Termination and Confluence Properties of Structured Rewrite Systems

Vom Fachbereich Informatik
der Universität Kaiserslautern
zur Verleihung des akademischen Grades
Doktor der Naturwissenschaften (Dr. rer. nat.)
genehmigte Dissertation

von

Dipl.-Inform. Bernhard Gramlich

Datum der wissenschaftlichen Aussprache: 19. Januar 1996

Dekan: Prof. Dr. Hans Hagen

Promotionskommission:

Vorsitzender: Prof. Dr. Jürgen Nehmer
Berichterstatter: Prof. Dr. Jürgen Avenhaus
Prof. Dr. Klaus E. Madlener

D 386

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Abstract Termination and confluence properties of term rewriting systems are of fundamental importance, both from a theoretical point of view and also concerning many practical applications in computer science and mathematics. We study structural aspects of termination and confluence properties of unconditional and conditional term rewriting systems. Two types of structural aspects are considered.

First we investigate single rewrite systems. In a systematic manner we analyze how restricted versions of termination and confluence relate to each other and to general termination and confluence. In particular we focus on innermost rewriting and its properties. Various structural syntactic and semantic conditions are isolated which guarantee equivalence of general termination (and confluence) and weakened versions of these properties.

In a second major part we consider structural aspects of combining systems. Here we investigate modularity and preservation properties of rewrite systems under various types of combinations. We survey known results in a unifying framework and develop new approaches, in particular for ensuring the preservation of termination under various types of combinations. The abstract results obtained provide a deeper insight into the striking phenomena that rewriting in combined systems may exhibit, and entail various interesting positive consequences.

Zusammenfassung Terminierungs- und Konfluenzeigenschaften von Termersetzungssystemen sind von fundamentaler Bedeutung sowohl in theoretischer Hinsicht als auch im Hinblick auf viele praktische Anwendungen in Informatik und Mathematik. Wir untersuchen strukturelle Aspekte von Terminierungs- und Konfluenzeigenschaften bei unbedingten und bedingten Termersetzungssystemen. Dabei werden zweierlei Arten von strukturellen Aspekten betrachtet.

Zunächst werden einzelne Ersetzungssysteme behandelt. In systematischer Art und Weise analysieren wir, wie abgeschwächte Formen von Termination und Konfluenz miteinander in Beziehung stehen, und welche Beziehungen zwischen ihnen und allgemeiner Termination und Konfluenz bestehen. Besondere Berücksichtigung findet dabei die 'innermost'-Reduktionsrelation, bei der minimale Teilterme, d.h. Teilterme an innnersten Stellen, ersetzt werden. Es werden verschiedene strukturelle, sowohl syntaktische als auch semantische, Bedingungen hergeleitet, die hinreichend sind für die Äquivalenz von allgemeiner Termination (und Konfluenz) und abgeschwächten Formen von Termination (und Konfluenz).

Im zweiten Teil betrachten wir strukturelle Aspekte bei der Kombination mehrerer Systeme. Dabei werden Modularitäts- und Erhaltungseigenschaften unter verschiedenen Kombinationstypen untersucht. Bekannte Ergebnisse werden in einem einheitlichen Rahmen systematisch präsentiert. Wir entwickeln wesentlich neue Ansätze zur Erhaltung von Termination unter verschiedenen Kombinationstypen. Die hierbei erzielten abstrakten Resultate ermöglichen eine tiefere Einsicht in die erstaunlichen Phänomene, die beim Termersetzen in kombinierten Systemen auftreten können. Ferner lassen sich viele interessante positive Konsequenzen und Erhaltungssätze elegant daraus herleiten.

Preface I am much indebted to my supervisors Jürgen Avenhaus and Klaus Madlener who gave me the opportunity to write this thesis. Their continuous support and encouragement as well as their critical comments, questions and hints concerning my work have always been very stimulating and helpful. Without their tenacious patience this thesis might have never been written.

Furthermore I thank my colleagues for creating a pleasant and constructive atmosphere and for their interest in my work. In particular, I appreciate many valuable discussions on various (not only scientific) topics with Thomas Deiß, Roland Fettig, Ulrich Kühler, Norbert Kuhn, Birgit Reinert, Andrea Sattler-Klein, Joachim Steinbach and Claus-Peter Wirth. Thanks also to Mauricio Ayala Rincon, Klaus Becker, Jörg Denzinger, Matthias Fuchs, Carlos Loría-Saénz and Rita Kohl for their helpfulness.

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And finally, I would like to apologize to all those who have suffered somehow during the last years because of all the time and energy I devoted to writing this thesis.

Contents

Chapter 1

Introduction and Overview

In this introduction we shall first sketch the background of this thesis, namely the field of term rewriting, its basic idea, some history and applications. Then the main motivations for this work, its context and goals pursued are briefly discussed. A detailed summary of the thesis and its main contributions follow. And finally, we mention some aspects concerning the presentation philosophy.

Term Rewriting: Basic Idea, History and Applications

Term rewriting systems provide an elegant, abstract and simple, yet powerful, computation mechanism. The basic idea is very simple: *replacement of equals by equals* by applying *symbolic equations* over symbolically structured objects, *terms*. Applying equations in one direction only immediately leads to the concept of (directed) *term rewriting*. Since different parts of structured objects, the *subterms*, can be replaced by applying different *term rewriting* or *rewrite rules*, this obviously leads to non-deterministic computations. Moreover, after one computation (*rewrite*) step the same kind of computation may be possible again. Hence two basic questions naturally arise:

- Do all computations eventually stop?
- If there exist diverging computations (i.e., computations which proceed along different branches) issuing from the same origin (starting term), can the corresponding different intermediate results be brought together again (by appropriate further computations)?

The first problem is usually called the *termination* problem, the second one the *confluence* problem. A term which cannot be rewritten any more is in *normal form* or *irreducible*. For terminating systems normal forms always exist (and usually can be constructively computed in finite time). If a system is both terminating and confluent, then normal forms always exist and are even unique, irrespective of the computation (rewriting) strategy. For this reason, too, termination is a very useful and desirable property. In the case of non-terminating but confluent systems, normal forms need not exist, however, if a normal form exists, it is still unique. This corresponds to the general observation that computations (described by programs in some computation formalism) need not always stop, but if they do, the final results should – hopefully and provably – be unique. In a non-terminating but confluent rewrite system one even knows that (intermediate results of) infinite diverging computations can be joined again. In some cases, non-termination is inherently unavoidable, in other cases it may be very difficult to verify the termination property. Hence the problem of proving confluence (with or without termination) is of fundamental importance, too. In fact, virtually any computation formalism which is based on rewriting systems heavily relies on (various versions of) the fundamental properties of termination and confluence.

If the objects of computation do not have any additional internal structure, one arrives at an abstracted version of *term rewriting systems* (TRSs), namely *abstract reduction (rewrite) systems* (ARSs). In fact, many basic properties and results for rewrite systems can already be formulated in this more abstract setting. Then it is instructive to see which properties and results for term rewriting systems do not depend on the additional structure which one encounters there, and which ones do indeed make essential use of the term structure. For instance, for abstract reduction systems it is well-known that, under termination, confluence is equivalent to local confluence (which means that all one-step divergences can be joined), via *Newman's Lemma* ([New42]). For term rewriting systems, local confluence can be characterized by confluence of *critical pairs* (which make essential use of the term structure) as expressed by the well-known *Critical Pair Lemma* ([KB70], [Hue80]). Hence, for (finite) terminating TRSs, this *critical pair test* yields decidability of confluence. For proving confluence of non-terminating ARSs or TRSs, however, one usually needs much stronger local confluence properties.

The first systematic investigations related to term rewriting date back to the first half of this century. At that time basic computation formalisms like λ-*calculus* ([Chu41], [Bar84]) and *combinatory logic* ([Cur30a; Cur30b], [CF58], [CHS72]) were developed and studied. Also, first fundamental properties of abstract reductions systems were established at that time ([CR36], [New42]). The field of term rewriting got a decisive impact by the pioneering paper of Knuth & Bendix ([KB70]) which paved the way for a systematic study of ways to *complete* non-confluent (terminating) term rewriting systems to confluent (terminating) ones via so-called *completion procedures*.

Term rewriting has various applications in many fields of computer science and mathematics. Almost any conceivable form of symbolic computation is amenable to term rewriting techniques. In particular, term rewriting has been successfully applied in the fields of functional programming, functional-logic programming, equational programming, unification theory, abstract data type specifications, program verification, transformation, implementation, optimization and synthesis, general and inductive theorem proving in equational and first-order logic.

Motivation and Goals

Our main motivations concerning the topics of this thesis are essentially two-fold. First of all, we think that still nowadays many basic features and phenomena in term rewriting are largely unexplored, and lack a satisfactory explanation. In other words, we think that much more work towards a better understanding of rewriting mechanisms is indispensable, and might turn out to be very fruitful in other contexts, too. For instance, the theory of *orthogonal*, i.e., *non-overlapping* and *left-linear*, rewrite systems is fairly well-explored, however, not much is actually known for less restricted classes of rewrite systems.

A second, more concrete motivation was provided by the project background in which I was working for the last years. Here the main theme was 'Equational Reasoning' with a special focus on algebraic specification and inductive theorem proving in an equational setting. Using term rewriting as operational semantics for equational specifications of functions many questions naturally arise. For instance, what are the exact relations to other, more operational formalisms for specifying functions and algorithms where a fixed evaluation strategy (most often a kind of innermost evaluation) is prescribed? How do the corresponding properties of and proof techniques for the different frameworks compare to each other? In a sense, the investigations to be presented in Chapter 3 correspond closely to these basic questions. Another related issue is the general difficulty of proving termination and confluence properties. In algebraic specification very often some kind of incremental or modular specification discipline is used, for obvious reasons. Hence, an investigation of modular aspects of such equational or rewrite specifications is quite natural and indeed necessary. Modularity and preservation properties of combination mechanisms for rewrite systems are extremely useful for analyzing complex specifications by appropriate decompositions (*divide and conquer*), and, from a dual point of view, for an incremental or modular construction of complex systems from smaller, less complicated ones. In fact, this is a general observation which applies to many fields and problems. In the general term rewriting setting, however, things are rather complicated. The systematic study of such modular aspects in this field started with the pioneering work of Toyama about ten years ago ([Toy87b; Toy87a]) who established the modularity of confluence for disjoint unions and discovered the non-modularity of termination. Subsequently, Middeldorp ([Mid90]) and others obtained further very interesting results. Meanwhile, the study of modular aspects of term rewriting has turned out to be a very active and fruitful field of research.

Our main goals pursued in this thesis can be summarized as follows. We wish to contribute to

- a deeper understanding of structural termination and confluence properties of term rewriting, and
- a thorough analysis of the crucial phenomena and (non-)modular aspects concerning the combination of rewrite systems.

