Dichotomy results for constraint satisfaction problems

Michael Kompatscher

michaelkompatscher@hotmail.com

Institut für Computersprachen
Technische Universität Wien

PhDs in Logic VII - 15/05/2015
Let $\mathcal{A}$ be a structure in a finite language $L$.

**CSP($\mathcal{A}$)**

*Instance:* $\psi = \exists x_1, \ldots, x_j \phi_1 \land \ldots \land \phi_n$ with $\phi_i$ atomic $L$-formulas

*Problem:* Is $\psi$ true in $\mathcal{A}$?
Let $\mathcal{A}$ be a structure in a finite language $L$

**CSP($\mathcal{A}$)**

*Instance:* $\psi = \exists x_1, ..., x_j \phi_1 \land ... \land \phi_n$ with $\phi_i$ atomic $L$-formulas

*Problem:* Is $\psi$ true in $\mathcal{A}$?

$\mathcal{A}$ is called the **template** of the CSP.
Let $\mathcal{A}$ be a structure in a finite language $L$

**CSP($\mathcal{A}$)**

**Instance:** $\psi = \exists x_1, \ldots, x_j \phi_1 \land \ldots \land \phi_n$ with $\phi_i$ atomic $L$-formulas

**Problem:** Is $\psi$ true in $\mathcal{A}$?

$\mathcal{A}$ is called the **template** of the CSP.

The input is called a **primitively positive sentence** (pp-sentence).
CSPs with finite templates

2-SAT

*Instance*: Variables $x_1, ..., x_n$ and a set of 2-clauses

*Problem*: Is there a truth assignment satisfying all clauses?
CSPs with finite templates

2-SAT

Instance: Variables $x_1, ..., x_n$ and a set of 2-clauses

Problem: Is there a truth assignment satisfying all clauses?

CSP(\{0, 1\}, R_1, R_2, R_3), where
CSPs with finite templates

2-SAT

*Instance:* Variables $x_1, ..., x_n$ and a set of 2-clauses

*Problem:* Is there a truth assignment satisfying all clauses?

$\text{CSP}([0, 1], R_1, R_2, R_3)$, where

\[
\begin{align*}
R_1(x, y) &\iff x \lor y \\
R_2(x, y) &\iff x \lor \neg y \\
R_3(x, y) &\iff \neg x \lor \neg y
\end{align*}
\]
CSPs with finite templates

1-in-3-SAT

*Instance*: Variables $x_1, \ldots, x_n$ and a set of 3-clauses

*Problem*: Is there a satisfying truth assignment, such that exactly one literal of every clause is true?

$CSP(\{0, 1\}, R)$, where $R = \{(0,0,1), (0,1,0), (0,0,1)\}$
1-in-3-SAT

**Instance:** Variables $x_1, ..., x_n$ and a set of 3-clauses

**Problem:** Is there a satisfying truth assignment, such that exactly one literal of every clause is true?

$\text{CSP}([0,1], R)$, where
CSPs with finite templates

1-in-3-SAT

Instance: Variables $x_1, ..., x_n$ and a set of 3-clauses

Problem: Is there a satisfying truth assignment, such that exactly one literal of every clause is true?

CSP($\{0, 1\}, R$), where

$$ R = \{(0, 0, 1), (0, 1, 0), (0, 0, 1)\} $$
Let $\Psi$ be a finite set of relations on $\{0, 1\}$
Let $\Psi$ be a finite set of relations on $\{0, 1\}$

**SAT($\Psi$)**

*Instance*: Variables $x_1, \ldots, x_n$ and a set of atomic formulas in $\Psi$

*Problem*: Is there a satisfying truth assignment?
Reduction

Let $\Psi$ be a finite set of relations on $\{0, 1\}$

$\text{SAT}(\Psi)$

Instance: Variables $x_1, ..., x_n$ and a set of atomic formulas in $\Psi$

Problem: Is there a satisfying truth assignment?

$\text{CSP}({0, 1}, \Psi)$.
Reduction

Let $\Psi$ be a finite set of relations on $\{0, 1\}$

$\text{SAT}(\Psi)$

*Instance:* Variables $x_1, \ldots, x_n$ and a set of atomic formulas in $\Psi$

*Problem:* Is there a satisfying truth assignment?

