\mathcal{C}}) \cup \mathcal{D}R^*(\hat{\mathcal{C}})\pi.$$

Proof:
(a) $R^*(\mathcal{C}) \subseteq_M R^*(\hat{\mathcal{C}}) \cup \mathcal{D}R^*(\hat{\mathcal{C}})\pi$:
It is enough to prove that $\mathcal{C} \subseteq_M R^*(\hat{\mathcal{C}}) \cup \mathcal{D}R^*(\hat{\mathcal{C}})\pi$ and that $R^*(\hat{\mathcal{C}}) \cup \mathcal{D}R^*(\hat{\mathcal{C}})\pi$ is deductively closed.

$\{C_1, \ldots, C_n\} \subseteq_M R^*(\hat{\mathcal{C}}) \cup \mathcal{D}R^*(\hat{\mathcal{C}})\pi$ is trivial.

We show that $D_j\pi \subseteq_M R^*(\hat{\mathcal{C}}) \cup \mathcal{D}R^*(\hat{\mathcal{C}})\pi$:
By definition of R^* we have $I \in R^*(\hat{\mathcal{C}})$ and therefore

$$D_j\pi \subseteq_M \mathcal{D}R^*(\hat{\mathcal{C}})\pi.$$

It remains to show that $R^*(\hat{\mathcal{C}}) \cup \mathcal{D}R^*(\hat{\mathcal{C}})\pi$ is deductively closed:
Clearly $R^*(\hat{\mathcal{C}})$ itself is deductively closed and consists of type Ia clauses only.

So let $E_1 \in R^*(\hat{\mathcal{C}})$ and $D_j E_2\pi \in \mathcal{D}R^*(\hat{\mathcal{C}})\pi$. Then

$$E_1 D_j E_2 \pi = D_j E_1 E_2 \pi$$

by point (a) of the commutation lemma. But $E_1 E_2 \in R^*(\hat{\mathcal{C}})$ and therefore $E_1 D_j E_2 \pi \in \mathcal{D}R^*(\hat{\mathcal{C}})\pi$.

We reverse the order of resolution:

$$D_j E_2 \pi E_1 = D_j E_2 E_1^{-1} \pi$$

by the commutation lemma point (c). From sublemma I (to be proved afterwards) we know that $E \in R^*(\hat{\mathcal{C}})$ implies $E^{-1} \in R^*(\hat{\mathcal{C}})$. Therefore $E_2 E_1^{-1} \in R^*(\hat{\mathcal{C}})$ and so

$$D_j E_2 E_1^{-1} \pi \in \mathcal{D}R^*(\hat{\mathcal{C}})\pi.$$

It remains to investigate resolutions within $\mathcal{D}R^*(\hat{\mathcal{C}})\pi$:
Let $D_i E_1 \pi, D_j E_2 \pi \in \mathcal{D}R^*(\hat{\mathcal{C}})\pi$. Applying the commutation lemma (point (c)) twice we get

$$D_i E_1 \pi D_j E_2 \pi = D_i E_1 D_j^{-1} \pi E_2 \pi = D_i E_1 D_j^{-1} E_2^{-1} \pi\pi.$$

But $\pi\pi = I$ and therefore

$$D_i E_1 \pi D_j E_2 \pi = D_i E_1 D_j^{-1} E_2^{-1}.$$

But now only type Ia clauses appear in the resolution product and the elements can be permuted (commutation lemma point (a)). So we get

$$D_i E_1 \pi D_j E_2 \pi = D_i D_j^{-1} E_1 E_2^{-1}.$$

By definition of $\hat{\mathcal{C}}$ we have $D_i D_j^{-1} \in \hat{\mathcal{C}}$ and by sublemma I $E_2^{-1} \in R^*(\hat{\mathcal{C}})$, so $E_1 E_2^{-1} \in R^*(\hat{\mathcal{C}})$. This eventually gives

$$D_i E_1 \pi \circ D_j E_2 \pi \in R^*(\hat{\mathcal{C}}).$$

This completes the proof of direction (a).

(b) $R^*(\hat{\mathcal{C}}) \cup \mathcal{D}R^*(\hat{\mathcal{C}})\pi \subseteq_M R^*(\mathcal{C})$:

(b1) $R^*(\hat{\mathcal{C}}) \subseteq_M R^*(\mathcal{C})$:

Clearly it is sufficient to show $\hat{\mathcal{C}} \subseteq_M R^*(\mathcal{C})$.

$\{C_1, \ldots, C_n\} \subseteq R^*(\mathcal{C})$ is trivial.

We prove $\{C_i^{-1}\} \subseteq_M R^*(\mathcal{C})$ for $i \in \{1, \ldots, n\}$. By definition of $\mathcal{C}$ and by deductive closure of $R^*(\mathcal{C})$ we have

$$(D_1\pi) C_i (D_1\pi) \in R^*(\mathcal{C}).$$

Iterated application of the commutation lemma gives

$$\begin{aligned} D_1 \pi C_i D_1 \pi &= D_1 C_i^{-1} D_1^{-1} \pi\pi = \\ D_1 C_i^{-1} D_1^{-1} &= D_1 D_1^{-1} C_i^{-1}. \end{aligned}$$

But $\{I\} \subseteq_M \{D_1 D_1^{-1}\}$ and therefore

$$\{C_i^{-1}\} \subseteq_M D_1 D_1^{-1} C_i^{-1} \in R^*(\mathcal{C}).$$

It remains to show that for $i, j \in \{1, \ldots, m\}$ $D_i D_j^{-1} \in R^*(\mathcal{C})$.

Clearly $D_i \pi D_j \pi \in R^*(\mathcal{C})$ by definition of $\mathcal{C}$ and by deductive closure. But

$$D_i \pi D_j \pi = D_i D_j^{-1} \pi\pi = D_i D_j^{-1}.$$

and $D_i D_j^{-1} \in \mathcal{C}$ by definition.

(b2) $\mathcal{D}R^*(\hat{\mathcal{C}})\pi \subseteq_M R^*(\mathcal{C})$:

Let us consider $D_j E \pi$ with $E \in R^*(\hat{\mathcal{C}})$. Then E is of type Ia and, by (b1), $\{E\} \subseteq_M R^*(\mathcal{C})$. Therefore the commutation lemma gives

$$D_j E \pi = E \circ (D_j \circ \pi).$$

But $D_j\pi \in \mathcal{C}$ by definition of $\mathcal{C}$ and so $D_j\pi \in R^*(\mathcal{C})$. Finally, as $R^*(\mathcal{C})$ is deductively closed, we obtain $ED_j\pi \in R^*(\mathcal{C})$.

(a) and (b) together yield $R^*(\mathcal{C}) =_M R^*(\hat{\mathcal{C}}) \cup \mathcal{D}R^*(\hat{\mathcal{C}})\pi$. $\diamond$

Sublemma I:
Let

$$\hat{\mathcal{C}} \quad = \quad \{C_1, \ldots, C_n, C_1^{-1}, \ldots, C_n^{-1}\} \cup \bigcup_{i,j=1}^{m} \{D_i D_j^{-1}\}.$$

s.t. the C_i and the D_j are of type Ia. Then for every $E \in R^*(\hat{\mathcal{C}})$ the element E^{-1} is in $R^*(\hat{\mathcal{C}})$.

Proof: By induction on the length k of resolution products.
$k = 1$: $C_i^{-1} \in \hat{\mathcal{C}}$ by definition. Moreover

$$(C_i^{-1})^{-1} = C_i \quad \text{and} \quad (D_i D_j^{-1})^{-1} = D_j D_i^{-1}.$$

(IH) Assume that for every product E of clauses $F_1, \ldots, F_l \in \hat{\mathcal{C}}$ for $l \leq k$ $E^{-1} \in R^*(\hat{\mathcal{C}})$.

case $k+1$:
$E = (F_1 \cdots F_k) \circ F_{k+1}$. By (IH) $(F_1 \cdots F_k)^{-1} \in R^*(\hat{\mathcal{C}})$ and $F_{k+1}^{-1} \in R^*(\hat{\mathcal{C}})$. Because $R^*(\hat{\mathcal{C}})$ is deductively closed we have

$$F_{k+1}^{-1}(F_1 \cdots F_k)^{-1} \in R^*(\hat{\mathcal{C}}).$$

But E^{-1} is just $F_{k+1}^{-1}(F_1 \cdots F_k)^{-1}$. $\diamond$

Example 3.1 Let C be the clause $\neg P(x,y) \vee P(y + 2\alpha + 3, x)$. Then $C = D_1\pi$ for

$$D_1 = \neg P(x,y) \vee P(x, y + 2\alpha + 3).$$

So let $\mathcal{C} = \{C\}$. Then $\hat{\mathcal{C}} = \{D_1 D_1^{-1}\}$ and the first representation lemma yields

$$R^*(\mathcal{C}) \quad =_M \quad R^*(\hat{\mathcal{C}}) \cup D_1 R^*(\hat{\mathcal{C}})\pi.$$

