A new proof of the existence of cores of oligomorphic structures

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Outline

1. Homomorphic equivalence and cores
2. Oligomorphic groups
3. The proof
4. Summary

Homomorphic equivalence

Definition

Two structures $\mathbb{A}$ and $\mathbb{B}$ are homomorphically equivalent, if there are homomorphisms $h\colon \mathbb{A} \to \mathbb{B}$ and $h' : \mathbb{B} \to \mathbb{A}$.

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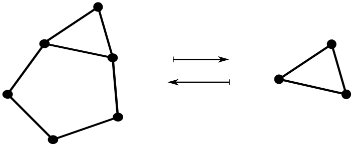

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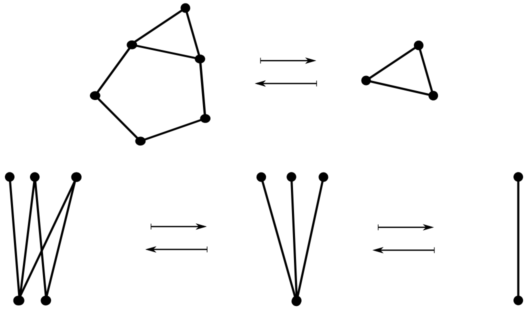

Cores of finite structures

Definition

For finite $\mathbb{A}$, we say $\mathbb{B}$ is a core of $\mathbb{A}$, if

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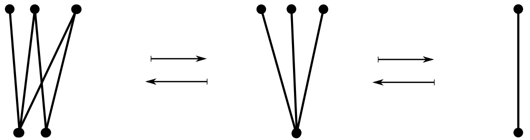
Proof: take endomorphism with minimal range



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We say $\tilde{\mathcal{M}}$ is a core.

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Example

The rational order $(\mathbb{Q}; <)$ is a core.

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but there is no "limit".

$\rightarrow$ we need compactness to obtain a core!

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- proof uses concepts from model theory
- condition on $\mathrm{Aut}(\mathbb{A})$, not $\mathrm{End}(\mathbb{A})$

Our Plan

For a finite transformation monoid $\mathcal{M}$ on A we found $B \subseteq A$ such that the monoid

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B needs to reflect the "minimal range" condition from the finite case.

A compactness argument

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- I is minimal with those properties

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(...show elements of $\tilde{\mathcal{M}}$ locally look like invertibles) $\square$

The result

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Theorem (Barto, K, Olšak, Pham, Pinsker '17)

Let $\mathbb{A}$ be a structure such that $\mathrm{End}(\mathbb{A})$ is weakly oligomorphic. Then $\mathbb{A}$ is homomorphically equivalent to a unique core $\mathbb{B}$. Moreover $\mathrm{Aut}(\mathbb{B})$ is oligomorphic.

The end

Thank you for your attention!