$\text{CSP}([0, 1], \Psi)$.

If $\Psi \subset \Psi'$ then $\text{CSP}([0, 1], \Psi)$ reduces to $\text{CSP}([0, 1], \Psi')$. 
Let $\Psi$ be a finite set of relations on $\{0, 1\}$

**SAT($\Psi$)**

*Instance*: Variables $x_1, \ldots, x_n$ and a set of atomic formulas in $\Psi$

*Problem*: Is there a satisfying truth assignment?

CSP($\{0, 1\}, \Psi$).

If $\Psi \subset \Psi'$ then CSP($\{0, 1\}, \Psi$) reduces to CSP($\{0, 1\}, \Psi'$).

If $R$ is pp-definable in $\Psi$ then CSP($\{0, 1\}, R, \Psi$) reduces to CSP($\{0, 1\}, \Psi$).
Reduction

1-in-3-SAT is NP-complete
Reduction

1-in-3-SAT is NP-complete

Every 3-clause has a pp-definition in $R = \{(0, 0, 1), (0, 1, 0), (0, 0, 1)\}$:
Reduction

1-in-3-SAT is NP-complete

Every 3-clause has a pp-definition in $R = \{(0, 0, 1), (0, 1, 0), (0, 0, 1)\}$:

$$y \leftrightarrow \neg x \iff \exists a, b \ R(a, a, b) \land R(b, x, y)$$
Reduction

1-in-3-SAT is NP-complete

Every 3-clause has a pp-definition in $R = \{(0, 0, 1), (0, 1, 0), (0, 0, 1)\}$:

\[
y \leftrightarrow \neg x \iff \exists a, b \ R(a, a, b) \land R(b, x, y)
\]

\[
x \lor y \lor z \iff \exists a, b, c \ R(\neg x, a, b) \land R(y, b, c) \land R(\neg z, a, c)
\]
1-in-3-SAT is NP-complete

Every 3-clause has a pp-definition in
\( R = \{(0, 0, 1), (0, 1, 0), (0, 0, 1)\} \):

\[
\begin{align*}
y &\leftrightarrow \neg x \iff \exists a, b \ R(a, a, b) \land R(b, x, y) \\
x \lor y \lor z &\iff \exists a, b, c \ R(\neg x, a, b) \land R(y, b, c) \land R(\neg z, a, c)
\end{align*}
\]

Classify templates up to primitive positive interdefinability.
Every computational problem $\text{CSP}(\{0, 1\}, \Psi)$ either reduces to one of 6 known P-problems, or is NP-complete.
Dichotomy conjecture

**Schaefer ’79**

Every computational problem $\text{CSP}(\{0, 1\}, \Psi)$ either reduces to one of 6 known P-problems, or is NP-complete.

**Conjecture (Feder and Vardi)**

Every constraint satisfaction problem with finite template $\mathcal{A}$ lies either in P or in NP-complete.
Dichotomy conjecture

Schaefer ’79

Every computational problem CSP(\{0, 1\}, \Psi) either reduces to one of 6 known P-problems, or is NP-complete.

Conjecture (Feder and Vardi)

Every constraint satisfaction problem with finite template \(\mathcal{A}\) lies either in P or is NP-complete.

Proven for \(|\mathcal{A}| \leq 3\) (Bulatov ’06).
CSPs with infinite templates
CSPs with infinite templates

Diophant

*Instance*: A finite set of equations in the language $+, -, 0, 1, *$

*Problem*: Is there a solution in $\mathbb{Z}$?

CSP($\mathbb{Z}, +, -, 0, 1, *, =$)
CSPs with infinite templates

**Diophant**

*Instance:* A finite set of equations in the language $+, -, 0, 1, *$

*Problem:* Is there a solution in $\mathbb{Z}$?

CSP($\mathbb{Z}, +, -, 0, 1, *, =$)

**Digraph acyclicity**

*Instance:* A finite directed graph $(G, E)$

*Problem:* Is $G$ acyclic?

CSP($\mathbb{Q}, <$)
Homogeneous structures

A structure is called **homogeneous** if every isomorphism between finitely generated substructures extends to an automorphism.
Homogeneous structures

A structure is called **homogeneous** if every isomorphism between finitely generated substructures extends to an automorphism.

