Completing labelled graphs to metric spaces

Michael Kompatscher
May 30, 2017

Joint work with
Andrés Aranda, David Bradley-Williams, Jan Hubička, Miltiadis Karamanlis, Matěj Konečný, Micheal Pawliuk
Completing labelled graphs to metric spaces

Michael Kompatscher
May 30, 2017

Joint work with
Andrés Aranda, David Bradley-Williams, Jan Hubička, Miltiadis Karamanlis, Matěj Konečný, Micheal Pawliuk
Ramsey Theory DocCourse

September - December 2016
Charles University in Prague

D. Barbosová
D. Coulson
T. Evans
J. Fox
R. Morris
L. Nguyen Van Thé
M. Pinsker
V. Rödl
S. Solecki
J. Sosnowski
S. Todorcevic
B. Weiss
1. Introduction

2. Cherlin’s census of metrically homogeneous graphs

3. The extension property for partial automorphisms (EPPA)

4. Results
Introduction
**Edge-labelled graphs**

Every metric space can be regarded as an edge-labelled, complete graph:

\[(G, d)\]

\[(G, \bar{d})\]

**Questions**

Given an edge-labelled graph \((G, d)\):

- Can \((G, d)\) be completed to a metric space \((G, \bar{d})\)?
- Is there an algorithm completing \((G, d)\)?
- Are there completion algorithms that preserves nice properties of the graph?

We consider metric spaces with distances 1, 2, \ldots, \delta.
By the triangle inequality, \((G, \bar{d})\) is not a metric space, if it contains a non-metric triangle.

More general, \((G, \bar{d})\) is not a metric space, if it contains a non-metric cycle.

Non-metric cycles are obstacles, i.e. as subgraphs of \((G, d)\) they prevent completion. In our setting there are only finite non-metric cycles.
Path completion

For a given edge-labelled graph \((G, d)\), and non-edge \((x, y)\), let \(d^+(x, y)\) be the minimal path length between \(x\) any \(y\):

\[
d^+(x, y) := \min(\delta, \min\{\sum_{i=0}^{k} d(u_i, u_{i+1}) : u_0 = x, u_{k+1} = y\})
\]

For all existing edges: \(d^+(x, y) = d(x, y)\).

Let us call \((G, d^+)\) the path completion of \((G, d)\).
Lemma
The following are equivalent:

- $(G, d^+)$ is a metric space
- $(G, d)$ can be completed to a metric space
- $(G, d)$ contains no non-metric cycles

Proof.
Assume that there is a non-metric triangle in $(G, d^+)$, i.e.
\[ d^+(u, v) + d^+(v, w) < d^+(u, w). \]

Then $d^+(u, w) = d(u, w)$ and there was already a non-metric cycle in $(G, d)$.  \[ \square \]
Lemma
The path completion maximizes distances: For \((G, d)\), let \((G, \bar{d})\) be any completion to a metric space. Then

\[
\bar{d}(x, y) \leq d^+(x, y) \leq \delta
\]

Proof.
Assume \(\bar{d}(x, y) > d^+(x, y)\). Then, \(\bar{d}(x, y)\) and the path witnessing \(d^+(x, y)\) form a non-metric cycle in \((G, \bar{d})\). \(\square\)
Lemma
For all edge-labelled \((G, d)\) we have \(\text{Aut}(G, d) \leq \text{Aut}(G, d^+)\)

Proof.
Let \(f \in \text{Aut}(G, d)\).
Note that \(u_0, u_1, \ldots, u_l\) is a path from \(x\) to \(y\), if and only if \(f(u_0), f(u_1), \ldots, f(u_l)\) is a path from \(f(x)\) to \(f(y)\).
So \(d^+(f(x), f(y)) = d^+(x, y)\), thus \(f \in \text{Aut}(G, d^+)\).

In general, completions do not have to preserve automorphisms.
Path completion implies amalgamation

Definition
We say that a class $C$ of structures has the **amalgamation property** if

$$\forall A, B_1, B_2 \in C, \forall \alpha_i : A \rightarrow B_i \exists C \in C, \beta_i : B_i \rightarrow C: \beta_2 \alpha_2 = \beta_1 \alpha_1.$$  

The class of metric spaces (distances $1, 2, \ldots, \delta$) has a canonical amalgamation:

Take $C = B_1 \cup B_2$ and form its path completions.
Let \((G, d^+)\) be the path completion of \((G, d)\):

- \((G, d^+)\) is metric if and only if \((G, d)\) does not contain non-metric cycles
- \(\text{Aut}(G, d) \leq \text{Aut}(G, d^+)\)
- \(d^+(x, y)\) is the maximal possible distance between \(x\) and \(y\)
- gives us a canonical amalgamation on metric spaces
Cherlin’s census of metrically homogeneous graphs
Cherlin’s census of metrically homogeneous graphs

Figure 1: Gregory Cherlin, likes to classify things
In ongoing work, Cherlin gives a (probably) complete list of amalgamation classes of metric spaces that contain all geodesics, i.e. triangles \((a, b, |b - a|)\).
Cherlin ’16

For parameters $(\delta, K_1, K_2, C_0, C_1)$ let $A_{K_1,K_2,C_0,C_1}^\delta$ be the class of metric spaces of diameter $\delta$ such that for all triangles $abc$ with $p = a + b + c$:

- $p < C_0$ if $p$ is even
- $p < C_1$ if $p$ is odd
- $2K_1 < p < 2K_2 + \min(a, b, c)$ if $p$ is odd

Then $A_{K_1,K_2,C_0,C_1}^\delta$ is an amalgamation class if and only if [see T-shirt].