But $R^*(\hat{\mathcal{C}}) =_M \{D_1 D_1^{-1}\}$; indeed,

$$E = D_1 D_1^{-1} = \neg P(x,y) \vee P(x, y + 2\alpha - 2\beta)$$

and $E \circ E \subseteq_M E$. Moreover

$$D_1 R^*(\hat{\mathcal{C}})\pi =_M \{D_1 E\pi\} =_M \{\neg P(x,y) \vee P(y + 2\alpha - 2\beta + 3, x)\}.$$

Therefore

$$R^*(\mathcal{C}) =_M \{\neg P(x,y) \vee P(x, y + 2\alpha - 2\beta),\ \neg P(x,y) \vee P(y + 2\alpha - 2\beta + 3, x)\}.$$

♯

Lemma 3.4 (representation lemma II) *Let $\mathcal{C}$ be a finite set of type-Ia clauses over a (single) predicate symbol. Then there exists a finite set $\mathcal{C}_*$ of type-Ia clauses s.t. $R^*(\mathcal{C}) =_M \mathcal{C}_*$.*

Proof: By induction on $|\mathcal{C}|$ (the number of clauses in $\mathcal{C}$).
(IB) $|\mathcal{C}| = 1$:
Then $\mathcal{C} = \{C\}$ for a clause C of the form

$$C \quad = \quad \neg P(x,y) \vee P(x, y + \sum_{i=1}^{n} r_i \alpha_i + p)$$

for $r_i, p \in \mathbf{Z}$ and meta-variables α_i. As C is a Krom-clause the self-resolvents of C can be represented as clause powers, i.e.

$$\begin{aligned} R^*(\{C\}) \quad &= \quad \{C^k \mid k \geq 0\} \text{ where} \\ C^k \quad &=_M \quad \neg P(x,y) \vee P(x, y + \sum_{i=1}^{n} r_i \alpha_i + pk) \text{ for } k \geq 1. \end{aligned}$$

Formally we have to distinguish between the number k and the number constant k appearing in C^k. But we prefer to write "k" both times in order to avoid overloading of the formalism. Let

$$D = \neg P(x,y) \vee P(x, y + \sum_{i=1}^{n} r_i \alpha_i + p(\beta + 1)),$$

where β is meta-variable different from the α_i. Then

$$R^*(\mathcal{C}) =_M \{I_P, D\}$$

and we may define $\mathcal{C}_* = \{I_P, D\}$. Clearly D can be normalized to a type Ia-clause D' for

$$D' \quad = \quad \neg P(x,y) \vee P(x, y + \sum_{i=1}^{n+1} r_i \alpha_i + p),$$

for $r_{n+1} = p$.

(IH) assume that for all $\mathcal{C}$ with $1 \leq |\mathcal{C}| \leq m$ and containing type-Ia clauses only there exist finite sets of type-Ia clauses $\mathcal{C}_*$ with $\mathcal{C}_* =_M R^*(\mathcal{C})$.

case $m+1$:
Let $\mathcal{C} = \{C_1, \ldots, C_{m+1}\}$ and $\mathcal{D} = \{C_1, \ldots, C_m\}$ and let D be a element in $R^*(\mathcal{C})$; then

$$D = C_1^1 \cdots C_{k_1}^1 C_{m+1} \cdots C_1^p \cdots C_{k_p}^p C_{m+1} C_1^{p+1} \cdots C_{k_{p+1}}^{p+1},$$

where $C_j^i \in \mathcal{D}$. Then D can be written as

$$D = E_1 C_{m+1} \cdots E_p C_{m+1} E_{p+1}$$

where the E_i are in $R^*(\mathcal{D})$ and (possibly) $E_1 = I$ and/or $E_{p+1} = I$. By the commutation lemma D can be rewritten into the form

$$D \quad = \quad E_1 \cdots E_{p+1} C_{m+1}^p.$$

In particular there exists an $E \in R^*(\mathcal{D})$ with $D = EC_{m+1}^p$. Therefore

$$R^*(\mathcal{C}) \subseteq R^*(\mathcal{D}) \circ R^*(\{C_{m+1}\})$$

and by $R^*(\mathcal{D}) \circ R^*(\{C_{m+1}\}) \subseteq R^*(\mathcal{C})$ (trivially)

$$R^*(\mathcal{C}) = R^*(\mathcal{D}) \circ R^*(\{C_{m+1}\}).$$

By (IH) there exist finite sets $\mathcal{D}_*$ and $\{C_{m+1}\}_*$ of Ia-clauses with

$$\mathcal{D}_* =_M R^*(\mathcal{D}) \quad \text{and} \quad \{C_{m+1}\}_* =_M R^*(\{C_{m+1}\}).$$

So we just define $\mathcal{C}_* = \mathcal{D}_* \circ \{C_{m+1}\}_*$. Then $\mathcal{C}_* =_M R^*(\mathcal{C})$ and $\mathcal{C}_*$, as a product of finite sets of type-Ia clauses, is a finite set of type-Ia clauses. $\diamond$

Lemma 3.5 (representation lemma III) *Let $\mathcal{C}$ be a finite set of type-Ia-IIa clauses over a (single) predicate symbol. Then there exists a finite set $\mathcal{C}_*$ of type Ia-IIa clauses s.t. $R^* =_M \mathcal{C}_*$.*

Proof: If $\mathcal{C}$ consists of type-Ia clauses only then we just apply representation lemma II. So we may assume that

$$\mathcal{C} \quad = \quad \{C_1, \ldots, C_n\} \cup \{D_1\pi, \ldots, D_m\pi\}$$

for $m > 0$ and C_i, D_j of type Ia. By representation lemma I we have

$$R^*(\mathcal{C}) = R^*(\hat{\mathcal{C}}) \cup \mathcal{D} R^*(\hat{\mathcal{C}})\pi$$

for

$$\hat{\mathcal{C}} \quad = \quad \{C_1, \ldots, C_n, C_1^{-1}, \ldots, C_n^{-1}\} \cup \bigcup_{i,j=1}^{m} \{D_i D_j^{-1}\}$$

and $\mathcal{D} = \{D_1, \ldots, D_m\}$.

But $\hat{\mathcal{C}}$ consists of type-Ia clauses only and so, by representation lemma II, there exists a finite set $\hat{\mathcal{C}}_*$ of type Ia clauses with $\hat{\mathcal{C}}_* =_M R^*(\hat{\mathcal{C}})$. Therefore we simply define

$$\mathcal{C}_* \quad = \quad \hat{\mathcal{C}}_* \cup \mathcal{D} \circ \hat{\mathcal{C}}_* \circ \pi.$$

By the closure lemma $\mathcal{C}_*$ is a finite set of type-Ia-IIa clauses and $\mathcal{C}_* =_M R^*(\mathcal{C})$. $\diamond$

Theorem 3.1 (representation theorem) *Let $\mathcal{C}$ be a finite set of type-Ia-IIa clauses. Then there exists a finite set $\mathcal{C}_*$ of type-Ia-IIa clauses s.t. $\mathcal{C}_* =_M R^*(\mathcal{C})$.*

Proof: By induction on $|\mathcal{C}|$.

(IB) $|\mathcal{C}| = 1$:

Then $\mathcal{C} = \{C\}$ for a clause C of type Ia-IIa. If C is of the "predicate type" $P \to Q$ for $P \neq Q$ then C does not resolve with itself and $R^*(\mathcal{C}) = \{C, I_P, I_Q\}$; here we simply define $\mathcal{C}_* = R^*(\mathcal{C})$.

If C is of predicate type $P \to P$ for a predicate symbol P then the set $\mathcal{C}_*$ exists by representation lemma III.

(IH) Assume that for all sets $\mathcal{C}$ of type-Ia-IIa clauses with $1 \leq |\mathcal{C}| \leq n$ there exist finite sets $\mathcal{C}_*$ of type-Ia-IIa clauses with $\mathcal{C}_* = R^*(\mathcal{C})$.

case $n+1$:
Let $\mathcal{C} = \{C_1, \ldots, C_{n+1}\}$ and $\mathcal{D} = \mathcal{C} - \{C_{n+1}\}$. By (IH) there exists a finite set of type-Ia-IIa clauses $\mathcal{D}$ with $\mathcal{D}_* =_M R^*(\mathcal{D})$.

We distinguish the following cases:

(a) C_{n+1} is of predicate type $P \to Q$ s.t. neither P nor Q occurs in $\mathcal{D}$.

We define $\mathcal{C}_* = \mathcal{D}_* \cup \{C_{n+1}\}_*$.

(b) C_{n+1} is of predicate type $P \to Q$ s.t. P occurs in $\mathcal{D}$, but Q does not.

Then $C_{n+1} \circ C_{n+1}$ and $C_{n+1} \circ \mathcal{D}$ are undefined. So we define

$$\mathcal{C}_* \;=\; \mathcal{D}_* \cup \mathcal{D}_* \circ C_{n+1}.$$

(c) C_{n+1} is of predicate type $P \to Q$ s.t. Q occurs in $\mathcal{D}$, but P does not.

This case is analogous to (b) and we define

$$\mathcal{C}_* \;=\; \mathcal{D}_* \cup C_{n+1} \circ \mathcal{D}_*.$$

(d) C_{n+1} is of predicate type $P \to Q$ s.t. both P and Q occur in $\mathcal{D}$.