Theorem (Fraïssé)

For every Fraïssé class there is a homogeneous structure, whose **age** consists of the elements of the class.
Homogeneous structures

A structure is called **homogeneous** if every isomorphism between finitely generated substructures extends to an automorphism.

**Theorem (Fraïssé)**

For every Fraïssé class there is a homogeneous structure, whose **age** consists of the elements of the class.

- \((\mathbb{N},=)\): finite sets
Homogeneous structures

A structure is called **homogeneous** if every isomorphism between finitely generated substructures extends to an automorphism.

**Theorem (Fraïssé)**

For every Fraïssé class there is a homogeneous structure, whose **age** consists of the elements of the class.

- $\langle \mathbb{N}, = \rangle$: finite sets
- Rational order $\langle \mathbb{Q}, < \rangle$: linear orders
Homogeneous structures

A structure is called **homogeneous** if every isomorphism between finitely generated substructures extends to an automorphism.

**Theorem (Fraïssé)**

For every Fraïssé class there is a homogeneous structure, whose **age** consists of the elements of the class.

- $(\mathbb{N}, =)$: finite sets
- rational order $(\mathbb{Q}, <)$: linear orders
- random graph $(V, E)$: finite graphs
**Temp-SAT(Ψ)**

### Digraph acyclicity

**Instance:** A finite directed graph \((G, E)\)

**Problem:** Is \(G\) acyclic?

\[\text{CSP}(\mathbb{Q}, <)\]
Digraph acyclicity

**Instance:** A finite directed graph \((G, E)\)

**Problem:** Is \(G\) acyclic?

**CSP\((\mathbb{Q}, <)\)**

\[
\text{Betw}(x, y, z) \iff x < y < z \lor z < y < x
\]
Temp-SAT(\(\Psi\))

### Digraph acyclicity

**Instance:** A finite directed graph \((G, E)\)

**Problem:** Is \(G\) acyclic?

\[
\text{CSP}(\mathbb{Q}, <)
\]

**Betw** \((x, y, z)\) : \(\iff x < y < z \lor z < y < x\)

### Betweenness

**Instance:** Given a set of variables and triples \((x, y, z)\)

**Problem:** Is there a linear order on the variables such that \(\text{Betw}(x, y, z)\) for all triples?

\[
\text{CSP}(\mathbb{Q}, \text{Betw})
\]
Temp-SAT(\(\Psi\))

Let \(\Psi\) be a set of relations definable from \(<\) without quantifiers.
Let $\Psi$ be a set of relations definable from $<$ without quantifiers.

**Temp-SAT($\Psi$)**

*Instance:* Variables $x_1, x_2, \ldots, x_n$ and statements $\psi_i$ about the variables, where each statement is in $\Psi$.

*Problem:* Is there a linear order satisfying all $\psi_i$?

**CSP(\mathbb{Q}, \Psi)**
Let $\Psi$ be a set of relations definable from $<$ without quantifiers.

**Temp-SAT($\Psi$)**

*Instance:* Variables $x_1, x_2, ..., x_n$ and statements $\psi_i$ about the variables, where each statement is in $\Psi$.

*Problem:* Is there a linear order satisfying all $\psi_i$?

**CSP($\mathbb{Q}, \Psi$)**

Classify all the reducts of $(\mathbb{Q}, <)$, up to pp-interdefinability.
Let $\mathcal{A}$ be a structure. Then $\text{Pol}(\mathcal{A})$ is the set of all homomorphisms

$$h : \mathcal{A}^n \rightarrow \mathcal{A}$$

for all $1 \leq n < \omega$. 
Polymorphism clones

Let $\mathcal{A}$ be a structure. Then $\text{Pol}(\mathcal{A})$ is the set of all homomorphisms

$$h : \mathcal{A}^n \to \mathcal{A}$$

for all $1 \leq n < \omega$.

An element of $\text{Pol}(\mathcal{A})$ is called polymorphism of $\mathcal{A}$. 
Polymorphism clones

Let $\mathcal{A}$ be a structure. Then $\text{Pol}(\mathcal{A})$ is the set of all homomorphisms

$$h : \mathcal{A}^n \rightarrow \mathcal{A}$$

for all $1 \leq n < \omega$. An element of $\text{Pol}(\mathcal{A})$ is called polymorphism of $\mathcal{A}$.