Structure of the Thesis

Essentially this thesis consists of two main parts, one dealing with structural termination (and confluence) properties of single rewrite systems (Chapter 2) and another one where properties of structured combinations of rewrite systems are investigated (Chapters 4 and 5).

In the second chapter we collect in a systematic form the necessary preliminaries needed later on. In Section 2.1 we first introduce the basic theory of abstract reduction systems (ARSs) which in essence are just sets equipped with a binary relation. The relevant properties of ARSs and the most important basic results are presented in a concise and self-contained manner. In Section 2.2 we provide the basic terminology of and theory about term rewriting systems (TRSs) which are ARSs with some additional structure. The format of term rewriting is extended in Section 2.3 where we consider *conditional term rewriting systems* (CTRSs). Here the applicability of rewrite rules is restricted by conditions which have to be recursively evaluated using the same rewrite mechanism. Basic problems with this extended notion of rewriting are also briefly touched. This introduction of abstract reduction systems and (unconditional as well as conditional) term rewriting focuses on known results, methods and techniques for verifying the fundamentally important properties of confluence and termination. Finally, in Section 2.4 different kinds of combinations of systems as well as basic problems arising are discussed. For the special case of disjoint unions we introduce the necessary terminology and basic theory. This is then extended to non-disjoint combinations of *constructor sharing* and of *composable* systems.

Chapter 3 constitutes the first major part of this thesis. We review in Section 3.1 known results on termination and confluence properties of orthogonal TRSs. Then we show in Section 3.2 how to generalize almost all of these results to non-overlapping, but not necessarily left-linear systems. We give various counterexamples showing that the preconditions of the obtained results cannot be dropped. Then, in Section 3.3 we relax the non-overlapping restriction by requiring the systems to be only overlaying and locally confluent. We show that the most important result for non-overlapping systems, namely the equivalence of innermost and general termination, does indeed also hold for this more general class of TRSs. In Section 3.4 we develop an alternate, incompatible approach for showing the equivalence of innermost and general termination, thereby generalizing most of the results of Section 3.2. In Section 3.5 we exclusively deal with properties of innermost rewriting and relate them to the corresponding properties of general rewriting, which again leads to a couple of new results and generalized versions of already known ones. Finally, in Section 3.6, we extend the previous analysis to conditional systems and show how to cope with the additional complications and problems arising there. In particular, we prove here a key lemma which expresses an interesting localized completeness property for conditional overlay systems without a full termination assumption.

In Chapter 4 the modularity of confluence and related properties is dealt with. We provide a comprehensive overview of known results and sketch the basic problems, ideas and proof techniques. The systematic and unifying presentation entails some slight improvements / simplifications of already known modularity results and proofs, respectively. Furthermore, we show by counterexamples that the properties *local confluence* and *joinability of all critical pairs*, which are well-known to be non-equivalent for conditional systems, are not even preserved under signature extensions. Finally, modularity of confluence properties for non-disjoint unions of conditional systems is briefly discussed.

Chapter 5 constitutes the second major part of this thesis. Modular aspects of termination properties are comprehensively treated here. An overview is provided in Section 5.1. First we give a brief historic account of the crucial papers, ideas, approaches and results that have been obtained up to date. Furthermore, basic counterexamples to the modularity of termination in the disjoint union case are collected in a systematic manner. We point out their characteristic features and develop a rough classification of corresponding successful approaches for obtaining positive modularity results for termination. Then, in Section 5.2, the known modularity results for weak termination, weak and strong innermost termination are recapitulated as well as their consequences, for instance concerning semi-completeness. Section 5.3 comprehensively deals with the modularity of various versions of general termination. First we show how, via an abstract structure theorem characterizing minimal counterexamples, many previous results can be generalized and presented in a unifying framework. This powerful abstract structure theorem entails a lot of derived results and criteria for modularity of termination. Then we show how, via a *modular* approach exploiting the modularity of innermost termination and the main results of Chapter 3, further interesting criteria for the preservation of termination and completeness can be obtained relatively easily. And finally, a third basic approach for ensuring modularity of termination is reviewed. In essence, it relies on commutation properties guaranteed by left-linearity and certain uniqueness properties of collapsing reduction. For all three approaches both symmetric and asymmetric criteria for the preservation of termination under disjoint unions are presented. In Section 5.4 it is shown how to extend many previously presented results to combinations of constructor sharing or even of composable systems. Special emphasis is put on the crucial differences of the latter more general combination mechanisms as compared to disjoint unions. And in fact, in most cases the basic ideas and proof techniques for the disjoint union case are also applicable in this more general setting, taking adequately the additional sources of complications into account. Section 5.5 summarizes the corresponding results for conditional rewrite systems. Here we demonstrate in particular that some intuitively appealing assertions are fallacious. Namely, we give counterexamples showing that weak termination as well as weak and strong innermost termination are not even preserved under signature extensions.

Finally, in chapter 6 we give an outline of issues that have not yet been explicitly treated or only touched. Crucial ideas and proof techniques which are generally applicable are mentioned. In Section 6.1 further more general, in particular hierarchical, types of combining rewrite systems are briefly discussed. We give pointers to the relevant literature and sketch basic new problems arising when one considers hierarchically structured combinations of systems. General aspects of combining abstract reduction systems and known approaches in this field are very briefly dealt with in Section 6.2. Topics and fields which are more or less closely related to the main themes in this thesis but which had to be neglected or omitted in the presentation are summarized in Section 6.3.

Appendix B constitutes an independent and self-contained investigation of a parameterized version of the well-founded induction principle. For the sake of readability of the main text the non-trivial and lengthy proof of the important Theorem 3.6.1 is postponed and given in Appendix A.[1]

Main Contributions

From a more abstract conceptual point of view, and corresponding to the overall structure of this thesis, we think that its main contributions are as follows.

- We have initiated a systematic study of relating termination and confluence properties of general rewriting and restricted versions of rewriting, in particular innermost rewriting, resulting in the discovery of some remarkable structural properties. These results entail new insights into the essence of sources for non-termination and non-confluence. Or, to put it positively, our analysis provides new techniques for verifying termination and confluence by a reduction to simpler problems.

- We have developed two new basic approaches for establishing modularity of termination. The first general approach proceeds by a careful analysis of minimal counterexamples and reveals an interesting structural property whose consequences subsume many results in a unifying framework. The second *modular* approach crucially relies on equivalence conditions for innermost and general termination and provides new powerful means for analyzing combined systems. Interestingly, the essential ideas of both approaches extend naturally, both to conditional systems as well as to combinations of systems that are more general than disjoint unions.

On a more concrete level we think that the most notable results obtained are the following:

- equivalence conditions for weak, weak innermost, strong innermost and general termination of rewrite systems (Theorems 3.2.11, 3.3.12, 3.4.11, 3.4.17, 3.4.33, 3.6.17, 3.6.19)

- a localized structural confluence property for conditional overlay system without a full termination assumption (Theorem 3.6.1)

- new results concerning confluence of innermost rewriting and its relation to general rewriting (Theorems / Corollaries 3.5.2, 3.5.4, 3.5.13)

- a general structure theorem characterizing non-modularity of termination (Theorems 5.3.8(a), 5.4.4, 5.5.31)

- modularity criteria for completeness (Theorems 5.3.31, 5.3.38, 5.3.42, 5.4.11, 5.4.12, 5.5.27, 5.5.37)

[1] Actually, subtle aspects in this proof motivated the study of conditions which guarantee correctness of parameterized versions of the well-founded induction principle.

- counterexamples showing that local confluence, joinability of critical pairs, weak termination, weak innermost termination and strong innermost termination of conditional rewrite systems are not even preserved under signature extensions (Examples 4.4.3, 4.4.4, 5.5.2, 5.5.3), and corresponding positive results describing better-behaved cases (Theorems 5.5.20, 5.5.21).

Style of Presentation

Some comments seem in order to ease reading and enhance comprehension of the underlying structure of the thesis and its form of presentation.

Unfortunately, we had to omit a lot of material in order to keep the thesis reasonable in size. For related own, partially joint work with others which is not or only briefly touched, the reader is referred to [MG93; MG95], [WG93; WG94], [Gra95b], [Gra95e], [GW96].

Own results presented in this thesis have already been partially published (for the corresponding most polished paper versions we refer to [Gra94a], [Gra93b], [Gra95a], [Gra95d], [Gra96b], [Gra95c], [Gra96a]).

Concerning the history of ideas and results, we have tried to the best of our knowledge to systematically label or name the origin of important concepts, ideas, definitions and results. Due to this reason we sometimes refer to early or preliminary paper versions. Furthermore, in order to improve readability and to be able to quickly recall things we often use keywords to label introduced concepts / definitions and presented results.

A major presentation problem we were faced with is due to the fact that in some sense the modularity issues treated here have a three-dimensional structure. One dimension is due to the properties considered (like termination, confluence, etc.), another one to the cases of unconditional and conditional TRSs, and a third one stems from the kind of union under consideration (disjoint / constructor sharing / composable systems). Our chosen linearization of this three-dimensional structure (which concerns Chapters 4 and 5), is roughly as follows. The first choice is the property (or class of properties) to be discussed, namely confluence (and related) properties and then termination properties. Secondly, we focus on the (less complicated) unconditional case, and finally we treat disjoint unions in detail. The latter decision has the advantage that the basic ideas, approaches and proof techniques (which are often already quite involved) can be more adequately exhibited than when starting from scratch within the more complicated setting. Consequently, for the remaining cases we only develop (or sometimes only sketch) the necessary adaptations.

Chapter 2

Preliminaries

In Section 2.1 we first introduce the basic theory of *abstract reduction systems* (ARSs) which in essence are just sets equipped with a binary relation. The relevant properties of ARSs and the most important basic results are presented in a concise and self-contained manner. In Section 2.2 we provide the basic terminology of and theory about *term rewriting systems* (TRSs) which are ARSs with some additional structure. The format of term rewriting is extended in Section 2.3 where we consider *conditional term rewriting systems* (CTRSs).

Here the applicability of (term) rewrite rules is restricted by a conjunction of conditions which have to be recursively evaluated using the same rewrite mechanism. Basic problems with this extended notion of rewriting are also briefly touched. This introduction of abstract reduction systems and (unconditional as well as conditional) term rewriting focuses on known results, methods and techniques for verifying the fundamentally important properties of confluence and termination. Finally, in Section 2.4 we give a compact introduction into the combination setting. Different kinds of combinations of systems are discussed as well as basic problems arising. For the special case of disjoint unions we then introduce the necessary terminology and basic theory. This is finally extended to non-disjoint combinations of *constructor sharing* and of *composable* systems.

Readers familiar with the theory of abstract reduction systems, term rewriting and modularity may skip these preliminaries for a first reading and consult them by need.

