Conditional Rewrite Systems under Signature Extensions: Some Counterexamples

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Extended Abstract

1 Introduction

Here we are interested in the question whether certain weak and strong termination and confluence properties of conditional term rewriting systems (CTRSs) are preserved under signature extensions. We shall focus on the most interesting type of CTRSs, namely join systems, where equality in the conditions of a rule

$$l \to r \Longleftarrow s_1 = t_1, \ldots, s_n = t_n$$

is (recursively) interpreted as joinability, i.e., as $\downarrow$. In the presentation which is based on some results of [5] we assume familiarity with the basic no(ta)tions and terminology of (conditional) term rewriting (cf. e.g. [1], [6]).

The termination and confluence properties considered include the following:

- (strong) termination
- weak termination
- weak innermost termination, i.e., weak termination of the innermost reduction relation
- (strong) innermost termination, i.e., termination of the innermost reduction relation
- confluence
- local confluence
- joinability of all (conditional) critical pairs

From a systematic point of view signature extensions constitute a very special type of combining (disjoint) systems. Starting with the pioneering work of Toyama ([10]), the general question which properties P of rewrite systems are preserved under (disjoint or less restrictive) combinations (and vice versa) has received a lot of attention. For surveys of achieved results in this field we refer to [7], [8], [5]. For instance, for unconditional term rewriting systems (TRSs) it is well-known that confluence is *modular*, i.e., preserved under disjoint unions (and vice versa), but termination is not modular ([9]). However, various restricted termination properties have turned out to be modular for TRSs: weak termination and weak innermost termination (cf. e.g. [2]) as well as (strong) innermost termination ([4]). This implies in particular that these properties of TRSs are also preserved under signature extensions. That termination of TRSs which is not modular in general is at least preserved under signature extensions is almost trivial ([3]). All confluence properties mentioned above are modular for TRSs (for local confluence which is equivalent to joinability of all critical pairs this is again almost trivial), hence also preserved under signature extensions. However, for CTRSs the analysis is considerably more complicated. Middeldorp ([7]) has shown that the modularity of confluence carries over to CTRSs. However, he also showed that local confluence (and joinability of critical pairs) is not modular:

Example 1.1 *([7]) Consider the disjoint CTRSs* $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ *given by*

$$\mathcal{R}_1 = \left\{ \begin{array}{lcl} f(x,y) \to x & \Leftarrow & x \downarrow z, z \downarrow y \\ f(x,y) \to y & \Leftarrow & x \downarrow z, z \downarrow y \end{array} \right\}$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{l} b \to a \\ b \to c \\ c \to b \\ c \to d \end{array} \right\}$$

over the signatures $\mathcal{F}_1 = \{f\}$ *and* $\mathcal{F}_2 = \{a, b, c, d\}$*, respectively. It is easy to see that both* $\mathcal{R}_1^{\mathcal{F}_1}$ *and* $\mathcal{R}_2^{\mathcal{F}_2}$ *have joinable (conditional) critical pairs and are locally confluent. However, their disjoint union* $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2} = (\mathcal{R}_1 \uplus \mathcal{R}_2)^{\mathcal{F}_1 \uplus \mathcal{F}_2}$ *is not locally confluent: The (new) instance*

$$a \;{}_{\mathcal{R}_1 \oplus \mathcal{R}_2}\!\leftarrow f(a,d) \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} d$$

of the critical peak between the first two rules of $\mathcal{R}_1$ *is not joinable, since both* a *and* d *are irreducible.*

Similarly, the properties of weak termination, weak and strong innermost termination are not modular for CTRSs in general.[1]

[1]For the first two properties this was shown in [7].

Example 1.2 *Consider the CTRSs* $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$

$$\{\ a \to a \quad \Longleftarrow \quad x \downarrow b \wedge x \downarrow c\ \}$$

and

$$\left\{ \begin{array}{l} G(x,y) \to x \\ G(x,y) \to y \end{array} \right\}$$

over $\mathcal{F}_1 = \{a, b, c\}$ *and* $\mathcal{F}_2 = \{G, A\}$*, respectively. Here, we have* $a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a$ *by applying the* $\mathcal{R}_1$*-rule (*x *is substituted by* $G(b,c)$*), but neither* $a \to_{\mathcal{R}_1} a$ *nor* $a \to_{\mathcal{R}_2} a$*. Hence,* a *is an* $\mathcal{R}_i$*-normal form (for* $i = 1, 2$*) but not a normal form w.r.t.* $\mathcal{R}_1 \oplus \mathcal{R}_2$*. Obviously, both* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *are strongly (hence also weakly and innermost) terminating but their disjoint union is not. For instance,* $a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} \ldots$ *is an infinite innermost derivation, and* a *does not have a normal form w.r.t.* $\mathcal{R}_1 \oplus \mathcal{R}_2$*.*

Below we give examples showing that the above non-modularity results for weak termination and confluence properties of CTRSs also hold for the very special case of signature extensions.