Let $D \in R^*(\mathcal{C})$. Then D can be written as a product in the form

$$D = E_1 C_{n+1} E_2 C_{n+1} \cdots E_m C_{n+1} E_{m+1}$$

for $E_1, \ldots, E_{m+1} \in R^*(\mathcal{D})$ and (possibly) $E_1 = I_P$ and/or $E_{m+1} = I_Q$.

Then all clauses $E_2, \ldots, E_m$ must be of the predicate type $Q \to P$, otherwise the product is undefined. In particular the clauses $C_{n+1}E_i$ are of predicate type $P \to P$ for $i = 2, \ldots, m$.

Let $\mathcal{D}_{Q\to P}$ be the subset of $\mathcal{D}_*$ describing all elements of predicate type $Q \to P$ in $R^*(\mathcal{D})$. Then $C_{n+1}\mathcal{D}_{Q\to P}$ is a finite set of type-Ia-IIa clauses of predicate type $P \to P$. By representation lemma III there exists a finite set $X_{P\to P}$ of type-Ia-IIa clauses with

$$X_{P\to P} =_M R^*(C_{n+1}\mathcal{D}_{Q\to P}).$$

But then $X_{P\to P}$ represents the subproduct $C_{n+1}E_2C_{n+1} \cdots C_{n+1}E_m$ of D.

Let $\mathcal{D}_{\to P}$ the subset of $\mathcal{D}_*$ representing clauses of predicate type $\to P$; $\mathcal{D}_{Q\to}$ is defined analogously. Then we define

$$\mathcal{C}_* \;=\; \mathcal{D}_* \cup \mathcal{D}_{\to P} X_{P\to P} C_{n+1} \mathcal{D}_{Q\to}.$$

By the arguments above $R^*(\mathcal{C}) \subseteq_M \mathcal{C}_*$; moreover $\mathcal{C}_*$ is defined via product and union from finite sets of type Ia-IIa clauses and thus is a finite set of Ia-IIa clauses by the closure theorem.

It remains to show that $\mathcal{C}_* \subseteq_M R^*(\mathcal{C})$. But this is straightforward as $\mathcal{D}_* \subseteq_M R^*(\mathcal{D})$, $\mathcal{D}_{\to P} \subseteq_M R^*(\mathcal{D})$, $\mathcal{D}_{Q\to} \subseteq_M R^*(\mathcal{D})$ and $R^*(\mathcal{D}) \subseteq_M R^*(\mathcal{C})$. Moreover $\{C_{n+1}\} \subseteq_M R^*(\mathcal{C})$ and, as $R^*(\mathcal{C})$ is deductively closed

$$\mathcal{D}_{\to P} X_{P\to P} C_{n+1} \mathcal{D}_{Q\to} \subseteq_M R^*(\mathcal{C}).$$

$\diamond$

Corollary 3.1 *Let $\mathcal{P}$ be a finite set of facts and $\mathcal{C}$ be a finite sets of Ia-IIa clauses. Then there exists a finite set of meta-facts $\mathcal{P}_*$ s.t.*

$$R_H^*(\mathcal{C} \cup \mathcal{P}) =_M \mathcal{C} \cup \mathcal{P}_*.$$

(R_H^* denotes deductive closure under hyperresolution)

Proof: As $\mathcal{C} \cup \mathcal{P} \in$ Krom $\cap$ Horn we have

$$R_H^*(\mathcal{C} \cup \mathcal{P}) \quad = \quad \mathcal{C} \cup \mathcal{P} \circ R^*(\mathcal{C}).$$

By the representation theorem there exists a finite set $\mathcal{C}_*$ of Ia-IIa clauses with $\mathcal{C}_* =_M R^*(\mathcal{C})$. But then also

$$\mathcal{P}\mathcal{C}_* =_M \mathcal{P}R^*(\mathcal{C}).$$

We define $\mathcal{P}_* = \mathcal{P}\mathcal{C}_*$; then $\mathcal{P}_*$ is a finite set of unit meta-clauses representing $\mathcal{P}R^*(\mathcal{C})$ and

$$R_H^*(\mathcal{C} \cup \mathcal{P}) =_M \mathcal{C} \cup \mathcal{P}_*.$$

$\diamond$

Comment:
Actually it is not necessary to reduce the clause syntax to type I and type II clauses in order to obtain termination. Clearly the meta-clauses rerpesenting clause powers can become very long, due to long linear combinations of meta-term variables. Thus reduction of type Ia to type I via greatest common divisor may be useful, but is not necessary for termination. A reduction to the syntax type

$$(*)\ \neg P(x, y) \vee P(x + \sum_{i=1}^{n} k_i\alpha_i + p, y + \sum_{j=1}^{m} l_j\beta_j + q)$$

where k_i, l_j are constants for natural numbers (instead of constants for integers) is possible too – without changing the algebraic properties. Similarly we may exchange x and y in the positive literal and obtain a new version of type-IIa clauses. Even for these new types all representation lemmas and the representation theorem hold.

4 Meta-Inference and Termination

The final aim is to prove that some refinement of meta-resolution terminates on the class $\mathcal{KH}_1$. To this aim it is enough to show that, on satisfiable sets, the inference machinery just produces the set of meta-clauses in the representation theorem.

Definition 4.1 (meta-subsumption) Let C, D be meta-clauses. We define $C \leq_{ss} D$ if there exists a substitution θ over variables and meta-variables and a subset D' of D s.t.

$$\mathrm{LIT}(D') \quad =_M \quad \mathrm{LIT}(C\theta).$$

The relation $\leq_{ss}$ is extended to sets of clauses in the standard way, i.e. $\mathcal{C} \leq_{ss} \mathcal{D}$ if for all $D \in \mathcal{D}$ there exists a $C \in \mathcal{C}$ s.t. $C \leq_{ss} D$. If $\mathcal{C} \leq_{ss} \mathcal{D}$ and $\mathcal{D} \leq_{ss} \mathcal{C}$ we write $\mathcal{C} =_{ss} \mathcal{D}$. $\sharp$

Definition 4.2 (forward subsumption) Let $\mathcal{C}, \mathcal{D}$ be sets of meta-clauses. $sf(\mathcal{C}, \mathcal{D})$ is the subset of clauses in $\mathcal{D}$ which is not subsumed by $\mathcal{C}$; more formally

$$sf(\mathcal{C}, \mathcal{D}) \quad = \quad \{D \in \mathcal{D} \mid \mathcal{C} \not\leq_{ss} \{D\}\}.$$

$\sharp$

Definition 4.3 (cycle operator) Let $cycle(C) = \{C\}$ if C is not a cycle clause and $cycle(C) = C_*$ otherwise. $\sharp$

Remark:
The concept of a cycle clause depends on the choice of the meta-term formalism and the calculus. Generally we denote any finite representation of $R^*(\{C\})$ in some meta-term syntax by C_*. If the rule-clauses are all Ia-IIa clauses then every clause of predicate type $P \to P$ for some p.s. P fulfils an obvious cycle condition. Indeed, due to the representation lemmas every "potentially cyclic" clause C defines a finite set of meta-clauses C_* representing $R^*(\{C\})$. $\sharp$

Definition 4.4 (deduction operators) Let $\mathcal{C}$ be a set of Krom-Horn clauses; then we define $\mathcal{C}_r$ as the set of rule clauses in $\mathcal{C}$. By $Res(\mathcal{C})$ we denote the set of all resolvents in $\mathcal{C}$ and by ϱ_H the operator of hyperresolution, i.e. $\varrho_H(\mathcal{C})$ is the set of all hyperresolvents definable in $\mathcal{C}$. The following "one-step" operator creates hyperresovents, rule resolvents and cycles:

$$\varrho_M(\mathcal{C}) \quad = \quad \varrho_H(\mathcal{C}) \cup Res(\mathcal{C}_r) \cup cycle(\mathcal{C}_r).$$

From ϱ_M we get the two monotone operators R_M and R_{M_s}, where R_{M_s} is R_M refined under forward subsumption.

$$\begin{aligned} R_M(\mathcal{C}) &= \mathcal{C} \cup \varrho_M(\mathcal{C}), \\ R_{M_s}(\mathcal{C}) &= \mathcal{C} \cup sf(\mathcal{C}, \varrho_M(\mathcal{C})). \end{aligned}$$

The deductive closure under R_M and R_{M_s} is defined by R^*_M and $R^*_{M_s}$:

$$\begin{aligned} R^*_M(\mathcal{C}) &= \bigcup_{i \geq 0} R^i_M(\mathcal{C}), \\ R^*_{M_s}(\mathcal{C}) &= \bigcup_{i \geq 0} R^i_{M_s}(\mathcal{C}). \end{aligned}$$

We say that (a monotone operator) R *converges* on $\mathcal{C}$ if there exists an i s.t. $R^i(\mathcal{C}) = R^{i+1}(\mathcal{C})$ (this clearly implies $R^*(\mathcal{C}) = R^i(\mathcal{C})$). We call R *complete* if for all unsatisfiable $\mathcal{C}$ $\Box \in R^*(\mathcal{C})$. $\sharp$

The following result is trivial:

Theorem 4.1 (completeness) *R_{M_s} is complete.*

Proof: $R_M(\mathcal{C})$ contains the set of clauses obtained by hyperresolution. But hyperresolution is complete and so R_M is complete. By definition of R_{M_s} we have $R^i_{M_s}(\mathcal{C}) \leq_{ss} R^i_M(\mathcal{C})$ and therefore $\Box \in R^i_M(\mathcal{C})$ implies $\Box \in R^i_{M_s}(\mathcal{C})$. $\Diamond$

Remark:
The completeness of R_{M_s} holds not only for sets of ordinary first-order clauses but for sets of meta-clauses as well – provided the meta-term formalism is capable of unification. This holds for much stronger meta-term formalisms (e.g. for R-terms) as those we need here.