**Question**

Is there an algorithm that completes edge-labelled graphs to $A_{K_1,K_2,C_0,C_1}^\delta$?
Cherlin’s census of metrically homogeneous graphs

Cherlin light

For parameters \((\delta, K, C)\) let \(A_{K,C}^\delta\) be the class of metric spaces of diameter \(\delta\) such that for all triangles \(abc\) with \(p = a + b + c\):

- \(p < C\)
- \(2K < p\) if \(p\) is odd

If \(C > 2\delta + K\), then \(A_{K,C}^\delta\) is an amalgamation class.

Question

Is there an algorithm that completes edge-labelled graphs to \(A_{K,C}^\delta\)?
Path completion fails for $A_{K,C}^\delta$

Adding big distances might introduce triangles of perimeter $> C$

**Example:** $\delta = 3$, $K = 1$, $C = 8$

Path completion (green) is not in $A_{1,8}^3$, while the red completion is!

$\rightarrow$ Idea: optimize not towards $\delta$, but to some $M < \delta$.

Only makes sense for $M \geq \frac{\delta}{2}$, $M \geq K$ and $M \leq \frac{C-\delta-1}{2}$. 
Completing triangles

Optimizing distances towards $\max(K, \frac{\delta}{2}) \leq M \leq \frac{C-\delta-1}{2}$.

Triangles $MMa$ are not forbidden due to the choice of $M$.

How to complete forks, i.e. triangles missing one edge?

\[
\begin{align*}
a < M & \quad b < M \\
a + b < M & \\
F^+ & \\
\hline
a > M & \quad b < M \\
a - b > M & \\
F^- & \\
\hline
a > M & \quad b > M \\
C - 1 - (a + b) < M & \\
F^C &
\end{align*}
\]
Generalized $M$-completion of $(G, d)$

Add all new edges of length $t(i)$ to $(G, d)$ in step $i$.

\[
\text{for } i = 0 \ldots \text{delta} - 1 \{
\begin{align*}
\text{if } t(i) > M & \text{ then complete all forks ab with } b-a = t(i) \\
\text{if } t(i) < M & \text{ then complete all forks ab with } b+a = t(i) \quad \text{ complete all forks ab with } C-b-a-1 = t(i) \\
\end{align*}
\}
\text{label remaining pairs by } M
Properties of the completion algorithm

Optimization lemma

Let \((G, d)\) be an edge-labelled graph, let \((G, \bar{d}) \in A_{K,C}^\delta\) be a completion and let \((G, d^M)\) be its \(M\)-completion. Then, for all \(x, y \in G\):

\[
\bar{d}(x, y) \geq d^M(x, y) \geq M \text{ or } \bar{d}(x, y) \leq d^M(x, y) \leq M.
\]
Properties of the completion algorithm

**Optimization lemma**

Let \((G, d)\) be an edge-labelled graph, let \((G, \bar{d}) \in \mathcal{A}_{K,C}^\delta\) be a completion and let \((G, d^M)\) be its \(M\)-completion. Then, for all \(x, y \in G\):

\[
\bar{d}(x, y) \geq d^M(x, y) \geq M \text{ or } \bar{d}(x, y) \leq d^M(x, y) \leq M.
\]

**Proposition**

Let \((G, d)\) be an edge-labelled graph, let \((G, \bar{d}) \in \mathcal{A}_{K,C}^\delta\) be a completion and let \((G, d^M)\) be its \(M\)-completion.

- \((G, d^M) \in \mathcal{A}_{K,C}^\delta\) and
- \(\text{Aut}(G, d) \leq \text{Aut}(G, d^M)\).
Properties of the completion algorithm

Optimization lemma
Let \((G, d)\) be an edge-labelled graph, let \((G, \bar{d}) \in A_{K, C}^\delta\) be a completion and let \((G, d^M)\) be its \(M\)-completion. Then, for all \(x, y \in G\):

\[
\bar{d}(x, y) \geq d^M(x, y) \geq M \text{ or } \bar{d}(x, y) \leq d^M(x, y) \leq M.
\]

Proposition
Let \((G, d)\) be an edge-labelled graph, let \((G, \bar{d}) \in A_{K, C}^\delta\) be a completion and let \((G, d^M)\) be its \(M\)-completion.