At first glance, these counterexamples may seem to be very surprising. In fact, they indicate once more the inherent complexity and intricacy of conditional rewriting.

2 Counterexamples for Signature Extensions

Example 2.1 (weak, weak innermost and strong innermost termination are not preserved under signature extensions)
Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ *given by*

$$\mathcal{R}_1 = \left\{ \begin{array}{rcl} f(g(x,y)) \to f(g(x,y)) & \Longleftarrow & x \downarrow z \wedge z \downarrow y \\ g(x,x) \to x & & \\ g(x,a) \to c & & \\ g(x,b) \to c & & \\ g(x,f(y)) \to c & & \\ g(x,g(y,z)) \to c & & \\ g(a,x) \to c & & \\ g(b,x) \to c & & \\ g(f(y),x) \to c & & \\ g(g(y,z),x) \to c & & \\ c \to a & & \\ c \to b & & \end{array} \right\}$$

over $\mathcal{F}_1 = \{f, g, a, b, c\}$ *and* $\mathcal{R}_2 = \emptyset$ *over* $\mathcal{F}_2 = \{G\}$ *(with* G *unary). It is straightforward to verify that* $\mathcal{R}_1^{\mathcal{F}_1}$ *is (strongly) innermost terminating (hence also weakly innermost terminating and weakly terminating). The crucial point* *is that an arbitrary infinite $\mathcal{R}_1^{\mathcal{F}_1}$- derivation must contain rewrite steps using the first rule. But whenever this rule is applicable, the contracted redex cannot be innermost, since some proper subterm must then be reducible by the remaining rules which constitute a terminating CTRS. Nevertheless, in the extended system $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1{}^{\mathcal{F}_1 \uplus \{G\}}$ we get the cyclic (hence infinite) innermost $\mathcal{R}^{\mathcal{F}}$-derivation*

$$f(g(G(a), G(b))) \to_{\mathcal{R}^{\mathcal{F}}} f(g(G(a), G(b))) \to_{\mathcal{R}^{\mathcal{F}}} f(g(G(a), G(b))) \to_{\mathcal{R}^{\mathcal{F}}} \ldots$$

by applying the first rule (instantiating the extra variable z by $G(c)$). Note moreover that there is no other way of reducing $f(g(G(a), G(b)))$ (all its proper subterms are in normal form w.r.t. $\mathcal{R}^{\mathcal{F}}$, and the second rule is clearly not applicable). Hence, $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1{}^{\mathcal{F}_1 \uplus \{G\}}$ is not innermost terminating (and also neither weakly innermost terminating nor weakly terminating).

Example 2.2 (local confluence and joinability of critical pairs are not preserved under signature extensions)
Consider the join CTRS $\mathcal{R}^{\mathcal{F}}$ with

$$\mathcal{R} = \left\{ \begin{array}{rcl} f(x,y,z) \to g(x) & \Longleftarrow & x \downarrow y, y \downarrow z \\ f(x,y,z) \to g(z) & \Longleftarrow & x \downarrow y, y \downarrow z \\ b \to a & & \\ b \to c & & \\ c \to b & & \\ c \to d & & \\ g(a) \to g(d) & & \\ f(a,x,y) \to f(d,x,y) & & \\ f(x,a,y) \to f(x,d,y) & & \\ f(x,y,a) \to f(x,y,d) & & \end{array} \right\}$$

and $\mathcal{F} = \{a, b, c, d, f, g\}$. It is not very difficult to show that $\mathcal{R}^{\mathcal{F}}$ has joinable critical pairs and is even locally confluent (but not confluent).