The main issue is to show termination of R_{M_s} on our class.

*** *lemma corresponding to representation lemma II* ***

Lemma 4.1 *Let $\mathcal{C} : \{C_1, \ldots, C_n\}$ be a set of Ia clauses over a predicate symbol P. Then*

$$\begin{aligned} R^*_{M_s}(\mathcal{C}) &= R^n_{M_s}(\mathcal{C}) \text{ and} && (1) \\ R^n_{M_s}(\mathcal{C}) &=_{ss} (C_1)_* \cdots (C_n)_*. && (2) \end{aligned}$$

Proof: It is sufficient to show

(a) $(C_1)_* \cdots (C_n)_* \subseteq R_M^n(\mathcal{C})$,

(b) $(C_1)_* \cdots (C_n)_* \leq_{ss} R_M^i(\mathcal{C})$ for all $i \geq 0$,

(c) $R_M^k(\mathcal{C}) \leq_{ss} (C_1)_* \cdots (C_k)_*$ for all $k \leq n$

By definition of the operator R_M we have $R_M^i(\mathcal{C}) =_{ss} R_{M_s}^i(\mathcal{C})$ for all i; together with (b) above this gives

$$(C_1)_* \cdots (C_n)_* \leq_{ss} R_M^{n+1}(\mathcal{C}).$$

From (c) we get $R_{M_s}^n(\mathcal{C}) \leq_{ss} (C_1)_* \cdots (C_n)_*$ and by transitivity of $\leq_{ss}$ $R_{M_s}^n(\mathcal{C}) \leq_{ss} R_{M_s}^{n+1}(\mathcal{C})$. But then

$$R_{M_s}^*(\mathcal{C}) = R_{M_s}^n(\mathcal{C}),\ (C_1)_* \cdots (C_n)_* =_{ss} R_M^n(\mathcal{C}).$$

We first prove (a):
As all clauses $C_1, \ldots, C_n$ are of type Ia the sets $(C_1)_* \cdots (C_n)_*$ are defined, where

$$cycle(\mathcal{C}) = \{(C_1)_*, \ldots, (C_n)_*\} \subseteq \varrho_M(\mathcal{C}).$$

By definition of R_M this implies

$$(C_1)_*(C_2)_* \subseteq R_M^2(\mathcal{C}), \ldots, (C_1)_* \cdots (C_n)_* \subseteq R_M^n(\mathcal{C}).$$

(b) We show $(C_1)_* \cdots (C_n)_* \leq_{ss} R_M^i(\mathcal{C})$ by induction on i:

$i = 0$:
Let $C_j \in \mathcal{C}$; then $(C_j)_* \subseteq (C_1)_* \cdots (C_n)_*$ because for all k $(C_k)_* = \{I_p, C_k^*\}$. This also implies

$$(C_j)_* = I_P^{j-1}(C_j)_* I_P^{n-j+1} \subseteq (C_1)_* \cdots (C_n)_*$$

and therefore

$$(C_1)_* \cdots (C_n)_* \leq_{ss} \{(C_1)_*, \ldots, (C_n)_*\}.$$

But for all i $(C_i)_* \leq_{ss} C_i$ and, finally,

$$(C_1)_* \cdots (C_n)_* \leq_{ss} \mathcal{C}.$$

(IH) Assume $(C_1)_* \cdots (C_n)_* \leq_{ss} R_M^i(\mathcal{C})$.

Let $C \in R_M^{i+1}(\mathcal{C}) - R_M^i(\mathcal{C})$; then either $C = EC_j$ or $C = E_*$ for some $E \in R_M^i(\mathcal{C})$, $C_j \in \mathcal{C}$.

$C = EC_j$:
By induction hypothesis we have

$$(C_1)_* \cdots (C_n)_* \leq_{ss} E$$

and therefore

$$(C_1)_* \cdots (C_n)_* C_j \leq_{ss} EC_j$$

Now, for every k, $(C_k)_*$ consists of two Ia-clauses and the commutation lemma can be applied. But then

$$(C_1)_* \cdots (C_n)_* C_j = (C_1)_* \cdots (C_{j-1})_* (C_j)_* C_j (C_{j+1})_* \cdots (C_n)_*.$$

By definition of the $*$-operator we have $(C_j)_* \leq_{ss} (C_j)_* C_j$ and so

$$(C_1)_* \cdots (C_n)_* \leq_{ss} (C_1)_* \cdots (C_n)_* C_j \leq_{ss} EC_j.$$

$C = (E)_*$:
By induction hypothesis we have

$$(C_1)_* \cdots (C_n)_* \leq_{ss} E$$

For every C_i $(C_i)_*$ is a set of Ia-clauses. By definition of the $*$-operator and by the commutation lemma we have $G_* F_* \leq_{ss} (G_* F_*)_*$ for all Ia-clauses G, F and therefore

$$(C_1)_* \cdots (C_n)_* \leq_{ss} ((C_1)_* \cdots (C_n)_*)_*.$$

Moreover $F \leq_{ss} G$ implies $F^* \leq_{ss} G^*$ and

$$(C_1)_* \cdots (C_n)_* \leq_{ss} E^*.$$

This concludes the proof of (b).

It remains to prove (c):
We have already seen that, for all $k \leq n$,

$$(C_1)_* \cdots (C_k)_* \subseteq R^k_M(\mathcal{C}).$$

But we also have $R^k_{M_s}(\mathcal{C}) \leq_{ss} R^k_M(\mathcal{C})$ and finally we see that

$$R^k_{M_s}(\mathcal{C}) \leq_{ss} (C_1)_* \cdots (C_k)_*.$$

$\diamond$

Our next step is to prove termination of R_{M_s} on finite sets of Ia-IIa clauses over a single predicate symbol. To this aim we need the following technical lemma:

Lemma 4.2 *Let $C_1, \ldots, C_n, D_1, \ldots, D_m$ be Ia clauses over a predicate symbol P and*

$$\mathcal{C} \;=\; \{C_1, \ldots, C_n, C_1^{-1}, \ldots, C_n^{-1}\} \cup \bigcup_{i,j \leq m} D_i D_j^{-1}.$$

Then $E \in R^*_M(\mathcal{C})$ implies $E^{-1} \in R^*_M(\mathcal{C})$.

Proof: We prove the result for $R^i_M(\mathcal{C})$ by induction on i.

$i = 0$:
trivial as, by definition, $E^{-1} \in \mathcal{C}$ for every $E \in \mathcal{C}$.

(IH) Assume that for $E \in R^i_M(\mathcal{C})$ we have $E^{-1} \in R^i_M(\mathcal{C})$.

Now let $E \in R^{i+1}_M(\mathcal{C}) - R^i_M(\mathcal{C})$. We distinguish two cases:

(a) $E = E_1 E_2$ for $E_1, E_2 \in R^i_M(\mathcal{C})$.
Then $(E_1 E_2)^{-1} = E_1^{-1} E_2^{-1}$.
By induction hypothesis $E_1^{-1}, E_2^{-1} \in R^*_M(\mathcal{C})$ and, as $R^*_M(\mathcal{C})$ is closed under resolution, $E^{-1} = E_1^{-1} E_2^{-1} \in R^*_M(\mathcal{C})$.

(b) $E \in F_*$ for $F \in R^i_M(\mathcal{C})$.
By definition of the $*$-operator and the inverse of Ia-clauses we have $(F_*)^{-1} = (F^{-1})_*$ and, by induction hypothesis, $F^{-1} \in R^*_M(\mathcal{C})$. But $R^*_M(\mathcal{C})$ is closed under *cycle* and so $(F_*)^{-1} = (F^{-1})_* \in R^*_M(\mathcal{C})$.