- \((G, d^M) \in A_{K, C}^\delta\) and
- \(\text{Aut}(G, d) \leq \text{Aut}(G, d^M)\).

\(\rightarrow\) there is a finite set \(O\) of cycles that are obstacles to the completion.
Summary

Let \((G, d^M)\) be the \(M\)-completion of \((G, d)\):

- \((G, d^M) \in \mathcal{A}_K^\delta, C\)
  \(\Leftrightarrow\) \((G, d)\) has a completion in \(\mathcal{A}_K^\delta, C\)
  \(\Leftrightarrow\) \((G, d)\) does not contain a cycle \(\in \mathcal{O}\)

- \(\text{Aut}(G, d) \leq \text{Aut}(G, d^M)\)

- The distance \(d^M(x, y)\) is the closest possible to \(M\)

- \(M\)-completion gives us a canonical amalgamation on \(\mathcal{A}_K^\delta, C\)
The extension property for partial automorphisms (EPPA)
The extension property for partial automorphisms (EPPA)

**Question**

Let $C$ be a class of finite structures. Given a $A \in C$ and a set $I$ of partial automorphisms of $A$. Is there a structure $A \leq B \in C$ such that $p \in I$ extends to an automorphism $f \in Aut(B)$?

We say $C$ has the extension property for partial automorphisms (EPPA) (or Hrushovski property) if the above is true for all $A \in C$. 
The extension property for partial automorphisms (EPPA)

Examples

The following classes have EPPA:

- Sets
- Graphs - Hrushovski 1992
- $K_n$-free graphs - Herwig 1998
- Generalized to model-theoretic constructions - Herwig, Lascar 2000
- Metric spaces with rational distances - Solecki 2005

Linear orders and partial orders do not have EPPA.
Let \((A, d) \in \mathcal{A}_{K,C}^\delta\) and \(p\) be a partial isomorphism.
Let $(A, d) \in \mathcal{A}^\delta_{K, C}$ and $p$ be a partial isomorphism.

We can form the infinite extension $\bigcup_{i \in \mathbb{Z}} A_i$.
Let \((A, d) \in \mathcal{A}_{K,C}^\delta\) and \(p\) be a partial isomorphism.

We can form the infinite extension \(\bigcup_{i \in \mathbb{Z}} A_i\).

For a finite extension, we have to identify \(A_n = A_0\).
Let \((A, d) \in \mathcal{A}_{K,C}^\delta\) and \(p\) be a partial isomorphism.

We can form the infinite extension \(\bigcup_{i \in \mathbb{Z}} A_i\).

For a finite extension, we have to identify \(A_n = A_0\).

If \(n\) big enough, we can complete to \((B, d) \in \mathcal{A}_{K,C}^\delta\).
Theorem (Herwig, Lascar ’00)

Let $\mathcal{O}$ be a finite set of relational structures, and let $\text{Forb}(\mathcal{O})$ be the class of all structures that contain no homomorphic images of structures in $\mathcal{O}$. Then $\text{Forb}(\mathcal{O})$ has EPPA.
EPPA by a result of Herwig, Lascar

Theorem (Herwig, Lascar ’00)
Let $\mathcal{O}$ be a finite set of relational structures, and let $\text{Forb}(\mathcal{O})$ be the class of all structures that contain no homomorphic images of structures in $\mathcal{O}$. Then $\text{Forb}(\mathcal{O})$ has EPPA.

Consequently $\mathcal{A}_{K,C}^\delta$ has EPPA:
Let $\mathcal{O}$ be set of obstacles (finitely many cycles) for completion $\mathcal{A}_{K,C}^\delta$. For $(A, d) \in \mathcal{A}_{K,C}^\delta$, form an EPPA-witness $(B, d) \in \text{Forb}(\mathcal{O})$. Then: $(B, d^M) \in \mathcal{A}_{K,C}^\delta$ and $\text{Aut}(B, d) \leq \text{Aut}(B, d^M)$. 
Results
Theorem AB-WHKKKP ’17

Let $\mathcal{A}_{K_1, K_2, c_0, c_1, S}^\delta$ be an amalgamation class in Cherlin’s catalogue.

Then

1. $\mathcal{A}_{K_1, K_2, c_0, c_1, S}^\delta$ has EPPA, canonical amalgamation and its expansion by linear orders has the Ramsey property,

2. or we are in one of two extremal cases.
Theorem AB-WHKKKP '17

Let $\mathcal{A}_{K_1,K_2,C_0,C_1,S}^\delta$ be an amalgamation class in Cherlin’s catalogue.

Then

1. $\mathcal{A}_{K_1,K_2,C_0,C_1,S}^\delta$ has EPPA, canonical amalgamation and its expansion by linear orders has the Ramsey property,

2. or we are in one of two extremal cases.

Remark

These properties imply several facts about the Fraïssé limits of $\mathcal{A}_{K_1,K_2,C_0,C_1,S}^\delta$ and their automorphism groups.
Thank you!