2.1 Abstract Reduction Systems

First let us remark that we assume familiarity with basic mathematical no(ta)tions, terminology, and facts about sets, relations, functions, pairs, tuples etc..

For more details concerning the following basic no(ta)tions and results for *abstract reduction systems* the reader is referred to [Klo92] (cf. also [Hue80], [New42], [Ros73], [Sta75]).

Definition 2.1.1 (abstract reduction system)
An *abstract reduction system* (ARS for short) is a pair $\mathcal{A} = \langle A, \rightarrow \rangle$ consisting of a (base) set A and a binary relation $\rightarrow \subseteq A \times A$ also called *reduction* or *rewrite relation*. Instead of $(a, b) \in \rightarrow$ we write $a \rightarrow b$. The (binary) identity relation $\{(a, a) \mid a \in A\}$ on A is denoted by id_A. A *reduction(sequence)* or *derivation* (in $\mathcal{A}$) is a (finite or infinite) sequence $a_1, a_2, a_3, \ldots$ of elements in A such that $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \ldots$.

Definition 2.1.2 Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS.

(1) $\rightarrow$ is the *one-step rewrite* relation with transitive closure $\rightarrow^+$ and transitive-reflexive closure $\rightarrow^*$. If $a \rightarrow^* b$ we say that a *reduces* or *rewrites* to b and that b is a *reduct* or *successor* of a. The n-fold composition $\rightarrow^n$ of $\rightarrow$ (for $n \geq 0$) is defined by $\rightarrow^0 := \mathrm{id}_A$ and $\rightarrow^{n+1} := \rightarrow^n \circ \rightarrow$.

(2) $\rightarrow^= := \rightarrow^{\leq 1} := \rightarrow \cup \rightarrow^0$ denotes the reflexive closure of $\rightarrow$.

(3) $\leftarrow := \rightarrow^{-1} := \{(b, a) \mid a \rightarrow b\}$ denotes the inverse of $\rightarrow$. Analogously, ${}^=\!\!\leftarrow$ and ${}^*\!\!\leftarrow$ stand for the reflexive and transitive-reflexive closure of $\leftarrow$, respectively.

(4) $\leftrightarrow := \rightarrow \cup \leftarrow$ denotes the symmetric closure of $\rightarrow$. The transitive-reflexive closure $\leftrightarrow^*$ of $\leftrightarrow$ is called *conversion* or *convertibility* relation.

(5) $\downarrow := \rightarrow^* \circ {}^*\!\!\leftarrow$ is the *joinability* or *common successor* relation. If $a \downarrow b$ then there exists some c with $a \rightarrow^* c \;{}^*\!\!\leftarrow b$. Such an element c is called a *common reduct* of a and b.

Note that the base set of the derived reduction relations $\rightarrow^+$, $\rightarrow^*$ etc. is tacitly assumed to be A thus yielding ARSs over the same base set.

Given some ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ one is often interested in properties of the reduction relation on some specific subset of A. This leads to the following concept.

Definition 2.1.3 (Sub-ARS)
Let $\mathcal{A} = \langle A, \rightarrow_\alpha \rangle$ and $\mathcal{B} = \langle B, \rightarrow_\beta \rangle$ be two ARSs. Then $\mathcal{A}$ is a *sub-ARS* of $\mathcal{B}$, if the following conditions are satisfied:

(1) $A \subseteq B$.

(2) $\rightarrow_\alpha$ is the restriction of $\rightarrow_\beta$ to A, i.e., $\forall a, a' \in A.\, a \rightarrow_\alpha a' \iff a \rightarrow_\beta a'$.

(3) A is closed under $\rightarrow_\beta$, i.e., $\forall a \in A \, \forall b \in B.\, a \rightarrow_\beta b \Longrightarrow b \in A$.

Of particular interest is the sub-ARS determined by an element a in an ARS.

Definition 2.1.4 (reduction graph)
Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS and $a \in A$. Then $\mathcal{G}(a)$, the *reduction graph* of a, is the smallest sub-ARS of $\mathcal{A}$ containing a, i.e., $\mathcal{G}(a) = \langle G(a), \rightarrow \cap (G(A) \times G(A)) \rangle$ with $G(a) = \{b \in A \mid a \rightarrow^* b\}$.

Next we introduce various basic properties of ARSs. If an ARS $\mathcal{A} = \langle A, \to\rangle$ has some property $\mathcal{P}$ (which does not directly depend on the base set A but only on $\to$[1]) we also say that $\to$ has property $\mathcal{P}$. In this case, or if A is clear from the context, we also write $\mathcal{P}(\to)$ or simply $\mathcal{P}$ instead of $\mathcal{P}(\mathcal{A})$. If $\mathcal{P}$ is a universally quantified property of $\mathcal{A}$ of the form $\forall a \in A.\, \mathcal{P}'(a)$ then, slightly abusing notation, we also write $\mathcal{P}(a)$ for the 'local' version $\mathcal{P}'(a)$ of the property $\mathcal{P}$. In order to make clear the respective reduction relation, we also write $\mathcal{P}(a, \to)$ instead of $\mathcal{P}(a)$. Instead of "$\langle A, \to\rangle$ is an ARS" we also say "$\to$ is a reduction relation on A".

Definition 2.1.5 (confluence properties)
Let $\mathcal{A} = \langle A, \to\rangle$ be an ARS.

(1) $\mathcal{A}$ is *confluent* (CONF) if ${}^*\!\!\leftarrow \circ \to^* \subseteq \to^* \circ {}^*\!\!\leftarrow$.

(2) $\mathcal{A}$ is *Church-Rosser* (CR) if $\leftrightarrow^* \subseteq \to^* \circ {}^*\!\!\leftarrow$.

(3) $\mathcal{A}$ is *locally confluent* or *weakly Church-Rosser* (WCR) if $\leftarrow \circ \to \subseteq \to^* \circ {}^*\!\!\leftarrow$.

(4) $\mathcal{A}$ is *strongly confluent* (SCR) if $\leftarrow \circ \to \subseteq \to^* \circ {}^=\!\!\leftarrow$.

(5) $\mathcal{A}$ is *subcommutative* ($\mathrm{WCR}^{\leq 1}$) if $\leftarrow \circ \to \subseteq \to^= \circ {}^=\!\!\leftarrow$.

(6) $\mathcal{A}$ is *uniformly confluent* (WCR^1) if $\leftarrow \circ \to \subseteq \mathrm{id}_A \cup (\to \circ \leftarrow)$.

(7) $\mathcal{A}$ has the *diamond property* ($\Diamond$) if $\leftarrow \circ \to \subseteq \to \circ \leftarrow$.

Note that if the reduction relation $\to$ of $\mathcal{A} = \langle A, \to\rangle$ is reflexive, then the diamond property, subcommutativity and uniform confluence of $\mathcal{A}$ are equivalent.

All confluence properties above except (2) specialize in a straightforward way to elements $a \in A$ by taking a to be the start of the diverging reductions. For instance, we say that $a \in A$ is confluent if for all $b, c \in A$ with $b \,{}^*\!\!\leftarrow a \to^* c$ there exists $d \in A$ with $b \to^* d \,{}^*\!\!\leftarrow c$.

Lemma 2.1.6 (characterizations of confluence)
Let $\mathcal{A} = \langle A, \to\rangle$ be an ARS. Then the following properties are equivalent:

(1) $\to$ is Church-Rosser.

(2) $\to$ is confluent.

(3) $\to^*$ has the diamond property.

(4) $\to^*$ is locally confluent.

(5) $\to^*$ is subcommutative.

(6) $\leftarrow \circ \to^* \subseteq \to^* \circ {}^*\!\!\leftarrow$.

Proof: The implications (1) $\Longrightarrow$ (2) $\Longrightarrow \ldots \Longrightarrow$ (6) are trivial. (6) $\Longrightarrow$ (1) follows from $\forall n \geq 0 : \leftrightarrow^n \subseteq \to^* \circ {}^*\!\!\leftarrow$ which is easily proved by induction on n. ∎

[1]More precisely, this means: $\mathcal{P}(\langle A, \to\rangle) \iff \mathcal{P}(\langle B, \to\rangle)$, where $B = \{a \in A \mid \exists b \in A.\, a \to b\} \cup \{b \in A \mid \exists a \in A.\, a \to b\}$.

Remark 2.1.7 Justified by the equivalence of (1) and (2) above and following a widespread convention we shall subsequently denote the confluence property by CR.

Remark 2.1.8 Note that due to the equivalences $\mathrm{CR}(\to) \iff \Diamond(\to^*) \iff \mathrm{CR}(\to^*)$ (by idempotence of $\cdot^*$), for proving confluence of $\to$ it suffices to prove confluence of $\leadsto$ for some reduction relation $\leadsto$ with the same transitive-reflexive closure, i.e., with $\leadsto^* = \to^*$.

The next result summarizes the relationships between the various confluence properties introduced in Definition 2.1.5 above. It provides interesting sufficient criteria for proving confluence by strengthened versions of local confluence.

Theorem 2.1.9 (confluence by strengthening local confluence)
Let $\mathcal{A} = \langle A, \to \rangle$ be an ARS. Then the following implications hold:

$$\Diamond \Longrightarrow \mathrm{WCR}^1 \Longrightarrow \mathrm{WCR}^{\leq 1} \Longrightarrow \mathrm{SCR} \Longrightarrow \mathrm{CR}.$$

Proof: All implications except the last one are trivial. For proving $\mathrm{SCR} \Longrightarrow \mathrm{CR}$ one shows $\forall n.\ \leftarrow \circ \to^n \subseteq \to^* \circ \ {}^=\!\!\leftarrow$ by induction on n which implies $\leftarrow \circ \to^* \subseteq \to^* \circ \ {}^*\!\!\leftarrow$. Applying Lemma 2.1.6 then yields CR. ■

Refining a given reduction relation may also be useful for proving confluence.

Definition 2.1.10 (compatible refinement, [Sta75])
An ARS $\langle A, \to_1 \rangle$ is called a *refinement* of an ARS $\langle A, \to \rangle$ if $\to \subseteq \to_1^*$. Such a refinement is called *compatible* if $\to_1^* \subseteq \to^* \circ \ {}^*\!\!\leftarrow$.

Compatibility of refinements can be characterized as follows.

Lemma 2.1.11 ([Sta75])
A refinement $\langle A, \to_1 \rangle$ of an ARS $\langle A, \to \rangle$ is compatible if and only if $\to_1 \circ \to^* \subseteq \to^* \circ \ {}^*\!\!\leftarrow$.