$\diamond$

*** *lemma corresponding to representation lemma III* ***

Lemma 4.3 *Let $\mathcal{C}$ be a (finite) set of Ia-IIa clauses over a predicate symbol P . Then there exists a p with $R^*_{M_s}(\mathcal{C}) = R^p_{M_s}(\mathcal{C})$.*

Proof: If $\mathcal{C}$ consists of Ia-clauses only then the result follows from Lemma 4.1. Thus we may assume that there are IIa clauses in $\mathcal{C}$ and, by definition of Ia and IIa, $\mathcal{C}$ is of the form

$$\mathcal{C} = \{C_1, \ldots, C_n\} \cup \{D_1\pi, \ldots, D_m\pi\}$$

where the C_i, D_j are Ia clauses, $m > 0$ and $\pi = \pi_P$. Let $\mathcal{D} = \{D_1, \dots, D_m\}$. Then $\mathcal{D}$ is a set of Ia clauses which plays a role in the syntactic form of $R^*_{M_s}(\mathcal{C})$. Indeed we will show

$$(I) \quad R^*_{M_s}(\mathcal{C}) =_{ss} R^*_{M_s}(\hat{\mathcal{C}}) \cup R^*_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi$$

for

$$\hat{\mathcal{C}} = \{C_1, \dots, C_n, C_1^{-1}, \dots, C_n^{-1}\} \cup \bigcup_{i,j \leq m} D_i D_j^{-1}.$$

Note that $\hat{\mathcal{C}}$ is a set of Ia clauses and so the deductive closure $R^*_{M_s}(\mathcal{C})$ can be expressed via a resolution product of (a deductive closure of) Ia clauses and IIa clauses.

Suppose now that we have already proven (I). Then, by Lemma 4.1, there exists a number q (in fact the number of clauses in $\hat{\mathcal{C}}$) with

$$R^*_{M_s}(\hat{\mathcal{C}}) = R^q_{M_s}(\hat{\mathcal{C}})$$

and therefore

$$R^*_{M_s}(\hat{\mathcal{C}}) \cup R^*_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi = R^q_{M_s}(\hat{\mathcal{C}}) \cup R^q_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi,$$

which is a finite set of Ia-IIa clauses. Now by

$$R^*_{M_s}(\hat{\mathcal{C}}) \leq_{ss} R^q_{M_s}(\hat{\mathcal{C}}) \cup R^q_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi$$

there exists a p such that

$$(II) \quad R^p_{M_s}(\mathcal{C}) \leq_{ss} R^q_{M_s}(\hat{\mathcal{C}}) \cup R^q_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi.$$

On the other hand we have

$$(III) \quad R^q_{M_s}(\hat{\mathcal{C}}) \cup R^q_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi \leq_{ss} R^{p+1}_{M_s}(\mathcal{C}).$$

But (II) and (III) together give $R^p_{M_s}(\mathcal{C}) \leq_{ss} R^{p+1}_{M_s}(\mathcal{C})$ which implies $R^*_{M_s}(\mathcal{C}) = R^p_{M_s}(\mathcal{C})$.

It remains to prove (I):

(a) $R^*_{M_s}(\mathcal{C}) \leq_{ss} R^*_{M_s}(\hat{\mathcal{C}}) \cup R^*_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi$:

Note that $R^*_M(\mathcal{C}) =_{ss} R^*_{M_s}(\mathcal{C})$; therefore it suffices to show by induction on i that

$$R^*_M(\mathcal{C}) \leq_{ss} R^i_{M_s}(\hat{\mathcal{C}}) \cup R^i_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi.$$

$i = 0$:
We have to prove that $R^*_M(\mathcal{C}) \leq_{ss} \hat{\mathcal{C}} \cup \hat{\mathcal{C}}\mathcal{D}\pi$.

$R^*_M(\mathcal{C}) \leq_{ss} \{C_1, \dots C_n\}$ ist trivial by $\{C_1, \dots, C_n\} \subseteq \mathcal{C}$. Now consider the clauses $C_i^{-1} \in \hat{\mathcal{C}}$. Obviously the product of three clauses in $\mathcal{C}$ is in $R^2_M(\mathcal{C})$, in particular $D_1\pi C_i D_1\pi \in R^2_M(\mathcal{C})$. By the commutation lemma we may tranform the product and obtain

$$D_1\pi C_i D_1\pi = C_i^{-1} D_1 D_1^{-1} \in R^2_M(\mathcal{C}).$$

Note that $m \geq 1$ and so $D_1\pi$ is an element of $\mathcal{C}$.
But $D_1 D_1^{-1} \leq_{ss} I_P$ and so $C_i^{-1} D_1 D_1^{-1} \leq_{ss} C_i^{-1}$. This clearly gives

$$R^2_M(\mathcal{C}) \leq_{ss} C_i^{-1}.$$

In order to show $R^*_M(\mathcal{C}) \leq_{ss} \hat{\mathcal{C}}$ it remains to deal with the clauses $D_i D_j^{-1}$. Again the product $D_i\pi D_j\pi$ is in $R_M(\mathcal{C})$ and commutation lemma yields

$$D_i\pi D_j\pi = D_i D_j^{-1}.$$

Puting things together we obtain

$$R^2_M(\mathcal{C}) \leq_{ss} \hat{\mathcal{C}}.$$

Now $\mathcal{D}\pi \subseteq \mathcal{C}$ and so $\hat{\mathcal{C}}\mathcal{D}\pi \subseteq R_M(\hat{\mathcal{C}} \cup \mathcal{C})$. But then, by $R^2_M(\mathcal{C}) \leq_{ss} \hat{\mathcal{C}}$, $R^3_M(\mathcal{C}) \leq_{ss} \hat{\mathcal{C}}\mathcal{D}\pi$. In total we get

$$R^*_M(\mathcal{C}) \leq_{ss} \hat{\mathcal{C}} \cup \hat{\mathcal{C}}\mathcal{D}\pi.$$

(IH) Assume that $R^*_M(\mathcal{C}) \leq_{ss} R^i_{M_s}(\hat{\mathcal{C}}) \cup R^i_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi$.

Let $E \in R^{i+1}_{M_s}(\hat{\mathcal{C}}) \cup R^{i+1}_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi$.
We distinguish two cases:

(1) $E \in R^{i+1}_{M_s}(\hat{\mathcal{C}}) - R^i_{M_s}(\hat{\mathcal{C}})$.
By definition of R_M we have $E \in cycle(R^i_{M_s}(\hat{\mathcal{C}})) \cup Res(R^i_{M_s}(\hat{\mathcal{C}}))$.
If $E \in cycle(R^i_{M_s}(\hat{\mathcal{C}}))$ then there exists a $G \in R^i_{M_s}(\hat{\mathcal{C}})$ with $E \in G_*$. By induction hypothesis there exists a clause H in $R^*_M(\mathcal{C})$ with $H \leq_{ss} G$. But then $H_* \in R^*_M(\mathcal{C})$ and $H_* \leq_{ss} G_*$.

If $E \in Res(R^i_{M_s}(\hat{\mathcal{C}}))$ then $E = G_1 G_2$ for clauses $G_1, G_2 \in R^i_{M_s}(\hat{\mathcal{C}})$. By induction hypothesis there are $H_1, H_2 \in R^*_M(\mathcal{C})$ with $H_1 \leq_{ss} G_1$ and $H_2 \leq_{ss} G_2$. But this implies $H_1 H_2 \leq_{ss} G_1 G_2$. By $H_1 H_2 \in R^*_M(\mathcal{C})$ we thus have $R^*_M(\mathcal{C}) \leq_{ss} E$.

(2) $E \in R^{i+1}_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi$.
From case (1) we know that $R^*_M(\mathcal{C}) \leq_{ss} R^{i+1}_{M_s}(\hat{\mathcal{C}})$. Then there exists a k with

$$R^k_M(\mathcal{C}) \leq_{ss} R^{i+1}_{M_s}(\hat{\mathcal{C}}) \text{ and } R^k_M(\mathcal{C})\mathcal{D}\pi \leq_{ss} R^{i+1}_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi.$$

But by $\mathcal{D}\pi \subseteq \mathcal{C}$ we have $R^k_M(\mathcal{C})\mathcal{D}\pi \subseteq R^{k+1}_M(\mathcal{C})$ and so

$$R^{k+1}_M(\mathcal{C}) \leq_{ss} R^{i+1}_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi.$$

This settles the case $i+1$ and the direction (a).

It remains to show

(b) $R^*_{M_s}(\hat{\mathcal{C}}) \cup R^*_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi \leq_{ss} R^*_{M_s}(\mathcal{C})$.

Let us write $\mathcal{R}$ as abbreviation for the set $R^*_M(\hat{\mathcal{C}}) \cup R^*_M(\hat{\mathcal{C}})\mathcal{D}\pi$.

Obviously (b) follows from the two properties (b1), (b2) below

(b1) $\mathcal{R} \leq_{ss} \mathcal{C}$ and

(b2) $R_M(\mathcal{R}) = \mathcal{R}$

Note that (b2) implies $R^*_M(\mathcal{R}) = \mathcal{R}$ an thus by (b1) $\mathcal{R} \leq_{ss} R^*_M(\mathcal{C})$. But $R^*_{M_s}(\mathcal{C}) =_{ss} R^*_M(\mathcal{C})$ and $\mathcal{R} =_{ss} R^*_{M_s}(\hat{\mathcal{C}}) \cup R^*_{M_s}(\hat{\mathcal{C}})\mathcal{D}\pi$.

(b1) is trivial as $\hat{\mathcal{C}} \cup \mathcal{D}\pi \subseteq \mathcal{R}$.
So it remains to prove (b2). By definition of R_M we have

$$R_M(\mathcal{R}) = \mathcal{R} \cup Res(\mathcal{R}) \cup cycle(\mathcal{R}).$$

So we have to prove $Res(\mathcal{R}) \subseteq \mathcal{R}$ and $cycle(\mathcal{R}) \subseteq \mathcal{R}$.

