Endomorphism monoids of $\omega$-categorical structures

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TACL - 24/06/2015
A structure is called \( \omega \text{-categorical} \) iff its theory has exactly one countable model.
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A countable structure $\mathcal{A}$ is $\omega$-categorical

- iff $\text{Aut}(\mathcal{A})$ is oligomorphic:
  Every action $\text{Aut}(\mathcal{A}) \curvearrowright A^n$ has only finitely many orbits.
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Countable, \( \omega \)-cat. structures \( \mathcal{A} \) and \( \mathcal{B} \) are interdefinable iff

\[
\text{Aut}(\mathcal{A}) = \text{Aut}(\mathcal{B})
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A surjective partial function \( I : \mathcal{A}^n \rightarrow \mathcal{B} \) is called an interpretation iff every preimage of a relation in \( \mathcal{B} \) is definable in \( \mathcal{A} \).
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**Theorem (Ahlbrandt and Ziegler ’86)**

Two countable \( \omega \)-categorical structures \( \mathcal{A}, \mathcal{B} \) are bi-interpretable iff

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- What about \( \text{Aut}(\mathcal{A}) \) as abstract group?
- Can we \textit{reconstruct} the topology of \( \text{Aut}(\mathcal{A}) \)?
Versions of interpretability

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- The **endomorphisms monoid** \( \text{End}(\mathcal{A}) \):
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Questions

Can we reconstruct the topology of a closed oligomorphic
- permutation group
- transformation monoid
- function clone
from its abstract algebraic structure?

No! (Evans + Hewitt '90; Bodirsky + Evans + Pinsker + MK '15)
Reconstruction

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So from now on work in $\text{ZFC}$.

**Profinite groups** are closed permutation groups where every orbits contains finitely many elements.

**Example (Witt ’54)**

There are two separable profinite groups $G$, $G'$ that are isomorphic, but not topologically isomorphic.
Encoding profinite groups with oligomorphic groups

Lift the result to oligomorphic groups:
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**Lemma (Hrushovski)**

There is a oligomorphic $\Phi$ such that for every separable profinite group $R$ there is an oligomorphic $\Sigma_R$:

1. $\Sigma_R/\Phi \cong_T R$.
2. $\Phi$ is the intersection of open subgroups of finite index in $\Sigma_R$. 

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Proof idea: $R \leq \prod_{n \geq 1} \text{Sym}(n)$. 

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*Proof idea:* $R \leq \prod_{n \geq 1} \text{Sym}(n)$.

Look at finite sets. Partition the $n$-tuples into partition classes $P^n_1, P^n_2, \ldots P^n_n$ for all $n \geq 1$. This gives us a Fraïssé-class.
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Let $\mathcal{A} = (A, (P^n_i)_{i,n})$ be the Fraïssé-limit; $\Phi = \text{Aut}(\mathcal{A})$
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This gives us $\Sigma/\Phi \cong^T \prod_{n \in \mathbb{N}} \text{Sym}(n)$. $\square$
Permutation groups

Idea

Use the encoding lemma to show:

\[ G \cong T \Rightarrow \Sigma G \cong T \Sigma G' \]

\[ G \cong G' \Rightarrow \Sigma G \cong \Sigma G' \]

Problem: We do not know if \( \Sigma G \cong \Sigma G' \) for \( G \cong G' \).

The real proof deviates from the above.
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Use the encoding lemma to show:

\[ G \not\cong_T G' \implies \Sigma_G \not\cong_T \Sigma_{G'} \]
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Lifting to the monoid closure

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**Lemma**

The quotient homomorphism \( \Sigma_R \to R \) extends to a continuous monoid homomorphism

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Let $\Sigma_R$ be the topological closure of $\Sigma_R$ in $\omega^\omega$.

**Lemma**

The quotient homomorphism $\Sigma_R \to R$ extends to a continuous monoid homomorphism $\Sigma_R \to R$ with kernel $\Phi$.

We get:

**Result for monoids**

$\Sigma_G$ and $\Sigma_{G'}$ are isomorphic, but not topologically isomorphic.
Oligomorphic clones

Observation

Let $I : \Gamma \to \Delta$ be a monoid homomorphism. If $I$ sends constants to constants, it has a natural extension to a clone homomorphism $	ext{Clo}(\Gamma) \to \text{Clo}(\Delta)$. 

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The clones $\text{Clo}(\Sigma G)$ and $\text{Clo}(\Sigma G')$ are isomorphic but not topologically isomorphic.

This answers a question by Bodirsky, Pinsker and Pongrácz.
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Thank you!