Proof: The "$\Longrightarrow$"-direction is trivial. For proving the converse "$\Longleftarrow$" one shows $\forall n \geq 0 : .\ \to_1^{n+1} = \to_1 \circ \to_1^n \subseteq \to^* \circ \ {}^*\!\!\leftarrow$ by an easy induction on n. ■

Theorem 2.1.12 (confluence by compatible refinements, [Sta75])
Let $\langle A, \to \rangle$ be an ARS. Then the following statements are equivalent:

(1) $\langle A, \to \rangle$ is confluent.

(2) There exists a compatible refinement $\langle A, \to_1 \rangle$ of $\langle A, \to \rangle$ which is confluent.

(3) Every compatible refinement $\langle A, \to_1 \rangle$ of $\langle A, \to \rangle$ is confluent.

Proof: Since every ARS is a compatible refinement of itself it suffices to show $\mathrm{CR}(\to) \iff \mathrm{CR}(\to_1)$, for an arbitrary compatible refinement $\langle A, \to_1 \rangle$ of $\langle A, \to \rangle$. Suppose $\mathrm{CR}(\to)$. Then ${}_1^*\!\!\leftarrow \circ \to_1^* \subseteq \to^* \circ \ {}^*\!\!\leftarrow \circ \to^* \circ \ {}^*\!\!\leftarrow \subseteq \to^* \circ \to^* \circ \ {}^*\!\!\leftarrow \circ \ {}^*\!\!\leftarrow = \to^* \circ \ {}^*\!\!\leftarrow \subseteq \to_1^* \circ \ {}_1^*\!\!\leftarrow$ by compatibility of $\to_1$, confluence of $\to$, and the refinement property of

$\to_1$. Conversely, suppose CR($\to_1$). Then we have ${}^*\!\!\leftarrow \circ \to^* \subseteq {}_1^*\!\!\leftarrow \circ \to_1^* \subseteq \to_1^* \circ {}_1^*\!\!\leftarrow$ by the refinement property of $\to_1$ and confluence of $\to_1$. Compatibility of $\to_1$ yields $\to_1^* \subseteq \to^* \circ {}^*\!\!\leftarrow$, hence ${}^*\!\!\leftarrow \circ \to^* \subseteq \to^* \circ {}^*\!\!\leftarrow \circ {}_1^*\!\!\leftarrow$. Finally, by the refinement property and compatibility of $\to_1$ we obtain ${}^*\!\!\leftarrow \circ \to^* \subseteq \to^* \circ {}^*\!\!\leftarrow \circ {}_1^*\!\!\leftarrow \subseteq \to^* \circ {}_1^*\!\!\leftarrow \subseteq \to^* \circ \to^* \circ {}^*\!\!\leftarrow = \to^* \circ {}^*\!\!\leftarrow$ as desired. ■

The condition for being a compatible refinement, $\to \subseteq \to_1^* \subseteq \to^* \circ {}^*\!\!\leftarrow$, is obviously satisfied if $\to_1^*$ is some intermediate reduction relation in between $\to$ and $\to^*$ (cf. Remark 2.1.8).

Corollary 2.1.13 An ARS $\langle A, \to \rangle$ is confluent if there exists a confluent ARS $\langle A, \to_1 \rangle$ satisfying $\to \subseteq \to_1^* \subseteq \to^*$.

For further interesting results on confluence of ARSs we refer to [Oos94a; Oos94b].

Next we introduce further basic properties of ARSs.

Definition 2.1.14 (termination properties)
Let $\mathcal{A} = \langle A, \to \rangle$ be an ARS.

(1) An element $a \in A$ is a *normal form* or *irreducible* (w.r.t. $\to$) if there is no $b \in A$ with $a \to b$. An element $a \in A$ *has a normal form* if $a \to^* b$ for some normal form b. The set of normal forms of $\mathcal{A}$ is denoted by NF($\mathcal{A}$) or NF($\to$) when A is clear from the context.

(2) $\mathcal{A}$ is *weakly normalizing* or *weakly terminating* (WN) if every $a \in A$ has a normal form.

(3) $\mathcal{A}$ is *strongly normalizing* or *terminating* or *Noetherian* (SN) if there are no infinite derivations $a_1 \to a_2 \to a_3 \to \dots$ in $\mathcal{A}$, i.e., every derivation eventually ends in some normal form.

(4) $\mathcal{A}$ is *well-founded* if every non-empty subset of A has a minimal element w.r.t. $\to$. Here, $a \in B \subseteq A$ is *minimal (in B w.r.t. $\to$)* if $a \to b$ implies $b \notin B$.

It is well-known that – by the Axiom of Choice – termination and well-foundedness are equivalent concepts. We shall use the latter notion mainly for ordering relations.

Definition 2.1.15 (properties related to termination)
Let $\mathcal{A} = \langle A, \to \rangle$ be an ARS.

(1) $\mathcal{A}$ is *inductive* (IND) if for every (possibly infinite) derivation $a_1 \to a_2 \to a_3 \to \dots$ in $\mathcal{A}$ there is an $a \in A$ such that $a_n \to^* a$ for all $n \geq 1$.

(2) $\mathcal{A}$ is *increasing* (INC) if there is a mapping $|.| : A \to \mathbb{N}$ such that $\forall a, b \in A.\ a \to b \Longrightarrow |a| < |b|$ (where $<$ is the usual ordering on the natural numbers).

(3) $\mathcal{A}$ is *finitely branching* if for all $a \in A$ the set of one-step reducts, $\{b \in A \mid a \to b\}$, is finite.

(4) $\mathcal{A}$ is *acyclic* if $\to^+$ is irreflexive.

A well-known criterion for termination of finitely branching reduction relations is the following.

Lemma 2.1.16 (König's Lemma)
Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be a finitely branching and acyclic ARS, and let $a \in A$. Then $G(a)$ is finite if and only if a is terminating (this is usually phrased as "a finitely branching tree is infinite if and only if it contains an infinite path").

Definition 2.1.17 (properties related to normal forms)
Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS.

(1) $\mathcal{A}$ has *unique normal forms* (UN) if different normal forms are not convertible, i.e., for all $a, b \in A$, if $a \leftrightarrow^* b$ and $a, b \in \mathrm{NF}(\mathcal{A})$ then $a = b$.

(2) $\mathcal{A}$ has *unique normal forms w.r.t. reduction* ($\mathrm{UN}^{\rightarrow}$) if no element of A reduces to different normal forms, i.e., for all $a, b, c \in A$, if $a \; {}^*\!\!\leftarrow c \rightarrow^* b$ and $a, b \in \mathrm{NF}(\mathcal{A})$ then $a = b$.

(3) $\mathcal{A}$ has the *normal form property* (NF) if every element of A convertible to a normal form reduces to that normal form, i.e., for all $a, b, c \in A$, if $a \leftrightarrow^* b$ and $b \in \mathrm{NF}(\mathcal{A})$ then $a \rightarrow^* b$.

Definition 2.1.18 (completeness properties)
An ARS is *complete* or *convergent* (COMP) if it is confluent and strongly normalizing. It is called *semi-complete* or *uniquely normalizing* if it is confluent and weakly normalizing.

The following lemma summarizes the basic relations between confluence and properties related to normal forms. Note that missing implications do not hold as is easily verified.

Lemma 2.1.19 (confluence vs. normal form properties)
For any ARS the following properties hold:

(1) $\mathrm{CR} \Longrightarrow \mathrm{NF} \Longrightarrow \mathrm{UN} \Longrightarrow \mathrm{UN}^{\rightarrow}$.

(2) $\mathrm{WN} \wedge \mathrm{UN}^{\rightarrow} \Longrightarrow \mathrm{CR}$.

Local confluence does in general not imply confluence, but only under the additional assumption of termination. This fundamentally important result, known as *Newman's Lemma*, is at the heart of many confluence proofs in the literature.

Theorem 2.1.20 (Newman's Lemma, [New42])
A terminating ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ is confluent if and only if it is locally confluent ($\mathrm{WCR} \wedge \mathrm{SN} \Longrightarrow \mathrm{CR}$).

Proof: This result was first shown in [New42], but with an unnecessarily complex proof. The following simple proof by well-founded induction (using $\rightarrow^+$ as the well-founded ordering) stems from [Hue80]. Let $P(x)$ be the predicate $\forall y, z.\, y \; {}^*\!\!\leftarrow x \rightarrow^* z \Longrightarrow y \downarrow z$. Suppose $y \; {}^m\!\!\leftarrow x \rightarrow^n y$. If $m = 0$ or $n = 0$ then $y \downarrow z$ is trivially satisfied. Otherwise, we may assume $y \; {}^*\!\!\leftarrow y_1 \leftarrow x \rightarrow z_1 \rightarrow^* z$ for some y_1, z_1. By local confluence there exists some u with $y_1 \rightarrow^* u \;{}^*\!\!\leftarrow z_1$. By induction hypothesis we have $P(y_1)$ yielding some v with $y \rightarrow^* v \;{}^*\!\!\leftarrow u$. Again by induction hypothesis we know $P(z_1)$. Hence there exists some w with $v \rightarrow^* w \;{}^*\!\!\leftarrow z$ which finally yields $y \downarrow z$ as desired. ■

Next we focus on abstract criteria for ensuring termination of ARSs. An easy result due to Nederpelt ([Ned73]) is the following.

Lemma 2.1.21 ([Ned73])
Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS. If $\mathcal{A}$ is increasing and inductive then it is terminating (INC $\wedge$ IND $\Longrightarrow$ SN).

Proof: Suppose $a_1 \rightarrow a_2 \rightarrow \ldots$ is an infinite derivation in $\mathcal{A}$. By IND there exists some $a \in A$ such that $a_n \rightarrow^* a$ for all $n \geq 1$. By INC there is a mapping $|.| : A \rightarrow \mathbb{N}$ with $|a_1| < |a_2| < \ldots$. Since we also have $|a_n| < |a|$ for all $n \geq 1$ this yields a contradiction. ■

Useful criteria for inferring termination from a combination of restricted confluence and termination properties are the following.

Theorem 2.1.22 ([Klo80])
Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS and $a \in A$. Suppose:

(1) $\mathcal{G}(a)$ is locally confluent (WCR($\mathcal{G}(a)$)), and

(2) a has a normal form b such that the length of reductions from a to b is bounded, i.e., $\exists n \in \mathbb{N}\, \forall m \in \mathbb{N}.\, a \rightarrow^m b \Longrightarrow m \leq n$.

Then $\mathcal{G}(a)$ is terminating (SN($\mathcal{G}(a)$)), and hence also confluent (CR($\mathcal{G}(a)$)).