(b21) $Res(\mathcal{R}) \subseteq \mathcal{R}$:
Let $E_1, E_2 \in \mathcal{R}$. If E_1 and E_2 are both in $R^*_M(\hat{\mathcal{C}})$ then clearly $E_1E_2 \in R^*_M(\hat{\mathcal{C}}) \subseteq \mathcal{R}$.

So let $E_1 \in R^*_M(\hat{\mathcal{C}})$ and $E_2 \in R^*_M(\hat{\mathcal{C}})\mathcal{D}\pi$. Then $E_2 = ED_i\pi$ for an $E \in R^*_M(\hat{\mathcal{C}})$ and a $D_i \in \mathcal{D}$. But then

$$E_1E_2 = E_1(ED_i\pi) = (E_1E)D_i\pi \in R^*_M(\hat{\mathcal{C}})\mathcal{D}\pi$$

and $E_1E_2 \in \mathcal{R}$. Now consider the product E_2E_1; note that E_1 and E_2 do not commute in general as E_2 is a IIa-clause. By the commutation lemma we have

$$E_2E_1 = ED_i\pi E_1 = EE_1^{-1}D_i\pi.$$

By Lemma 4.2 we get $E_1^{-1} \in R^*_M(\hat{\mathcal{C}})$ and thus also $EE_1^{-1} \in R^*_M(\hat{\mathcal{C}})$ and $E_2E_1 \in R^*_M(\hat{\mathcal{C}})\mathcal{D}\pi$.

If E_1 and E_2 are both in $R^*_M(\hat{\mathcal{C}})\mathcal{D}\pi$ then there are $E, F \in R^*_M(\hat{\mathcal{C}})$ and $D_i, D_j \in \mathcal{D}$ with $E_1 = ED_i\pi$ and $E_2 = FD_j\pi$. Again the commutation lemma yields

$$E_1E_2 = ED_i\pi FD_j\pi = EF^{-1}D_iD_j^{-1}.$$

Now $D_iD_j^{-1} \in \hat{\mathcal{C}}$, and by Lemma 4.2 $F^{-1} \in R^*_M(\hat{\mathcal{C}})$. This eventually gives

$$E_1E_2 \in R^*_M(\hat{\mathcal{C}}) \subseteq \mathcal{R}$$

Note that in the last case $E_1E_2 = E_2E_1$ as the resolvent can be written as a product of Ia clauses.

(b22) $cycle(\mathcal{R}) \subseteq \mathcal{R}$:
Let $E \in \mathcal{R}$; we compute E_*.
If $E \in R^*_M(\hat{\mathcal{C}})$ then also $E_* \subseteq R^*_M(\hat{\mathcal{C}}) \subseteq \mathcal{R}$.
If $E \in R^*_M(\hat{\mathcal{C}})\mathcal{D}\pi$ then $E = FD_i\pi$ for some $F \in R^*_M(\hat{\mathcal{C}})$ and $D_i \in \mathcal{D}$. So $E_* = (FD_i\pi)_*$. Now

$$(FD_i\pi)_* = (FF^{-1})_*(D_iD_i^{-1})_* \cup (FF^{-1})_*(D_iD_i^{-1})_*FD_i\pi.$$

It is enough to show that

$$(FF^{-1})_*(D_iD_i^{-1})_* \subseteq R^*_M(\hat{\mathcal{C}})$$

because then also

$$(FF^{-1})_*(D_iD_i^{-1})_*FD_i\pi \subseteq R^*_M(\hat{\mathcal{C}})\mathcal{D}\pi$$

and $(FD_i\pi)_* \subseteq \mathcal{R}$.

By Lemma 4.2 we have $F^{-1} \in R^*_M(\hat{\mathcal{C}})$ and therefore $FF^{-1} \in R^*_M(\hat{\mathcal{C}})$. But R_M is closed under *cycle* and so $(FF^{-1})_* \subseteq R^*_M(\hat{\mathcal{C}})$ (note that FF^{-1} may be different from I_P if there are meta-terms in F). $D_iD_i^{-1} \in \hat{\mathcal{C}}$ and thus $(D_iD_i^{-1})_* \subseteq R_M(\hat{\mathcal{C}})$, finally resulting in

$$(FF^{-1})_*(D_iD_i^{-1})_* \subseteq R^*_M(\hat{\mathcal{C}}).$$

This concludes the proof of (b22). ◇

*** *lemma corresponding to the representation theorem* ***

Lemma 4.4 *Let $\mathcal{C}$ be a (finite) set of Ia-IIa clauses. Then $R^*_{M_s}(\mathcal{C})$ is finite.*

Proof: By induction on $|\mathcal{C}|$.
(IB) $|\mathcal{C}| = 1$:
Then $\mathcal{C} = \{C\}$ where C is a Ia or a IIa clause. If C if of type $P \to Q$ for different predicate symbols P and Q then C is not a cycle clause. Therefore $Res(\{C\}) = \emptyset$ and $cycle(\mathcal{C}) = \{C, I_P, I_Q\}$ (note that, as all clauses are rules, ϱ_H yields the empty set anyway). Thus trivially $R^*_{M_s}(\mathcal{C})$ is finite. If C is of type $P \to P$ then $R^*_{M_s}(\mathcal{C})$ is finite by Lemma 4.3.

(IH) Assume that for all sets $\mathcal{C}$ of type Ia-IIa clauses with $|\mathcal{C}| \leq n$ $R^*_{M_s}(\mathcal{C})$ is finite.

case $n+1$:
Let $\mathcal{C} = \mathcal{D} \cup \{C_{n+1}\}$, where $\mathcal{D} = \{C_1, \ldots, C_n\}$. By induction hypothesis there exists a finite set of Ia-IIa clauses $\mathcal{E}$ with $\mathcal{E} = R^*_{M_s}(\mathcal{D})$.
We distinguish the following cases:

(a) C_{n+1} is of type $P \to Q$ where neither P nor Q occurs in in $\mathcal{D}$.

In this case $R^*_{M_s}(\mathcal{C}) = R^*_{M_s}(\mathcal{D}) \cup R^*_{M_s}(\{C_{n+1}\})$. Then, by induction hypothesis, $R^*_{M_s}(\mathcal{C}) = \mathcal{E} \cup \mathcal{F}$ for some finite set of Ia-IIa clauses $\mathcal{F}$.

(b) C_{n+1} is of type $P \to Q$ where P occurs in $\mathcal{D}$ but Q does not.

We show that

$$R^*_{M_s}(\mathcal{C}) \quad =_{ss} \quad \mathcal{E} \cup \mathcal{E}\{C_{n+1}\}_*.$$

Clearly this subsumption equivalence yields a number j with $R^*_{M_s}(\mathcal{C}) = R^j_{M_s}(\mathcal{C})$.

(b1) $R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{E} \cup \mathcal{E}\{C_{n+1}\}_*$:

Note that $\mathcal{E} = R^*_{M_s}(\mathcal{D})$ and $\mathcal{D} \subseteq \mathcal{C}$, so $R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{E}$ is trivial. But $C_{n+1} \in \mathcal{C} \subseteq R^*_{M_s}(\mathcal{C})$ and by $R^*_{M_s}(\mathcal{C}) \leq_{ss} cycle(R^*_{M_s}(\mathcal{C}))$ we obtain $R^*_{M_s}(\mathcal{C}) \leq_{ss} \{C_{n+1}\}_*$. Moreover

$$R^*_{M_s}(\mathcal{C}) \leq_{ss} R^*_{M_s}(\mathcal{C})R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{E}\{C_{n+1}\}_*.$$

(b2) $\mathcal{E} \cup \mathcal{E}\{C_{n+1}\}_* \leq_{ss} R^*_{M_s}(\mathcal{C})$:

We prove the relation for $R^i_{M_s}(\mathcal{C})$ by induction on i.

$i = 0$:

$\mathcal{E} \leq_{ss} \mathcal{D}$, $\{C_{n+1}\}_* \leq_{ss} C_{n+1}$ and $I_P \in \mathcal{E}$; thus clearly

$\mathcal{E} \cup \mathcal{E}\{C_{n+1}\}_* \leq_{ss} \mathcal{C}$.

(IH-b2) $\mathcal{E} \cup \mathcal{E}\{C_{n+1}\}_* \leq_{ss} R^i_{M_s}(\mathcal{C})$:

We hav to prove

$$\mathcal{E} \cup \mathcal{E}\{C_{n+1}\}_* \quad \leq_{ss} \quad Res(R^i_{M_s}(\mathcal{C})) \cup cycle(R^i_{M_s}(\mathcal{C})).$$

Let $CD \in Res(R^i_{M_s}(\mathcal{C}))$ and $E_1, E_2 \in \mathcal{E} \cup \mathcal{E}\{C_{n+1}\}_*$ s.t. $E_1 \leq_{ss} C_1$ and $E_2 \leq_{ss} C_2$. Then also E_1E_2 is defined and $E_1E_2 \leq_{ss} CD$. So it is sufficient to prove

$$\mathcal{E} \cup \mathcal{E}\{C_{n+1}\}_* \quad \leq_{ss} \quad E_1E_2.$$

If $E_1, E_2 \in \mathcal{E}$ then, by $\mathcal{E} \leq_{ss} \mathcal{E}\mathcal{E}$, we have $\mathcal{E} \leq_{ss} E_1E_2$.