Example

$$(x, y) \mapsto \min(x, y) \in \text{Pol}(\mathbb{Q}, <)$$ since

$a < x, b < y \Rightarrow \min(a, b) < \min(x, y)$
Polymorphism clones

Let $\mathcal{A}$ be a structure. Then $Pol(\mathcal{A})$ is the set of all homomorphisms

$$h : \mathcal{A}^n \rightarrow \mathcal{A}$$

for all $1 \leq n < \omega$.

An element of $Pol(\mathcal{A})$ is called polymorphism of $\mathcal{A}$.

Example

$$(x, y) \mapsto \min(x, y) \in Pol(\mathbb{Q}, <) \text{ since } a < x, b < y \Rightarrow \min(a, b) < \min(x, y)$$

$$\min \notin Pol(\mathbb{Q}, \text{Betw}) \text{ since } \text{Betw}(-1, 0, 1), \text{Betw}(2, 0, -1), \neg \text{Betw}(-1, 0, -1)$$
Polymorphism clones

Theorem (Bodirsky + Nešetřil, ’03)
Let $\mathcal{A}$ be $\omega$-categorical or finite. A relation is pp-definable in $\mathcal{A}$, iff it is preserved by all polymorphisms of $\mathcal{A}$.
Theorem (Bodirsky + Nešetřil, ’03)

Let $\mathcal{A}$ be $\omega$-categorical or finite. A relation is pp-definable in $\mathcal{A}$, iff it is preserved by all polymorphisms of $\mathcal{A}$.

The complexity of CSP($\mathcal{A}$) only depends on Pol($\mathcal{A}$).
Theorem (Bodirsky + Nešetřil, ’03)

Let $\mathcal{A}$ be $\omega$-categorical or finite. A relation is pp-definable in $\mathcal{A}$, iff it is preserved by all polymorphisms of $\mathcal{A}$.

The complexity of CSP($\mathcal{A}$) only depends on Pol($\mathcal{A}$).
Theorem (Bodirsky + Nešetřil, ’03)

Let $\mathcal{A}$ be $\omega$-categorical or finite. A relation is pp-definable in $\mathcal{A}$, iff it is preserved by all polymorphisms of $\mathcal{A}$.

The complexity of CSP$(\mathcal{A})$ only depends on Pol$(\mathcal{A})$. 

\[ \mathcal{A} = (A, R_1, \ldots) \]

\[ \text{Pol}(\mathcal{A}) \]
Theorem (Bodirsky + Kára, ’10)

Let $\mathcal{Q}_\Psi$ be a reduct of $(\mathbb{Q}, <)$. If $\text{Pol}(\mathcal{Q}_\Psi)$ contains one of the operators $ll$, $min$, $mi$, $mx$, their duals, or a constant operation, then it is tractable. Otherwise it lies in NP-complete.
Dichotomy for Graph-SAT
Current and future research

- Dichotomy for Poset-SAT(ψ)
Current and future research

- Dichotomy for Poset-SAT(Ψ)
- More general: C-SAT(Ψ), for Fraïssé-class C
Current and future research

- Dichotomy for Poset-SAT(Ψ)
- More general: $C$-SAT(Ψ), for Fraïssé-class $C$
- How to determine closed clones containing a given polymorphism clone?
Current and future research

- Dichotomy for Poset-SAT(Ψ)
- More general: $C$-SAT(Ψ), for Fraïssé-class $C$
- How to determine closed clones containing a given polymorphism clone?
  - Results from Ramsey theory and topological dynamics
Current and future research

- Dichotomy for Poset-SAT(Ψ)
- More general: $C$-SAT(Ψ), for Fraïssé-class $C$
- How to determine closed clones containing a given polymorphism clone?
  - Results from Ramsey theory and topological dynamics
- Compare the complexity of CSPs on different domains
Current and future research

- Dichotomy for Poset-SAT(Ψ)
- More general: $C$-SAT(Ψ), for Fraïssé-class $C$
- How to determine closed clones containing a given polymorphism clone?
  - Results from Ramsey theory and topological dynamics
- Compare the complexity of CSPs on different domains
  - Polymorphism clones as topological objects
Thank you!