Proof: We sketch the elegant proof idea detailed in [Klo80]. For reasoning by contradiction suppose there is an infinite derivation $D : a = a_1 \rightarrow a_2 \rightarrow \ldots$. Let $B := \{c \in G(a) \mid c \rightarrow^* b\}$. Then, by assumption (2), D must leave B eventually, i.e.,

$$\exists k \in \mathbb{N}\, \forall j \geq k.\, a_j \notin B. \quad (*)$$

Now define for every $c \in B$ the natural number

$$|c| := \max\{m \mid c \rightarrow^m b\}.$$

By assumption (2), $|c|$ is well-defined. Note that for all $c, c' \in B$:

$$c \rightarrow c' \Longrightarrow |c| > |c'|.$$

Now one can prove by (course-of-values) induction on $|c|$ (using local confluence below a) that B is closed under reduction, i.e.,

$$c \in B \wedge c \rightarrow c' \Longrightarrow c' \in B.$$

This gives a contradiction with (*) above, hence we are done. ■

Lemma 2.1.23 ([Klo80])
Let $\mathcal{A} = \langle A, \to \rangle$ be an ARS. If $\mathcal{A}$ is locally confluent, weakly normalizing and increasing then it is strongly normalizing (WCR $\wedge$ WN $\wedge$ INC $\Longrightarrow$ SN).

Proof: This follows from Theorem 2.1.22 since INC (together with WN) implies the boundedness condition in hypothesis (2) of Theorem 2.1.22. ■

Another very useful consequence of Theorem 2.1.22 which will be exploited later on in Chapter 3 is the following result.

Lemma 2.1.24 (termination by weak termination and uniform confluence, [New42])
Let $\mathcal{A} = \langle A, \to \rangle$ be an ARS. If $\mathcal{A}$ is uniformly confluent and weakly normalizing then it is strongly normalizing (WCR^1 $\wedge$ WN $\Longrightarrow$ SN).

Proof: Here the boundedness condition in hypothesis (2) of Theorem 2.1.22 is satisfied since all reductions of some element to its (unique) normal form must have the same length. ■

Remark 2.1.25 (local versions of properties)
Note that the properties CR, WCR, SCR, WCR$^{\leq 1}$, WCR^1, $\Diamond$, WN, SN, IND, INC, FB, UN, UN$^{\to}$, NF are all preserved downwards w.r.t. taking sub-ARSs, e.g. if $\mathcal{A}$ is a sub-ARS of $\mathcal{B}$ and $\mathcal{B}$ is SN, then $\mathcal{A}$ is also SN. Furthermore note that there are obvious relationships between global and local versions of most of these properties. To be more precise, for an ARS $\langle A, \to \rangle$ and an element $a \in A$ consider the following local termination and confluence properties: SN(a) (every derivation issuing from a eventually ends in some normal form), WN(a) (a has a normal form), CR(a) ($\forall b, c \in G(a).\, b \ {}^*\!\!\leftarrow a \to^* c \Longrightarrow b \to^* \circ \ {}^*\!\!\leftarrow c$), WCR($a$) ($\forall b, c \in G(a).\, b \leftarrow a \to c \Longrightarrow b \to^* \circ \ {}^*\!\!\leftarrow c$). The other local versions of confluence properties, WCR^1 etc., are defined analogously. Then we clearly have

$$\mathrm{SN}(\mathcal{G}(a)) \iff \mathrm{SN}(a).$$
$$\mathrm{WN}(\mathcal{G}(a)) \Longrightarrow \mathrm{WN}(a) \text{ but } \mathrm{WN}(a) \not\Longrightarrow \mathrm{WN}(\mathcal{G}(a)).$$

And moreover

$$\mathrm{CR}(\mathcal{G}(a)) \iff \mathrm{CR}(a).$$
$$\mathrm{WCR}(\mathcal{G}(a)) \Longrightarrow \mathrm{WCR}(a) \text{ but } \mathrm{WCR}(a) \not\Longrightarrow \mathrm{WCR}(\mathcal{G}(a)).$$
$$\mathrm{WCR}^1(\mathcal{G}(a)) \Longrightarrow \mathrm{WCR}^1(a) \text{ but } \mathrm{WCR}^1(a) \not\Longrightarrow \mathrm{WCR}^1(\mathcal{G}(a)), \text{ etc..}$$

The most important criteria for local versions of confluence (not to be confused with local confluence!) and termination used subsequently are the following.

Theorem 2.1.26 (local version of Newman's Lemma, cf. Theorem 2.1.20)
Let $\mathcal{A} = \langle A, \to \rangle$ be an ARS and $a \in A$. Then we have: WCR($\mathcal{G}(a)$) $\wedge$ SN(a) $\Longrightarrow$ CR(a) ($\iff$ CR($\mathcal{G}(a)$)).

Lemma 2.1.27 (local version of Lemma 2.1.24)
Let $\mathcal{A} = \langle A, \to \rangle$ be an ARS and $a \in A$. Then we have: $\mathrm{WCR}^1(\mathcal{G}(a)) \wedge \mathrm{WN}(a) \Longrightarrow \mathrm{SN}(a)$ ($\Longleftrightarrow$ $\mathrm{SN}(\mathcal{G}(a))$).

Note that the last result is not a direct consequence of Lemma 2.1.24 but follows from Theorem 2.1.22 or by a direct proof showing that whenever $\mathrm{WCR}^1(\mathcal{G}(a))$ and $\mathrm{WN}(a)$ with b some normal form of a, then every reduction sequence issuing from a eventually ends in b, and all these derivations have the same length.

Finally let us recall some basic definitions and facts about orderings which are ARSs with special properties, and basic extension constructions for ARSs.

Definition 2.1.28 (orderings)

(1) An ARS $\langle A, > \rangle$ is a *(strict or irreflexive) partial ordering* if $>$ is irreflexive and transitive.

(2) An ARS $\langle A, \gtrsim \rangle$ is a *quasi-ordering* if $\gtrsim$ is reflexive and transitive. Given a quasi-ordering $\langle A, \gtrsim \rangle$ its *associated partial ordering* (or its *strict part*) is defined by $a > b$ if $a \gtrsim b$ but $b \not\gtrsim a$. Its *associated equivalence* $\sim$ is given by: $a \sim b$ if both $a \gtrsim b$ and $b \gtrsim a$.

(3) An ARS $\langle A, \geq \rangle$ is a *reflexive partial ordering* if it is an anti-symmetric quasi-ordering. Given a reflexive partial ordering $\langle A, \geq \rangle$ its *associated partial ordering* (or its *strict part*) is defined by $a > b$ if $a \geq b$ but $b \neq a$.

Definition 2.1.29 (lexicographic product, multiset extension)

(1) The *lexicographic product* of n ARSs $\langle A_i, >_i \rangle$, $1 \leq i \leq n$, is the ARS $\langle A_1 \times \ldots \times A_n, >^{lex} \rangle$ defined by $(a_1, \ldots, a_n) >^{lex} (b_1, \ldots, b_n)$ if $a_j >_j b_j$ for the least $j \in \{1, \ldots, n\}$ with $a_j \neq b_j$.

(2) A *multiset* over a set A is an unordered collection of elements of A in which elements may have multiple occurrences. To distinguish between sets and multisets we use brackets instead of braces for the latter. Furthermore we use standard set notation for operations on multisets, with the obvious meaning. The set of all *finite* multisets over A is denoted by $\mathcal{M}(A)$. Formally, a multiset may be viewed as a mapping from A to $\mathbb{N}$ which indicates how many copies of each element are in the multiset.

(3) The *multiset extension* of an ARS $\langle A, > \rangle$ is the ARS $\langle \mathcal{M}(A), >^{mul} \rangle$ with $>^{mul}$ defined as follows: $M_1 >^{mul} M_2$ if there exist multisets $X, Y \in \mathcal{M}(A)$ such that

- $[] \neq X \subseteq M_1$,
- $M_2 = (M_1 \setminus X) \cup Y$,
- $\forall y \in Y \, \exists x \in X. \, x > y$.

Lexicographic products as well as multiset extensions enjoy many nice properties. For instance, they preserve (ir)reflexivity, transitivity and totality, hence in particular (strict) partial orderings. The fundamentally important property is the preservation of well-foundedness which is non-trivial, in particular for multiset extensions.

Theorem 2.1.30 (lexicographic products preserve well-foundedness)
A lexicographic product of well-founded ARSs is again well-founded.

Theorem 2.1.31 (multiset extensions preserve well-foundedness, [DM79])
The multiset extension $\langle \mathcal{M}(A), >^{mul} \rangle$ of an ARS $\langle A, > \rangle$ is well-founded if and only if $\langle A, > \rangle$ is well-founded.

As a nice application of multiset extension consider the following (proof of a) strengthening of Newman's Lemma 2.1.20.

Theorem 2.1.32 ([WB86])
Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS and let $>$ be a well-founded partial ordering such that $\rightarrow \subseteq >$. Then $\mathcal{A}$ is confluent if and only if for every local divergence $b \leftarrow a \rightarrow c$ there exists a conversion $b = d_1 \leftrightarrow d_2 \leftrightarrow \ldots \leftrightarrow d_{m-1} \leftrightarrow d_m = c$ such that $a > d_k$ for all $k \in \{1, \ldots, m\}$.

Proof: For an arbitrary conversion $a_1 \leftrightarrow \ldots \leftrightarrow a_m$ define its complexity to be the multiset $[a_1, \ldots, a_m]$. Then, by induction over the complexity of conversions using the multiset extension $>^{mul}$ as well-founded ordering, one easily shows that every *peak* $a_{i-1} \leftarrow a_i \rightarrow a_{i+1}$ can be turned into a *valley* $a_{i-1} \rightarrow^* \circ {}^*\!\leftarrow a_{i+1}$ thus yielding $a_1 \downarrow a_m$. ∎

2.2 Term Rewriting Systems

Now we introduce the basic concepts of term rewriting systems which are ARSs with some additional structure. More precisely, the domain of interest consists of a set of inductively constructed objects (terms) and the reduction relation is induced by a set of generic rewrite rules. We focus on those aspects of term rewriting systems which will be needed later on. For a more exhaustive treatment of term rewriting systems and equational reasoning we refer to the surveys [HO80], [DJ90], [Klo92], [AM90], [Pla94] and [Nip94]. Suggestions for fixing some standard notations for rewriting theory are given in [DJ91]. Here we focus in particular on confluence criteria (Section 2.2.1) and termination criteria (Section 2.2.2) for term rewriting systems.

Definition 2.2.1 (signature, terms)

(1) A *signature (vocabulary, alphabet)* is a countable set $\mathcal{F}$ of function symbols. Associated with every $f \in \mathcal{F}$ is a natural number denoting its *arity*, i.e. the number of its arguments. Hence $\mathcal{F} = \bigcup_{n \geq 0} \mathcal{F}^n$ where $\mathcal{F}^n$ denotes the set of all function symbols of arity n. Function symbols of arity 0 are called *constants*. To indicate the arity n of some $f \in \mathcal{F}^n$ we sometimes also write f^n.