Now let $E_1 \in \mathcal{E}$, $E_2 \in \mathcal{E}\{C_{n+1}\}_*$. Then $E_1E_2 \in \mathcal{E}\mathcal{E}\{C_{n+1}\}_*$ and, by $\mathcal{E}\{C_{n+1}\}_* \leq_{ss} \mathcal{E}\mathcal{E}\{C_{n+1}\}_*$ we get $\mathcal{E}\{C_{n+1}\}_* \leq_{ss} E_1E_2$. Note that C is of type $P \to Q$ where Q is not in $\mathcal{D}$ and thus E_2E_1 is only defined if $E_2 \in \mathcal{E}$. This shows

$$\mathcal{E} \cup \mathcal{E}\{C_{n+1}\}_* \quad \leq_{ss} \quad Res(R^i_{M_s}(\mathcal{C})).$$

Now let $C_* \subseteq cycle(R^i_{M_s}(\mathcal{C}))$. By (IH-b2) there exists an $E \in \mathcal{E} \cup \mathcal{E}\{C_{n+1}\}_*$ with $E \leq_{ss} C$. Then also $E_* \leq_{ss} C_*$. So it is enough to show that $\mathcal{E} \cup \mathcal{E}\{C_{n+1}\}_* \leq_{ss} E_*$.

If $E \in \mathcal{E}$ then, by $\mathcal{E} \leq_{ss} cycle(\mathcal{E})$, $\mathcal{E} \leq_{ss} E_*$. If $E \in \mathcal{E}\{C_{n+1}\}_* - \mathcal{E}$ then E is not a cycle clause and

$E_* \subseteq \{FC_{n+1}, I_P, I_Q, I_R\}$

for some predicate symbol R in $\mathcal{E}$ and $F \in \mathcal{E}$. We see that, also in this case $\mathcal{E} \cup \mathcal{E}\{C_{n+1}\}_* \leq_{ss} E_*$ and so $\mathcal{E} \cup \mathcal{E}\{C_{n+1}\}_* \leq_{ss} cycle(R^i_{M_s}(\mathcal{C}))$. This concludes the proof of (b).

(c) C_{n+1} is of type $P \to Q$ where Q occurs in $\mathcal{D}$ but P does not.

This case is completely symmetric to (b).

(d) C_{n+1} is of type $P \to Q$ where both P and Q occur in $\mathcal{D}$.

For any set of Ia-IIa clauses $\mathcal{F}$ we write $\mathcal{F}_{P\to Q}$ for the subset of clauses in $\mathcal{F}$ having type $P \to Q$. Moreover we define

$$\begin{aligned} \mathcal{F}_{P\to} &= \bigcup\{\mathcal{F}_{P\to Q} \mid Q \in \mathrm{PS}(\mathcal{F})\}, \\ \mathcal{F}_{\to P} &= \bigcup\{\mathcal{F}_{Q\to P} \mid Q \in \mathrm{PS}(\mathcal{F})\}. \end{aligned}$$

Now C_{n+1} is of type $P \to Q$ and thus the set of clauses $C_{n+1}\mathcal{E}_{Q\to P}$ is of type $P \to P$. By Lemma 4.3 there exists a finite set of Ia-IIa clauses $\mathcal{X}_P$ s.t.

$$\mathcal{X}_P = R^*_{M_s}(C_{n+1}\mathcal{E}_{Q\to P}).$$

We define

$$\mathcal{F} \quad = \quad \mathcal{E}_{\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to} \cup \mathcal{E}.$$

Then $\mathcal{F}$ is a finite set of Ia-IIa clauses. Our aim is to show

$$(*) \quad R^*_{M_s}(\mathcal{C}) =_{ss} \mathcal{F}.$$

Then by definition of R_{M_s} there exists a nuber p with $R^*_{M_s}(\mathcal{C}) = R^p_{M_s}(\mathcal{C})$, which is exactly what we want to show. So it remains to prove $(*)$.

(d1) $R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{F}$.

Note that for all sets of clauses $\mathcal{C}_1, \mathcal{C}_2$, $R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{C}_1$, $R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{C}_2$ implies $R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{C}_1\mathcal{C}_2$. Using this property we can reduce $R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{F}$ to

1. $R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{E}$,
2. $R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{E}_{\to P}$,
3. $R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{X}_P$,
4. $R^*_{M_s}(\mathcal{C}) \leq_{ss} C_{n+1}$ and
5. $R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{E}_{Q\to}$.

The cases 1,2,4 and 5 are trivial. But 4 and $R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{E}_{Q\to P}$ imply $R^*_{M_s}(\mathcal{C}) \leq_{ss} C_{n+1}\mathcal{E}_{Q\to P}$ and, as $R^*_{M_s}(\mathcal{C})$ is deductively closed, we have 3. So the case (d1) is settled.

(d2) $\mathcal{F} \leq_{ss} R^*_{M_s}(\mathcal{C})$.

The proof of (d2) can be reduced to proofs of (I), (II) and (III) below.

(I) $\mathcal{F} \leq_{ss} \mathcal{C}$,

(II) $\mathcal{F} \leq_{ss} Res(\mathcal{F})$,

(III) $\mathcal{F} \leq_{ss} cycle(\mathcal{F})$.

(II) + (III) guarantee that $\mathcal{F} \leq_{ss} R^*_{M_s}(\mathcal{F})$; together with (I) this gives $\mathcal{F} \leq_{ss} R^*_{M_s}(\mathcal{C})$.

(I) is easy to show:

Clearly $\mathcal{E} \leq_{ss} \mathcal{D}$ and thus $\mathcal{F} \leq_{ss} \mathcal{D}$. Moreover $\mathcal{E}_{P\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to Q} \subseteq \mathcal{F}$, $I_P \in \mathcal{E}_{P\to P} \cap \mathcal{X}_P$ and $I_Q \in \mathcal{E}_{Q\to Q}$. But $I_P C_{n+1} I_Q = C_{n+1}$ and therefore $\mathcal{F} \leq_{ss} C_{n+1}$.

We prove (II):

Let $E_1, E_2 \in \mathcal{F}$. We have to show that $\mathcal{F} \leq_{ss} E_1E_2$. If $E_1, E_2 \in \mathcal{E}$ then

$\mathcal{F} \leq_{ss} \mathcal{E} \leq_{ss} E_1E_2$.

So let us assume that that $E_1 \in \mathcal{E}$ and $E_2 \in \mathcal{E}_{\to P}\mathcal{X}_p C_{n+1}\mathcal{E}_{Q\to}$. Then there are predicate symbols R, S, T with

$E_1 \in \mathcal{E}_{R\to S}$, $E_2 \in \mathcal{E}_{S\to T}$.

Now

$$E_1E_2 \quad \in \quad \mathcal{E}_{R\to S}\mathcal{E}_{S\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to T}$$

and

$$\mathcal{E}_{R\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to T} \quad \leq_{ss} \quad \mathcal{E}_{R\to S}\mathcal{E}_{S\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to T}.$$

But also

$$\mathcal{E}_{R\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to T} \quad \subseteq \quad \mathcal{E}_{\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to}.$$

But $\mathcal{E}_{\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to}$ is a subset of $\mathcal{F}$ and therefore $\mathcal{F} \leq_{ss} E_1E_2$.

The case $E_1 \in \mathcal{E}_{\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to}$, $E_2 \in E$ is completely symmetric.

Now let $E_1, E_2 \in \mathcal{E}_{\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to}$.
The resolution E_1E_2 is only defined if E_1 is of some type $R \to S$ and E_2 of type $S \to T$ and therefore

$$E_1E_2 \in \mathcal{E}_{R\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to S}\mathcal{E}_{S\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to T}.$$

Now

$\mathcal{E}_{Q\to P} \leq_{ss} \mathcal{E}_{Q\to S}\mathcal{E}_{S\to P}$

and therefore

$$\mathcal{E}_{R\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to T} \leq_{ss} E_1E_2.$$

But

$$\begin{aligned}\mathcal{X}_P &= R^*_{M_s}(C_{n+1}\mathcal{E}_{Q\to P}) \text{ and so}\\ \mathcal{X}_P &\le_{ss} \mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to P}.\end{aligned}$$

This immediately yields

$$\mathcal{E}_{R\to P}\mathcal{X}_P\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to T} \le_{ss} \mathcal{E}_{R\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to T}$$

and by $\mathcal{X}_P \le_{ss} \mathcal{X}_P\mathcal{X}_P$

$$\mathcal{E}_{R\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to T} \le_{ss} E_1E_2.$$

But $\mathcal{E}_{R\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to T}$ is a subset of $\mathcal{E}_{\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to}$ which, in turn, is a subset of $\mathcal{F}$. So $\mathcal{F}$ subsumes E_1E_2 and (II) is proved.