Now we add a fresh unary function symbol G, i.e. we consider $\mathcal{R}^{\mathcal{F}'}$ with $\mathcal{F}' = \mathcal{F} \uplus \{G\}$. Then joinability of critical pairs and hence local confluence is lost. To wit, consider for instance the term $f(G(a), G(b), G(d))$ which reduces to two distinct normal forms by one $\mathcal{R}^{\mathcal{F}'}$-step, respectively :

$$g(G(a)) \leftarrow f(G(a), G(b), G(d)) \to g(G(d))$$

Clearly both $g(G(a))$ and $g(G(d))$ are irreducible. This divergence corresponds to an instance of the critical pair between the first two rules, namely

$$\langle g(x) = g(z) \rangle \Longleftarrow x \downarrow y, y \downarrow z\,.$$

Over the old signature $\mathcal{F}$ every substitution σ which is feasible for this critical pair satisfies $\sigma(g(x)) \downarrow \sigma(g(z))$ whereas this is not the case for the mixed substitution $\tau = \{x \mapsto G(a), y \mapsto G(b), z \mapsto G(d)\}$. Hence the critical pair above is not joinable any more over the extended signature $\mathcal{F}'$.

Note that in the above example $\mathcal{R}^{\mathcal{F}}$ is obviously non-terminating. This is not essential in the following sense. We may replace the 'non-terminating part' of $\mathcal{R}^{\mathcal{F}}$

$$\left\{\begin{array}{l} b \to a \\ b \to c \\ c \to b \\ c \to d \end{array}\right\}$$

which has joinable critical pairs, hence is locally confluent (it is an unconditional TRS!), by a terminating CTRS with joinable critical pairs which is not confluent, hence necessarily not locally confluent. To this end, we can take for instance the system

$$\left\{\begin{array}{rcl} h(x) \to k(b) & \Longleftarrow & k(x) \downarrow h(b) \\ k(a) \to h(a) & & \\ a \to b & & \end{array}\right\}$$

which has the desired properties (in particular, it is not locally confluent: $h(b) \leftarrow h(a) \to k(b)$ but $h(b)$ and $k(b)$ are irreducible). Then the remaining construction of $\mathcal{R}^{\mathcal{F}}$ is adapted accordingly.

Example 2.3 (joinability of critical pairs is not even preserved for terminating CTRSs under signature extensions)
Consider the CTRS $\mathcal{R}^{\mathcal{F}}$ with

$$\mathcal{R} = \left\{\begin{array}{rcl} f(x,y,z) \to g(x) & \Longleftarrow & x \downarrow y, y \downarrow z \\ f(x,y,z) \to g(z) & \Longleftarrow & x \downarrow y, y \downarrow z \\ h(x) \to k(b) & \Longleftarrow & k(x) \downarrow h(b) \\ k(a) \to h(a) & & \\ a \to b & & \\ g(h(b)) \to g(k(b)) & & \\ h(h(b)) \to h(k(b)) & & \\ k(h(b)) \to k(k(b)) & & \\ f(h(b),x,y) \to f(k(b),x,y) & & \\ f(x,h(b),y) \to f(x,k(b),y) & & \\ f(x,y,h(b)) \to f(x,y,k(b)) & & \end{array}\right\}$$

and $\mathcal{F} = \{a,b,f,g,h,k\}$. It is easy to verify that $\mathcal{R}^{\mathcal{F}}$ is terminating. With some effort one can show that $\mathcal{R}^{\mathcal{F}}$ has joinable critical pairs. But $\mathcal{R}^{\mathcal{F}}$ is not (locally) confluent since we have $h(b) \leftarrow h(a) \to k(b)$ with both $h(b)$ and $k(b)$ irreducible.

Now we add a fresh unary function symbol G, i.e. we consider $\mathcal{R}^{\mathcal{F}'}$ with $\mathcal{F}' = \mathcal{F} \uplus \{G\}$. The new system $\mathcal{R}^{\mathcal{F}'}$ is still terminating. But joinability of critical pairs is lost. Consider the critical pair between the first two rules of $\mathcal{R}^{\mathcal{F}'}$,

$$\langle g(x) = g(z) \rangle \Longleftarrow x \downarrow y, y \downarrow z\,,$$

and the $\mathcal{F}'$-substitution $\tau = \{x \mapsto G(h(b)), y \mapsto G(h(a)), z \mapsto G(k(b))\}$. The corresponding instance of the critical peak is

$$g(G(h(b))) \leftarrow f(G(h(b)), G(h(a)), G(k(b))) \rightarrow g(G(k(b)))\,,$$

due to $G(h(b)) \downarrow G(h(a)) \downarrow G(k(b))$, but $g(G(h(b)))$ and $g(G(k(b)))$ are not joinable since they are both irreducible (in $\mathcal{R}^{\mathcal{F}'}$).

In the talk, these counterexamples and some related positive results will be presented and discussed in more detail.

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