(2) The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of *terms* over a signature $\mathcal{F}$ and some countably infinite set $\mathcal{V}$ of *variables* with $\mathcal{F} \cap \mathcal{V} = \emptyset$ is the smallest set containing $\mathcal{V}$ such that $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ whenever $f \in \mathcal{F}^n$ and $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ for $i = 1, \ldots, n$. If $c \in \mathcal{F}$ is a

constant we write c instead of $c()$. In examples we shall use sometimes mixed infix and prefix notation as well as omit braces for unary operators. Terms containing no variables are called *ground* or *closed* terms. The subset of all ground terms of $\mathcal{T}(\mathcal{F},\mathcal{V})$ is denoted by $\mathcal{T}(\mathcal{F})$.

In the sequel we shall assume that some set $\mathcal{T}(\mathcal{F},\mathcal{V})$ of terms over $\mathcal{F}$ and $\mathcal{V}$ is given. Furthermore $\mathcal{F}$ is supposed to contain at least one constant which entails that the set $\mathcal{T}(\mathcal{F})$ of ground terms is non-empty. [2]

Definition 2.2.2 (root symbol, size, depth)
Let $\mathcal{T}(\mathcal{F},\mathcal{V})$ be given, and $t \in \mathcal{T}(\mathcal{F},\mathcal{V})$. The set of all function symbols occurring in t is denoted by $Fun(t)$, the set of all variables in t by $Var(t)$. The *root symbol* of t is defined by $root(t) = t$ if $t \in \mathcal{V}$ and $root(t) = f$ if $t = f(t_1,\ldots,t_n)$. The number of occurrences of a symbol $sym \in \mathcal{F} \cup \mathcal{V}$ in t is denoted by $|t|_{sym}$. The *size* of t, denoted by $|t|$, is the number of all occurrences of symbols from $\mathcal{T}(\mathcal{F},\mathcal{V})$ in t. The *depth* of a term t is the number of symbols on a longest 'path' of t from its root to a 'leaf' (in the usual tree representation), i.e.: $depth(t) = 1$ if t is a variable or a constant, and $depth(f(t_1,\ldots,t_n)) = 1 + \max_{i\in\{1,\ldots,n\}}\{depth(t_i)\}$.

Definition 2.2.3 (positions, prefix ordering, subterms, replacement)

(1) The set $Pos(t)$ of *positions (occurrences)* of a term $t \in \mathcal{T}(\mathcal{F},\mathcal{V})$ is defined by

$$Pos(t) = \begin{cases} \{\lambda\}, \text{ if } t \in \mathcal{V} \cup \mathcal{F}^0 \\ \{\lambda\} \cup \{ip \,|\, p \in Pos(t_i), 1 \le i \le n\}, \\ \qquad \text{if } t = f(t_1,\ldots,t_n), n \ge 1\,. \end{cases}$$

Hence, positions are sequences of natural numbers with λ denoting the empty sequence (concatenation is denoted by juxtaposition).

(2) For $p \in Pos(t)$ the *subterm (occurrence)* t/p of t at position p is given by

$$t/p = \begin{cases} t, \text{ if } p = \lambda \\ t_i/q, \text{ if } p = iq, 1 \le i \le n, t = f(t_1,\ldots,t_n)\,. \end{cases}$$

In the latter case the subterms $t_i = t/i$, $1 \le i \le n$, are called *immediate* or *direct* subterms of t. Any subterm t/p of t with $p \ne \lambda$ is a *proper* subterm of t. We use $\trianglelefteq$ (with inverse $\trianglerighteq$) and $\triangleleft$ (with inverse $\triangleright$) for denoting the subterm and proper subterm ordering relation, respectively. For subterms s/p, s/q of s with $p \le q$ we say that s/p is a *superterm* of s/q (in s). In this case we call s/p a *proper* superterm of s/q (in s) if $p < q$. And, if s/q is an immediate subterm of s/p (in s) we call s/p the[3] *immediate* superterm of s/q (in s).

(3) The sets of *variable* and of *non-variable positions* of t are given by

$$\mathcal{V}Pos(t) = \{p \in Pos(t) \,|\, t/p \in \mathcal{V}\}$$

[2]This is not a severe restriction and sometimes simplifies the discussion by excluding degenerate cases.

[3]Note that the immediate superterm of a some subterm (in a given term) is uniquely determined provided it exists.

and

$$\mathcal{F}Pos(t) = \{p \in Pos(t) \,|\, root(t/p) \in \mathcal{F}\} \ ,$$

respectively.

(4) Positions are partially ordered by the *prefix ordering* $\geq$ where $p \geq q$ if $p = qr$ for some r. If $p \geq q$ we say that p is *below* q or q is *above* p. If $p \geq q$ and $p \neq q$ we say that p is *strictly below* q or q *strictly above* p (denoted by $p > q$). If neither $p \geq q$ nor $q \geq p$ then p and q are *disjoint (parallel, independent)*, which is also denoted by $p|q$. For $p \geq q$ we define $p \backslash q = r$ where r is given by $p = qr$.

(5) Replacing in $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ the subterm at position $p \in Pos(t)$ by $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined by

$$t[p \leftarrow s] = \begin{cases} s, \text{ if p} = \lambda \\ f(t_1, \ldots, t_i[q \leftarrow s], \ldots, t_n), \\ \quad \text{if p} = \text{iq}, 1 \leq \text{i} \leq \text{n}, \text{t} = \text{f}(\text{t}_1, \ldots, \text{t}_\text{n}) \ . \end{cases}$$

Parallel replacement of subterms of t at mutually disjoint positions $p_1, \ldots, p_n \in Pos(t)$ by respective terms $s_1, \ldots, s_n$ is denoted by $t[p_1 \leftarrow s_1] \ldots [p_n \leftarrow s_n]$ or $t[p_i \leftarrow s_i \,|\, 1 \leq i \leq n]$.

In order to be able to speak about terms having a certain shape the notion of context is very useful.

Definition 2.2.4 (contexts)
A *context* is a 'term with holes' where holes are represented by a distinguished variable $\Box$ (which is exclusively used for that purpose).[4] Following [Mid90] we distinguish between three kinds of contexts. This allows for a compact reasoning in proofs later on concerning the treatment of many degenerate cases of contexts.

- $C[, \ldots,]$ denotes a context containing at least one occurrence of $\Box$.
- $C\langle, \ldots, \rangle$ denotes a context containing zero or more occurrences of $\Box$.
- $C\{, \ldots, \}$ denotes a context containing zero or more occurrences of $\Box$ which is different from $\Box$.

Contexts of the form $C[, \ldots,]$ are called *strict*. If $C[, \ldots,]$ is a (strict) context with $n \geq 1$ occurrences of $\Box$ and $t_1, \ldots, t_n$ are terms, then $C[t_1, \ldots, t_n]$ is the result of replacing (from left to right) the occurrences of $\Box$ by $t_1, \ldots, t_n$. A context $C[, \ldots,]$ containing exactly one occurrence of $\Box$ is also denoted by $C[]$. Note that s is a subterm of t if and only if there exists a context $C[]$ with $t = C[s]$. If we want to indicate the position p of s in t we write $t = C[s]_p$. Similarly, $C[t_1, \ldots, t_n]_{p_1, \ldots, p_n}$ indicates the positions of the subterms $t_1, \ldots, t_n$. Slightly abusing notation we write $t = C[s]_P$ for $C[s, \ldots, s]_{p_1, \ldots, p_n}$ if $P = \{p_1, \ldots, p_n\}$ is the set of all (mutually disjoint) positions of

[4]Alternately, one may also view $\Box$ as a special fresh constant which is always tacitly available. More formally, contexts can be viewed as 'terms' with bound variables (cf. [DJ91]). But in order to avoid corresponding lambda-notations like $\lambda x. C[x]$ we stick to the notation presented here which will be convenient for our purposes.

$\Box$ in $C[,\ldots,]$. The corresponding notations for $C\langle,\ldots,\rangle$ and $C\{,\ldots,\}$ are defined analogously. A context $C[,\ldots,] \neq \Box$ is said to be *non-empty*.

Definition 2.2.5 (substitution)
A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$ such that its *domain* $Dom(\sigma) = \{x \in \mathcal{V} \mid \sigma(x) \neq x\}$ is finite. The *variable range* of a substitution σ is given by $\mathcal{V}Ran(\sigma) = \bigcup_{x \in Dom(\sigma)} Var(\sigma(x))$. The *empty* substitution is denoted by ϵ (with $Dom(\epsilon) = \emptyset$). Substitutions are (uniquely) extended to morphisms from $\mathcal{T}(\mathcal{F},\mathcal{V})$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$ via $\sigma(f(t_1,\ldots,t_n)) = f(\sigma(t_1),\ldots,\sigma(t_n))$ for $f \in \mathcal{F}^n$ and $t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{V})$. Instead of $\sigma(s)$ we shall also write σs. The *composition* $\sigma \circ \tau$ of two substitutions σ and τ is defined by $(\sigma \circ \tau)(x) = \sigma(\tau(x))$. We call $\sigma(t)$ an *instance* of t. Two terms s and t are *unifiable* if there is a substitution σ such that $\sigma(s) = \sigma(t)$.[5] Such a σ is a *unifier* of s and t. It is a *most general unifier (mgu)* (of s and t) if for every unifier τ there exists a substitution ρ with $\tau = \rho \circ \sigma$.

Definition 2.2.6 (term rewriting system)
A *term rewriting system* or *rewrite system* (TRS for short) is a pair $(\mathcal{F},\mathcal{R})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R} \subseteq \mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V})$ of (*rewrite* or *reduction*) *rules* (l,r), denoted by $l \to r$ with $l \notin \mathcal{V}$ and $Var(r) \subseteq Var(l)$. Instead of $(\mathcal{F},\mathcal{R})$ we also write $\mathcal{R}^{\mathcal{F}}$ or simply $\mathcal{R}$ when $\mathcal{F}$ is clear from the context or irrelevant. The set of left- and right-hand sides of $\mathcal{R}$ are given by $lhs(\mathcal{R}) = \{l \mid l \to r \in \mathcal{R}\}$ and $rhs(\mathcal{R}) = \{r \mid l \to r \in \mathcal{R}\}$.

Remark 2.2.7 (restrictions on rewrite rules)
Note that the restrictions imposed on rewrite rules $l \to r$ which forbid variable left hand sides and extra variables on the right hand side are quite natural and not severe. In particular, concerning termination, they only exclude trivial cases. However, in a more general equational setting it makes sense to lift these restrictions in order to be get nice symmetry properties (cf. [Ges90] for a more detailed discussion of this aspect).

Furthermore note that rewrite rules are implicitly universally quantified. Hence, equality of rewrite rules is interpreted modulo variable renamings. When considering distinct rewrite rules we may always assume w.l.o.g. that they do not have common variables.

