Modularity of Termination of Left-Linear Rewrite Systems Revisited

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Abstract. We investigate modularity of termination of left-linear rewrite systems in the constructor sharing case. More precisely, we study cases where both involved system are left-linear, as well as other related cases where the conditions on the involved systems systems (like left-linearity) are not the same. Several tempting approaches to generalize the well-known deep modularity results of [21,22], [8,18] are discussed and falsified by counterexamples. By a thorough analysis of the crucial notions of *uniquely collapsing* and *consistency w.r.t. reduction* and of the role of left-linearity in *collapsing* and, more generally, *shared symbol lifting* reductions we finally show how to generalize certain modularity results for termination in the case of disjoint unions to combinations of constructor sharing systems. The problem in the symmetric case (where both systems are left-linear) remains open, whereas we are able to extend a couple of asymmetric modularity results from [4] to combinations of constructor sharing systems.

1 Background

Modularity of termination of rewrite systems has been extensively and thoroughly investigated in the past, under various types of combinations. Starting with the pioneering work of Toyama [20,19], especially with the fact that termination is in general not even modular for disjoint unions, a whole series of (positive and negative) results have been obtained so far (cf. e.g. [17], [13,14], [5,6], [2,3], [15,16], [1]). Subsequently, most of these results for disjoint unions – whether symmetric or asymmetric preservation criteria - have also been extended to more general types of non-disjoint combinations, namely *constructor sharing* systems, *composable* ones, and (certain) *hierarchical* combinations.

Somehow surprisingly, there is one major exception with the extension of basic results from the disjoint union case, namely for the case of left-linear systems. Toyama, Klop and Barendregt [21,22] have shown by a very sophisticated analysis of *collapsing reduction*:

(1) **Termination is modular for left-linear confluent systems**.

Later, independently Marchiori and Schmidt-Schauss & Panitz ([8], [18]) succeeded in giving a simplified proof of this deep result. Actually, their proof(s)[1] rely on the same basic observations (about the uniqueness of certain collapsing reductions) as the proof of [21,22]. However, the actual construction of [8], [18] for transforming an assumed minimal counterexample in the union to a smaller one is somewhat easier (to understand) than the proof in [21,22].[2] Moreover, they have shown that instead of confluence the weaker assumption of *consistency w.r.t. reduction ($CON^{\rightarrow}$)* suffices, too, i.e.:

(2) **Termination is modular for left-linear systems satisfying $\mathbf{CON}^{\rightarrow}$**.

The property $CON^{\rightarrow}$ says that a reduction of a term to two distinct variables is impossible, and means that in disjoint unions the *collapsing of some mixed term* to some of its *aliens* (or *principal subterms*) is only possible in a very restricted way.

[1] which are not identical but very similar
[2] Yet, it should be noted that in both papers [8], [18] these transformations are not really formally defined, but rather sketched. A completely formal presentation would most probably be much more involved.

Strikingly, and in contrast to most of the other modularity results for termination, none of the latter results (for left-linear systems) have been extended later on to non-disjoint unions. Perhaps this is partially due to the fact that the authors in [18] mentioned

> "The proof technique in this paper appears not to be extendable to the sum of term rewriting systems where there may be common constructors, i.e., function symbols that do not appear at the top in the left-hand sides of rules ...

Furthermore they argue that there are simple counterexamples to (2) above if shared constructors are permitted, e.g. the following variant of Toyama's basic counterexample (cf. [20,19]):

$$R_1 : h(a,b,x) \to h(x,x,x) \qquad\qquad R_2 : D \to a, D \to b$$

These two systems only share the common constructors a, b. Both are left-linear, terminating, and $\mathrm{CON}^{\to}$. Yet, their union allows a cycle:

$$h(D,D,D) \to h(a,D,D) \to h(a,b,D) \to h(D,D,D)$$

So, there seems to be no hope to extend (1), (2) above to non-disjoint combinations of rewrite systems.

The properties of being *uniquely collapsing*, *collapsing confluent* (cf. [4]) and left-linear are crucial in the analysis of the above *symmetric* case where both systems are supposed to be left-linear:

- Intuitively, a system is *uniquely collapsing* if whenever there is a reduction $s \to^* x$ from a term to a variable, then the *predecessor* of x in s is unique.
- A rewrite system is *collapsing confluent* whenever $s \to^* x$ and $s \to^* t$ imply $t \to^* x$, i.e., reduction to a variable remains always possible.

These properties are also crucial for various other *asymmetric* modularity results on termination of disjoint unions, e.g., when only one of the systems is supposed to be left-linear. Here we will consider the following known results in this area ([4]:

(3) **If one system is non-collapsing and the other one is both uniquely collapsing and collapsing confluent, then their disjoint union is terminating.**

(4) **If one system is non-collapsing and left-linear and the other one is uniquely collapsing, then their disjoint union is terminating.**

2 Our Approach

Regarding (1), (2), we will show that an extension attempt as above is too native to work. From the extension of other basic results about modularity of termination for disjoint unions to constructor sharing systems (and more general combinations) it is also well known that the additional complications caused by the increased amount of *shared information* in the union (especially the increasing possibilities of inferences between reduction steps in one system and steps in the other system) have to be taken appropriately into account.

We do this concerning (2) above, by a thorough analysis of the role of the preconditions *left-linearity*, *consistency w.r.t. reduction ($CON^{\to}$)* and of related properties (cf. [4]). We present a couple of tempting approaches and conjectures of how to adapt the preconditions of (2) in such a way that the extension to constructor sharing systems holds, and also provide counterexamples to most of them. Based on this analysis, we finally arrive at an extended version of (2) for systems with shared constructors. However, so far we have neither been able to prove it nor to discover a counterexample. [3] Hence, this remains an open problem.

