Endomorphisms of ω-categorical structures

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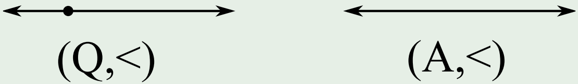

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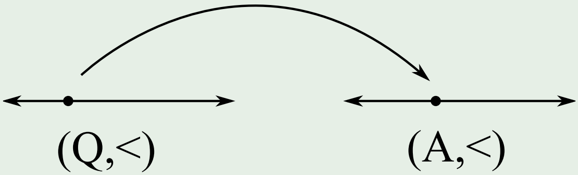

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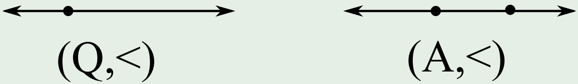

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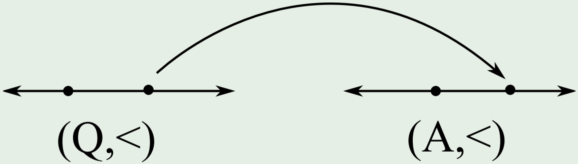

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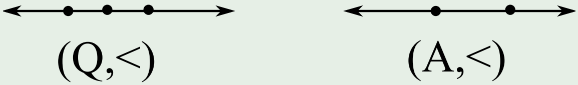

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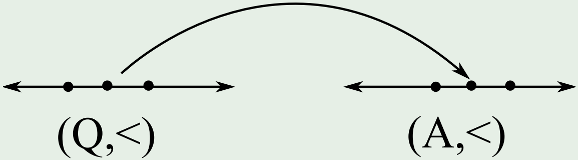

First-order interdefinability

A countable structure **A** is ω-categorical iff its automorphism group is an oligomorphic permutation group:
Aut(**A**) has finitely many orbits on A^n, for all $n \in \mathbb{N}$.

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Theorem (Ryll-Nardzewski)

Let **A** and **B** be two ω-categorical structures on the same domain. Then Aut(**A**) = Aut(**B**) as permutation groups iff **A** and **B** are first-order interdefinable.

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The partial map:

$$I : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$$
$$(x, y) \mapsto \frac{x}{y}$$

is an interpretation.

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for sufficiently large k.

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Theorem (Ahlbrandt+Ziegler)

Let $\mathbf{A}$ and $\mathbf{B}$ be two ω-categorical structures. Then $\mathrm{Aut}(\mathbf{A}) \cong^T \mathrm{Aut}(\mathbf{B})$ iff $\mathbf{A}$ and $\mathbf{B}$ are first-order bi-interpretable.

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- Can we „reconstruct“ $\mathbf{A}$ from the algebraic structure of $\mathrm{Aut}(\mathbf{A})$?
- Can we reconstruct the topology of a closed oligomorphic permutation group from its algebraic structure?

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	acting on A	topologically
Aut(**A**)	first-order interdefinable	first-order bi-interpretable
End(**A**)	*positive existentially* interdefinable	*positive existentially* bi-interpretable
Pol(**A**)	*primitive positive* interdefinable	*primitive positive* bi-interpretable

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No!

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A structure $\mathbf{A}$ has the small index property if every subgroup in $\mathrm{Aut}(\mathbf{A})$ with countable index is open.

If $\mathbf{A}$ has the small index property, then *every* isomorphism

$$I : \mathrm{Aut}(\mathbf{A}) \to \mathrm{Aut}(\mathbf{B})$$

is a homeomorphism.

The small index property

Example (Hrushovski)

There is an ω-categorical structure without the small index property.

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Proof idea: We look at all finite sets with n-ary relations $R_1^n(\bar{x})$, $R_2^n(\bar{x})$ that partition the n-tuples, for all $n \in \mathbb{N}$. This gives us a Fraïssé-class.

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Let $\mathbf{F}' = (F, (R_i^n)_{i,n})$ be the Fraïssé-limit of this class.

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Let $E^n(\bar{x}, \bar{y}) :\Leftrightarrow R_1^n(\bar{x}) \leftrightarrow R_1^n(\bar{y})$, and $\mathbf{F} = (F, (E_n)_{n \in \mathbb{N}})$.

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Hence **F** has not the small index property. □

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For every separable profinite group H there are oligomorphic permutation groups Φ and Σ, such that $\Sigma/\Phi \cong^T H$.

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Strategy:

Find two profinite groups that are isomorphic, but not topologically isomorphic.

A pathological profinite group

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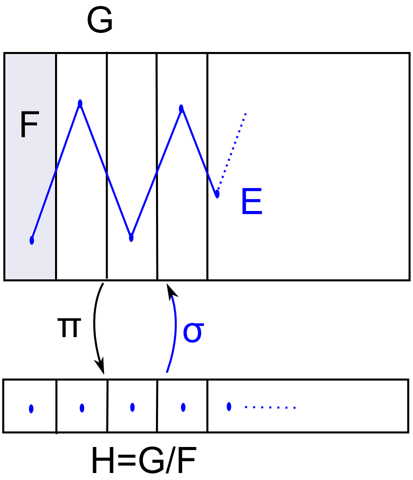

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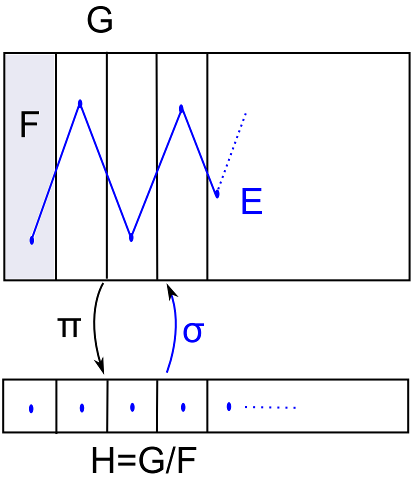

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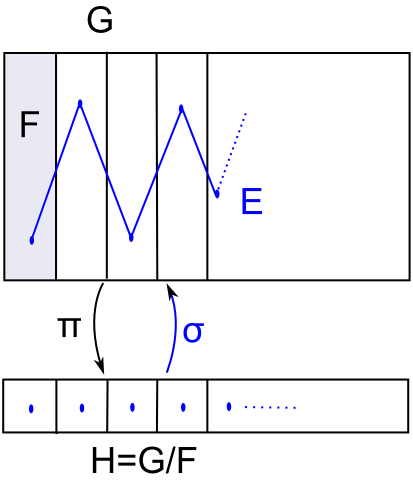

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- Take closed oligomorphic groups Σ, Φ on a set A, such that $\Sigma/\Phi \cong^T H$
- Expand A with C:
  - The action of Σ on $A \cup C$ is not continuous.
  - The closure of Σ in $\mathrm{Sym}(A \cup C)$ gives us an oligomorphic Γ with

$$\Gamma \cong \Sigma \times F$$

but not as topological groups!

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We can adapt the proof for the endomorphism monoids.

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Again let Λ act on $A \cup C$ and let Ω be its closure. Then Ω is oligomorphic and

$$\Omega \cong \Lambda \times F$$

as monoids, but not as topological monoids!

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The isomorphism $\Omega \to \Lambda \times F$ naturally extends to a clone isomorphism

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But these clones are not topologically isomorphic, since Ω and $\Lambda \times F$ are not.

Thank you!