Definition 2.2.8 (rewrite relation)
A binary relation $\to$ on terms is a *rewrite relation* if it is closed under contexts and substitutions, i.e. if $s \to t$ then $C[s] \to C[t]$ and $\sigma s \to \sigma t$ for all contexts $C[]$ and substitutions σ.

In the literature closure under contexts is also called *monotonicity* or *replacement property*. Closure under substitutions is also phrased as *stability*, *compatibility* or *full invariance property*.

Definition 2.2.9 (rewrite relation induced by a TRS)
The *rewrite* or *reduction relation* $\to_{\mathcal{R}}$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ induced by a TRS $\mathcal{R}^{\mathcal{F}}$ is the smallest

[5] See e.g. [BS94] for a recent survey on unification theory.

rewrite relation containing $\mathcal{R}$. Equivalently, $s \to_{\mathcal{R}} t$ if there exists a substitution σ and a context $C[]$ such that $s = C[\sigma l]$ and $t = C[\sigma r]$. We say that s *rewrites* or *reduces* to t by *contracting* redex σl. Here a *redex* (reducible expression) is an instance of a left-hand side of a rewrite rule.[6] We call $s \to t$ a *rewrite step* or *reduction step*. In order to indicate the position p of the contracted redex, the corresponding substitution σ and the applied rule $l \to r$, we also use notations like $s \to_p t$, $s \to_{p,l\to r} t$ and $s \to_{p,\sigma,l\to r} t$. A step of the form $s \to_\lambda t$ is called a *root reduction step*.

By associating with every TRS $\mathcal{R}^{\mathcal{F}}$ the ARS $\langle \mathcal{T}(\mathcal{F},\mathcal{V}), \to_{\mathcal{R}}\rangle$ all notions defined in the preceding section for ARSs carry over to TRSs. If $\to_{\mathcal{R}}$ has a certain property we also equivalently say that $\mathcal{R}$ has that property.

Note that rewriting a term in a TRS involves essentially two kinds of non-determinism, namely the choice of the redex to be contracted and the choice of the rule to be applied. Non-determinism of the first kind can be eliminated or restricted by imposing certain (position selection) strategies on rewriting. We consider here two basic variants where either minimal or maximal redexes are chosen for replacement.

Definition 2.2.10 (innermost/outermost reduction)
Let $\mathcal{R}$ be a TRS. If $s = C[\sigma l]_p \to C[\sigma r]_p = t$ (for some $C[]_p$, σ and $l \to r \in \mathcal{R}$) such that s/q is irreducible for every $q \in Pos(s)$ with $q > p$ (i.e., every proper subterm of σl is irreducible), we write $s \underset{i}{\longrightarrow} t$. If $s = C[\sigma l]_p \to C[\sigma r]_p = t$ (for some $C[]_p$, σ and $l \to r \in \mathcal{R}$) such that s/q is not a redex for every $q \in Pos(s)$ with $q < p$, we write $s \underset{o}{\longrightarrow} t$. The relations $\underset{i}{\longrightarrow}$ and $\underset{o}{\longrightarrow}$ are called *innermost* and *outermost reduction*, respectively. We also use notations like $s \underset{i}{\longrightarrow}_{p,\sigma,l\to r} t$, $s \underset{i}{\longrightarrow}_p t$, $s \underset{o}{\longrightarrow}_{p,\sigma,l\to r} t$, $s \underset{o}{\longrightarrow}_p t$ for indicating the position of the innermost/outermost redex to be contracted, the applied substitution and rule. $\mathcal{R}$ is *weakly innermost normalizing* (WIN) or *weakly innermost terminating* if the innermost reduction relation $\underset{i}{\longrightarrow}$ is weakly normalizing. $\mathcal{R}$ is *strongly innermost normalizing* (SIN) or *innermost terminating* if $\underset{i}{\longrightarrow}$ is strongly normalizing. *Weak* and *strong outermost normalization* are defined analogously. $\mathcal{R}$ is said to be *innermost/outermost confluent* if innermost/outermost reduction is confluent.

Let us remark that innermost and outermost reduction can of course be seen as abstract reduction relations, but they are no rewrite relations in the sense of 2.2.8. Innermost reduction is not closed under substitutions, and outermost reduction is neither closed under substitutions nor under contexts.

Innermost confluence is interesting, since denotational or operational semantics of computational formalisms is often defined by a kind of innermost evaluation procedure, which is similar to innermost reduction. For instance, the *call by value* parameter passing style in functional programming languages essentially means innermost evaluation. Outermost reduction can be viewed as a kind a lazy computation. Hence, outermost confluence is interesting for ensuring well-definedness of lazy semantics.

[6]Note that in order to specify uniquely the *contractum* of (i.e., the result of contracting) a term $t = \sigma l$ into $t' = \sigma r$, one needs in general σ, l and r (or, equivalently, t and the applicable rule $l \to r$). However, for *non-overlapping* (cf. Definition 2.2.16) TRSs the contractum of a (root-reducible) term as well as the applied rule are unique.

Before proceeding with criteria and techniques for proving confluence and termination of TRSs let us introduce some more basic syntactic terminology needed later on.

Definition 2.2.11 (syntactical properties of rules, TRSs)
A term is *linear* if every variable occurs at most once in it. A rule $l \to r$ is

(1) *left-linear* if l is linear,

(2) *right-linear* if r is linear,

(3) *linear* if both l and r are linear,

(4) *non-erasing* (or *variable-preserving*) if $Var(l) = Var(r)$,[7]

(5) *collapsing* if $r \in \mathcal{V}$,

(6) *duplicating* if there is a variable $x \in \mathcal{V}$ with $|l|_x < |r|_x$.

A TRS $\mathcal{R}$ is left-linear (LL) / right-linear (RL) / linear / non-erasing (NE) if all its rules have the respective property. It is collapsing (COL) / duplicating (DUP) if one of its rewrite rules has the respective property. It is *non-collapsing* or *collapse-free* (NCOL) if it is not collapsing, and *non-duplicating*[8] (NDUP) if it is not duplicating.

$\mathcal{R}$ is called *interreduced* (or *irreducible*) if, for every rule $l \to r \in \mathcal{R}$, r is irreducible w.r.t. $\mathcal{R}$ and l is irreducible w.r.t. $\mathcal{R} \setminus \{l \to r\}$.

Function symbols of a TRS can always be classified syntactically as follows.

Definition 2.2.12 (constructor, defined function symbol)
For any TRS $\mathcal{R}^{\mathcal{F}}$ the rules of $\mathcal{R}$ partition $\mathcal{F}$ into a set $\mathcal{D} = \{root(l) \,|\, l \to r \in \mathcal{R}\}$ of *defined* (function) symbols and a set $\mathcal{C} = \mathcal{F} \setminus \mathcal{D}$ of *constructors* (with $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$).

Restricting the function symbols in left-hand side arguments to be constructors leads to the following.

Definition 2.2.13 (constructor system)
A TRS $\mathcal{R}^{\mathcal{F}}$ with $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ (as above) is a *constructor system* (CS) if every left-hand side $f(t_1, \ldots, t_n)$ of a rule in $\mathcal{R}$ satisfies $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{C}, \mathcal{V})$.

Any TRS is a 'TRS with constructors (and defined symbols)' in the sense of Definition 2.2.12 but not necessarily a constructor system. Rewrite specification with constructor systems corresponds to the intuition that functions are defined by declaring their behaviour on data arguments (constructor terms). This is usually meant when one speaks of imposing a constructor discipline for specification.

[7] Consequently, $l \to r$ is called *erasing* if $Var(l) \setminus Var(r) \neq \emptyset$.
[8] called *conservative* in [FJ95].

2.2.1 Confluence Criteria

Local confluence for TRSs can be characterized by considering critical local divergences of rewrite steps. For this purpose one needs the following concept discovered by Knuth and Bendix in their pioneering paper [KB70].

Definition 2.2.14 (critical peak, critical pair)

(1) Let $l_1 \to r_1$ and $l_2 \to r_2$ be two rules of a TRS $\mathcal{R}$ that have no variables in common. Let $p \in \mathcal{F}Pos(l_1)$ such that l_1/p and l_2 are unifiable with mgu σ. Then the divergence $(\sigma l_1)[p \leftarrow \sigma r_2]\ {}_p{\leftarrow} \sigma l_1 \to_\lambda \sigma r_1$ is a *critical peak* of $\mathcal{R}$, determined by *overlapping* $l_2 \to r_2$ into $l_1 \to r_1$ at position p. The pair of reducts $\langle (\sigma l_1)[p \leftarrow \sigma r_2], \sigma r_1 \rangle$ is the corresponding *critical pair*. If the two rules are renamed versions of the same rule we do not consider the case $p = \lambda$ (which gives only rise to improper divergences).

(2) A critical peak $s\ {}_p{\leftarrow} u \to_\lambda t$ (and its corresponding critical pair $\langle s, t\rangle$ of $\mathcal{R}$) is *joinable* if $s \downarrow_\mathcal{R} t$. It is called *trivial* as well as the corresponding critical pair, if $s = t$.

(3) The set of all critical pairs between rules of $\mathcal{R}$ is denoted by CP($\mathcal{R}$). Joinability of all critical pairs of $\mathcal{R}$ is abbreviated by JCP($\mathcal{R}$) or simply JCP.

Remark 2.2.15 (critical peaks / pairs are asymmetric)
Observe the asymmetry in the definition of critical peaks / pairs. This entails in particular, that for a critical overlay $t_1\ {}_\lambda{\leftarrow} s \to_\lambda t_2$ we always get another critical overlay $t_2\ {}_\lambda{\leftarrow} s \to_\lambda t_1$, hence also two corresponding critical pairs, namely $\langle t_1, t_2\rangle$ and $\langle t_2, t_1\rangle$. Moreover we note, that a critical pair may correspond to (i.e., be obtained from) several distinct critical peaks (taking into account the position of the inside rewrite step and the applied rules). For the sake of readability we dispense here with a completely formal definition of critical peaks which is straightforward.

Definition 2.2.16 (non-overlapping, orthogonal, overlay system)
Let $\mathcal{R}$ be a TRS.

(1) $\mathcal{R}$ is *non-overlapping* (NO) or *non-ambiguous* if CP($\mathcal{R}$) $= \emptyset$. $\mathcal{R}$ is *weakly non-overlapping* (WNO) or *weakly non-ambiguous* if all its critical pairs are trivial.

(2) $\mathcal{R}$ is *orthogonal* (ORTH) if it is non-overlapping and left-linear. $\mathcal{R}$ is *weakly orthogonal* (WORTH) if it is weakly non-overlapping and left-linear.

(3) $\mathcal{R}$ is an *overlay system* (OS) or *overlaying* if every critical pair of $\mathcal{R}$ originates from an *overlay*, i.e., by overlapping rules at root position.