On the positive side, we show that the concepts of *uniquely collapsing* reduction and *collapsing confluence* can be extended in a natural way to the constructor sharing case such that for instance the asymmetric results (3) and (4) above can indeed be extended to this case.

We think that our results obtained are not only interesting from the point of view of providing new termination criteria, but possibly even more regarding the structural analysis of (uniqueness of) *collapsing* or, more generally, *shared symbol lifting* reductions in combinations of systems with shared constructors.

References

1. Thomas Arts and Jürgen Giesl. Modularity of termination using dependency pairs. In Tobias Nipkow, editor, *Proc. 9th Int. Conf. on Rewriting Techniques and Applications (RTA'98)*, LNCS 1379, pages 226–240, Tsukuba, Japan, 1998. Springer-Verlag.
2. Bernhard Gramlich. Generalized sufficient conditions for modular termination of rewriting. *Applicable Algebra in Engineering, Communication and Computing*, 5:131–158, 1994.
3. Bernhard Gramlich. Abstract relations between restricted termination and confluence properties of rewrite systems. *Fundamenta Informaticae*, 24:3–23, 1995.
4. Bernhard Gramlich. Modular aspects of rewrite-based specifications. In Francesco Parisi-Presicce, editor, *Recent Trends in Algebraic Development Techniques, 12th Int. Workshop, WADT 97, Tarquinia, Italy, June 1997, Selected Papers*, LNCS 1376, pages 253–268. Springer-Verlag, 1998.
5. Masahito Kurihara and Azuma Ohuchi. Modularity of simple termination of term rewriting systems with shared constructors. *Theoretical Computer Science*, 103:273–282, 1992.
6. Masahito Kurihara and Azuma Ohuchi. Modularity in noncopying term rewriting. *Theoretical Computer Science*, 152(1):139–169, December 1995.
7. Massimo Marchiori. Modularity of $UN^{\rightarrow}$ for left-linear term rewriting systems. Technical Report CS-R9433, CWI (Centre for Mathematics and Computer Science), Amsterdam, May 1994.
8. Massimo Marchiori. Modularity of completeness revisited. In Jieh Hsiang, editor, *Proc. 6th Int. Conf. on Rewriting Techniques and Applications (RTA'95)*, LNCS 914, pages 2–10, Kaiserslautern, Germany, April 1995. Springer-Verlag.
9. Massimo Marchiori. On the modularity of normal forms in rewriting. *Journal of Symbolic Computation*, 22(2):143–154, 1996.
10. Massimo Marchiori. The theory of vaccines. In Pierpaolo Degano, Robert Gorrieri, and Alberto Marchetti-Spaccamela, editors, *Proc. 24th Int. Conf. on Automata, Languages and Programming*, Lecture Notes in Computer Science, pages 660–670, Bologna, Italy, July 1997. Springer-Verlag.
11. Massimo Marchiori. Bubbles in modularity. *Theoretical Computer Science*, 192(1):31–54, 1998.
12. Aart Middeldorp. Modular aspects of properties of term rewriting systems related to normal forms. In N. Dershowitz, editor, *Proc. 3rd Int. Conf. on Rewriting Techniques and Applications*, LNCS 355, pages 263–277. Springer-Verlag, 1989.
13. Aart Middeldorp. A sufficient condition for the termination of the direct sum of term rewriting systems. In *Proceedings of the 4th IEEE Symposium on Logic in Computer Science*, pages 396–401, Pacific Grove, 1989.
14. Aart Middeldorp. *Modular Properties of Term Rewriting Systems.* PhD thesis, Free University, Amsterdam, 1990.
15. Enno Ohlebusch. On the modularity of termination of term rewriting systems. *Theoretical Computer Science*, 136:333–360, 1994.

[3] In essence, a proof in the spirit of [18] (still) seems to fail, since the *deletion part* of the *pile-and-delete* technique of [18], [22] appears not to work anymore (as argued already in [18]).

16. Enno Ohlebusch. Termination is not modular for confluent variable-preserving term rewriting systems. *Information Processing Letters*, 53:223–228, March 1995.
17. Michaël Rusinowitch. On termination of the direct sum of term rewriting systems. *Information Processing Letters*, 26:65–70, 1987.
18. Manfred Schmidt-Schauß, Massimo Marchiori, and Sven Eric Panitz. Modular termination of r-consistent and left-linear term rewriting systems. *Theoretical Computer Science*, 149(2):361–374, October 1995.
19. Yoshihito Toyama. Counterexamples to termination for the direct sum of term rewriting systems. *Information Processing Letters*, 25:141–143, 1987.
20. Yoshihito Toyama. On the Church-Rosser property for the direct sum of term rewriting systems. *Journal of the ACM*, 34(1):128–143, 1987.
21. Yoshihito Toyama, Jan Willem Klop, and Henk Pieter Barendregt. Termination for the direct sum of left-linear term rewriting systems. In N. Dershowitz, editor, *Proc. 3rd Int. Conf. on Rewriting Techniques and Applications (RTA'89)*, LNCS 355, pages 477–491. Springer-Verlag, 1989.
22. Yoshihito Toyama, Jan Willem Klop, and Henk Pieter Barendregt. Termination for direct sums of left-linear complete term rewriting systems. *Journal of the ACM*, 42(6):1275–1304, November 1995.