(III): We have to show $\mathcal{F} \le_{ss} \mathit{cycle}(\mathcal{F})$.

Let E be a clause in $\mathcal{F}$. If E is in $\mathcal{E}$ then clearly $\mathcal{E} \le_{ss} \mathit{cycle}(E) = E_*$ and $\mathcal{F} subsumes E_*$. So the only interesting case is

$E \in \mathcal{E}_{R\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to R}$

for some predicate symbol R. Then $E = E_1XC_{n+1}E_2$ for $E_1 \in \mathcal{E}_{R\to P}$, $X \in \mathcal{X}_P$ and $E_2 \in \mathcal{E}_{Q\to R}$. Then

$$EE = E_1XC_{n+1}E_2E_1XC_{n+1}E_2.$$

Note that $E : E_2E_1$ is of type $Q \to P$ and $C_{n+1}EX$ is of type $P \to P$. The cycle operator on E thus leads to

$$E_* = E_1X(C_{n+1}EX)_*C_{n+1}E_2 \cup \{I_R\}.$$

We write E^* for $E_1X(C_{n+1}EX)_*C_{n+1}E_2$.

Now $X \in \mathcal{X}_P$ and $E \in \mathcal{E}_{Q\to P}$ implies $\mathcal{X}_P \le_{ss} C_{n+1}EX$ by definition of $\mathcal{X}_P$. Let Y be an element in $\mathcal{X}_P$ with $Y \le_{ss} C_{n+1}EX$. Then also $Y_* \le_{ss} (C_{n+1}EX)_*$. But Y_* is a subset of X_P and $\mathcal{X}_P \le_{ss} X\mathcal{X}_P$; therefore we obtain

$$\begin{aligned}E_1\mathcal{X}_P C_{n+1}E_2 &\le_{ss} E^* \text{ and more general}\\ \mathcal{E}_{R\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to R} &\le_{ss} E^*.\end{aligned}$$

But $\mathcal{E}_{R\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to R} \subseteq \mathcal{E}_{\to P}\mathcal{X}_P C_{n+1}\mathcal{E}_{Q\to}$ which is a subset of $\mathcal{F}$. Moreover $\mathcal{F} \le_{ss} I_R$. Putting things together we finally obtain

$\mathcal{F} \leq_{ss} E_*$.

This concludes the proof of (III).

$\diamond$

The following lemma shows that, in computing a deductive closure of rules and facts, it is sufficient to compute the closure of rules and apply the resulting meta-clauses to facts afterwards.

Proposition 4.1 *Let $\mathcal{C}$ be a set of Ia-IIa clauses and $\mathcal{P}$ be a set of facts. Then*

$$R^*_{M_s}(\mathcal{C} \cup \mathcal{P}) \quad =_{ss} \quad \mathcal{P}R^*_{M_s}(\mathcal{C}) \cup R^*_{M_s}(\mathcal{C}).$$

Proof: The direction

$R^*_{M_s}(\mathcal{C} \cup \mathcal{P}) \leq_{ss} \mathcal{P}R^*_{M_s}(\mathcal{C}) \cup R^*_{M_s}(\mathcal{C})$

is trivial as $\mathcal{P}R^*_{M_s}(\mathcal{C}) \cup R^*_{M_s}(\mathcal{C})$ is a subset of $R^*_{M}(\mathcal{C})$. Thus it remains to show that

$$\mathcal{P}R^*_{M_s}(\mathcal{C}) \cup R^*_{M_s}(\mathcal{C}) \quad \leq_{ss} \quad R^*_{M_s}(\mathcal{C} \cup \mathcal{P}).$$

But this property follows essentially from the invariance of $\mathcal{P}R^*_{M_s}(\mathcal{C}) \cup R^*_{M_s}(\mathcal{C})$ under R_M; more precisely we have to prove the following four points:

(1) $\mathcal{P}R^*_{M_s}(\mathcal{C}) \cup R^*_{M_s}(\mathcal{C}) \leq_{ss} \mathcal{C} \cup \mathcal{P}$,

(2) $\mathcal{P}R^*_{M_s}(\mathcal{C}) \cup R^*_{M_s}(\mathcal{C})$ is closed under ϱ_H,

(3) $(\mathcal{P}R^*_{M_s}(\mathcal{C}) \cup R^*_{M_s}(\mathcal{C}))_r$ is closed under Res,

(4) $(\mathcal{P}R^*_{M_s}(\mathcal{C}) \cup R^*_{M_s}(\mathcal{C}))_r$ is closed under *cycle*.

(1) is trivial.

(3) is trivial too as the rule clauses in $\mathcal{P}R^*_{M_s}(\mathcal{C}) \cup R^*_{M_s}(\mathcal{C})$ are just $R^*_{M_s}(\mathcal{C})$ an $R^*_{M_s}(\mathcal{C})$ is closed under Res by definition.

In the same way also (4) is trivial as *cycle* just applies to $R^*_{M_s}(\mathcal{C})$.

The only interesting case is (2):
There are no negative clauses in $\mathcal{P}R^*_{M_s}(\mathcal{C}) \cup R^*_{M_s}(\mathcal{C})$, so ϱ_H only defines simple resolutions among facts and rules. Now let

$P \in \mathcal{P}R^*_{M_s}(\mathcal{C})$ and $\neg P' \vee Q \in R^*_{M_s}(\mathcal{C})$.

Then $C : P \circ \neg P' \vee Q \in \mathcal{P}R^*_{M_s}(\mathcal{C})R^*_{M_s}(\mathcal{C})$. But

$$\mathcal{P}R^*_{M_s}(\mathcal{C}) \quad \leq_{ss} \quad \mathcal{P}R^*_{M_s}(\mathcal{C})R^*_{M_s}(\mathcal{C}),$$

and so $\mathcal{P}R^*_{M_s}(\mathcal{C}) \leq_{ss} C$. $\diamond$

Remark:
Proposition 4.1 holds for all sets of facts, independent of their syntactic structure.

Theorem 4.2 (termination theorem) *Let $\mathcal{C}$ be a set of Ia-IIa clauses, $\mathcal{P}$ be a set of facts and $\mathcal{G}$ be a set of goals. Then $R^*_{M_s}(\mathcal{C} \cup \mathcal{P} \cup \mathcal{G})$ is finite.*

Proof: (a) $\mathcal{C} \cup \mathcal{P} \cup \mathcal{G}$ is unsatisfiable:
Then, by the completeness of R_{M_s}, there exists an i with $\Box \in R^i_{M_s}(\mathcal{C} \cup \mathcal{P} \cup \mathcal{G})$. But then, by definition of R_{M_s},

$$\begin{aligned} R^{i+1}_{M_s}(\mathcal{C} \cup \mathcal{P} \cup \mathcal{G}) &= R^i_{M_s}(\mathcal{C} \cup \mathcal{P} \cup \mathcal{G}), \\ R^*_{M_s}(\mathcal{C} \cup \mathcal{P} \cup \mathcal{G}) &= R^i_{M_s}(\mathcal{C} \cup \mathcal{P} \cup \mathcal{G}). \end{aligned}$$

(b) $\mathcal{C} \cup \mathcal{P} \cup \mathcal{G}$ is satisfiable.
By definition of R_M we have $R_M(\mathcal{D}) = \mathcal{D} \cup \varrho_M(\mathcal{D})$ where

$$\varrho_M(\mathcal{D}) = \varrho_H(\mathcal{D}) \cup Res(\mathcal{D}_r) \cup cycle(\mathcal{D}_r).$$

Note that $\varrho_H(\mathcal{P} \cup \mathcal{G}) = \emptyset$: As ϱ_H is the hyperresolution operator, $\mathcal{P}$ is a set of facts and $\mathcal{G}$ consists of strictly negative clauses, $\varrho_H(\mathcal{P} \cup \mathcal{G}) \neq \emptyset$ implies $\varrho_H(\mathcal{P} \cup \mathcal{G}) = \{\Box\}$; but this is impossible by the correctness of resolution and by satisfiability of $\mathcal{C} \cup \mathcal{P} \cup \mathcal{G}$. So there are no inferences whatsoever on the set $\mathcal{G}$ and we obtain

$$R^*_{M_s}(\mathcal{C} \cup \mathcal{P} \cup \mathcal{G}) \quad = \quad R^*_{M_s}(\mathcal{C} \cup \mathcal{P}) \cup \mathcal{G}.$$

By Proposition 4.1 we get

$$R^*_{M_s}(\mathcal{C} \cup \mathcal{P}) = \mathcal{P}R^*_{M_s}(\mathcal{C}) \cup R^*_{M_s}(\mathcal{C}).$$

By Lemma 4.4 $R^*_{M_s}(\mathcal{C})$ is finite and therefore $R^*_{M_s}(\mathcal{C} \cup \mathcal{P} \cup \mathcal{G})$ is a finite set of clauses.

$\diamond$

Corollary 4.1 R_{M_s} *decides the class of Krom-Horn clauses $\mathcal{C}$, where the rules are Ia-IIa clauses and facts and goals are unrestricted.*

Proof: obvious.

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