Note that for instance any constructor system is an overlay system, but not necessarily vice versa.

Lemma 2.2.17 (Critical Pair Lemma, [Hue80])
A TRS is locally confluent if and only if all its critical pairs are joinable (WCR $\Longleftrightarrow$ JCP).

Combining Newman's Lemma and the Critical Pair Lemma yields the following fundamental result.

Theorem 2.2.18 ([KB70])
A terminating TRS is confluent (hence complete) if and only if all its critical pairs are joinable (SN $\Longrightarrow$ [CR $\iff$ JCP]).

This result provides the basis for transforming a terminating TRS into an equivalent confluent one by so-called *completion procedures* which – roughly speaking – try to resolve critical pair divergences by adding appropriate new rewrite rules.

Proving confluence of TRSs without termination is in general much more difficult. Below we shall present two such criteria which rely on the corresponding results for ARSs.

First we note that even the absence of critical pairs which guarantees local confluence, due to Lemma 2.2.17, is not sufficient for confluence.

Example 2.2.19 ([Hue80])
The TRS

$$\mathcal{R} = \begin{cases} f(x,x) \to a \\ f(x,g(x)) \to b \\ c \to g(c) \end{cases}$$

has no critical pairs, hence is non-overlapping and thus locally confluent. But for instance the term $f(c,c)$ has two distinct normal forms:

$$a \leftarrow f(c,c) \to f(c,g(c)) \to b\,.$$

So $\mathcal{R}$ lacks unique normal forms and is not confluent.

Another interesting counterexample due to Barendregt and Klop is the following.

Example 2.2.20 ([Klo80])
The TRS

$$\mathcal{R} = \begin{cases} f(x,x) \to a \\ g(x) \to f(x,g(x)) \\ b \to g(b) \end{cases}$$

is non-overlapping and has unique normal forms. But we have

$$g(b) \to f(b,g(b)) \to f(g(b),g(b)) \to a$$

and hence

$$g(b) \to g(g(b)) \to^* g(a)\,.$$

Since a and $g(a)$ are not joinable, $\mathcal{R}$ is not confluent.

Note that both TRSs in the examples above contain non-linear rules. By forbidding such rules and imposing a kind of strong joinability requirement on critical pairs confluence can be guaranteed.

Definition 2.2.21 (strongly closed)
A TRS $\mathcal{R}$ is *strongly closed* if for every critical pair $\langle s, t\rangle$ there exist terms u and v such that $s \to^* u \;{}^{=}\!\!\leftarrow t$ and $s \to^= v \;{}^{*}\!\!\leftarrow t$.

Theorem 2.2.22 (linearity plus strong closedness implies confluence, [Hue80])
A left- and right-linear strongly closed TRS is strongly confluent, hence confluent.

Without right-linearity, strong closedness is not sufficient to ensure confluence, as shown by the following nice counterexample due to J.-J. Lévy.

Example 2.2.23 ([Hue80])
The TRS $\mathcal{R}$ given by

$$\mathcal{R} = \begin{cases} f(a,a) \to g(b,b) & g(b,b) \to f(a,a) \\ a \to a' & b \to b' \\ f(x,a') \to f(x,x) & g(x,b') \to g(x,x) \\ f(a',x) \to f(x,x) & g(b',x) \to g(x,x) \end{cases}$$

is left-linear, strongly closed and locally confluent, but not right-linear and not confluent. We have for instance $f(a',a') \leftrightarrow^* g(b',b')$ with $f(a',a') \not\downarrow g(b',b')$.

One way to get rid of the rather unnatural right-linearity condition in Theorem 2.2.22 above is to look for another reduction relation with the same transitive-reflexive closure (as that of $\to$) and to change the closure property for critical pairs appropriately in order to obtain strong confluence of the new reduction relation (cf. Remark 2.1.8 and Theorem 2.1.9). For that purpose we need the following definitions.

Definition 2.2.24 (parallel reduction)
Given a TRS $\mathcal{R}$ its induced *parallel reduction* relation $\mathrel{+\!\!\!\!\!\mapsto}_{\mathcal{R}}$ (or simply $\mathrel{+\!\!\!\!\!\mapsto}$) is defined as the smallest reflexive relation containing $\to_{\mathcal{R}}$ and verifying

$$s_1 \mathrel{+\!\!\!\!\!\mapsto} t_1 \wedge \ldots\ s_n \mathrel{+\!\!\!\!\!\mapsto} t_n \implies f(s_1, \ldots, s_n) \mathrel{+\!\!\!\!\!\mapsto} f(t_1, \ldots, t_n)$$

for all $f \in \mathcal{F}$ (n-ary). If s reduces to t by a parallel step contracting the redexes in s at some set $P = \{p_1, \ldots, p_k\}$ of parallel positions from P, this is also denoted by $s \mathrel{+\!\!\!\!\!\mapsto}_P t$. Writing $s \mathrel{+\!\!\!\!\!\mapsto}_{\geq p} t$ means that all redexes contracted are at pairwise disjoint positions below $p \in Pos(s)$.

Clearly, parallel and ordinary reduction are related as follows: $\to \subseteq \mathrel{+\!\!\!\!\!\mapsto} \subseteq \to^*$, hence also $\to^* = \mathrel{+\!\!\!\!\!\mapsto}^*$. Thus, for showing confluence of $\to$, it suffices to prove (strong) confluence of $\mathrel{+\!\!\!\!\!\mapsto}$.

Definition 2.2.25 (parallel closed)
A TRS is called *parallel closed* if every critical pair $\langle s, t\rangle$ satisfies $s \mathrel{+\!\!\!\!\!\mapsto} t$.

Theorem 2.2.26 (left-linearity plus parallel closedness implies confluence, [Hue80])
If $\mathcal{R}$ is a left-linear and parallel closed TRS, then $\mathrel{+\!\!\!\!\!\mapsto}_{\mathcal{R}}$ is subcommutative, and hence $\to$ (or $\mathcal{R}$) is confluent.

Proof: The ingenious proof in [Hue80] proceeds by considering parallel one-step divergences of the form $t \;{}_P\!\!\leftarrow\!\!\!\!\|\; s \;\|\!\!\!\!\rightarrow_Q\; u$. By induction over the sum of the sizes of those 'P- and Q-redexes' which are affected in both steps, and by case analysis according to the positions in P, Q, it is shown that $\|\!\!\!\!\rightarrow$ is subcommutative ($\mathrm{WCR}^{\leq 1}(\|\!\!\!\!\rightarrow)$), hence confluent (by Theorem 2.1.9) which implies confluence of $\rightarrow$ (by Remark 2.1.8) since $\|\!\!\!\!\rightarrow^* = \rightarrow^*$. ∎

This result is important in practice. An immediate consequence is the following older result.

Theorem 2.2.27 (confluence by orthogonality, [Ros73])
Every orthogonal TRS is confluent.

Before proceeding let us elaborate a bit on variations of critical pair conditions for ensuring confluence of left-linear TRSs. Huet's result above states that

(1) $s \;\|\!\!\!\!\rightarrow\; t$ for every critical pair $\langle s, t\rangle$ of $\mathcal{R}$

suffices for ensuring confluence. Surprisingly, it still seems to be unknown whether any of the following conditions also suffices (see [DJK91], Problem 13 of J.-J. Lévy).

(2) $s \;\|\!\!\!\!\rightarrow\; t$ or $t \;\|\!\!\!\!\rightarrow\; s$ for every critical pair $\langle s, t\rangle$ of $\mathcal{R}$.

(3) $s \rightarrow^= t$ or $t \rightarrow^= s$ for every critical pair $\langle s, t\rangle$ of $\mathcal{R}$.

(4) $t \;\|\!\!\!\!\rightarrow\; s$ for every critical pair $\langle s, t\rangle$ of $\mathcal{R}$.

(5) $t \rightarrow^= s$ for every critical pair $\langle s, t\rangle$ of $\mathcal{R}$.

Clearly, the following implications hold between these conditions: (5) $\Longrightarrow$ (3) $\Longrightarrow$ (2), (5) $\Longrightarrow$ (4) $\Longrightarrow$ (2). We remark that due to Theorem 2.2.22 and Theorem 2.2.18 potential counterexamples for (3) and (5) would have to be (besides left-linear) non-right-linear, non-terminating and non-orthogonal. Moreover, a potential proof for (2) or (4) is not possible via showing strong confluence (or even subcommutativity) of parallel reduction as in Theorem 2.2.26. This can be seen from the following example.

Example 2.2.28 The TRS

$$\mathcal{R} = \begin{cases} f(a, x) \rightarrow f(a, g(x)) \\ \quad a \rightarrow b \\ \; g(x) \rightarrow x \end{cases}$$

is left- (and even right-) linear and satisfies conditions (2) and (4) above – the only critical peak $f(b, x) \;{}_1\!\!\leftarrow f(a, x) \rightarrow_\lambda f(a, g(x))$ is joinable by one parallel step from right to left: $f(a, g(x)) \|\!\!\!\!\rightarrow f(b, x)$. Confluence of $\mathcal{R}$ cannot be shown using any of the results above. In particular, $\mathcal{R}$ is non-terminating and neither parallel closed nor strongly closed. Moreover, parallel reduction is also not strongly confluent. For instance we have: $f(b, b) \leftarrow\!\!\!\!\| \; f(a, a) \|\!\!\!\!\rightarrow f(a, g(a))$, where $f(a, g(a))$ reduces in two parallel steps to $f(b, b)$ but there is no term t with $f(b, b) \;\|\!\!\!\!\rightarrow^*\; t \;{}^{=}\!\!\leftarrow\!\!\!\!\|\; f(a, g(a)$.

On the positive side we mention one slight generalization of Theorem 2.2.26 which is indeed possible, using a restricted variant of condition (2).

Theorem 2.2.29 (a slightly generalized version of Theorem 2.2.26)
Let $\mathcal{R}$ be a left-linear TRS satisfying the following condition. For every critical peak $s\ {}_p{\leftarrow} u \rightarrow_\lambda t$ of $\mathcal{R}$ we have: $s \mathrel{\Vert\!\!\!\longrightarrow} t$ or $t \mathrel{\Vert\!\!\!\longrightarrow}_{\geq p} s$. Then $\mathrel{\Vert\!\!\!\longrightarrow}_{\mathcal{R}}$ is subcommutative, and hence $\rightarrow$ (or $\mathcal{R}$) is confluent.

Proof (idea): We can use essentially the same inductive proof as for Theorem 2.2.26 but have to consider an additional case when analysing one-step $\mathrel{\Vert\!\!\!\longrightarrow}$-divergences. More precisely, the argumentation is as follows: First we remark that it suffices to analyse one-step $\mathrel{\Vert\!\!\!\longrightarrow}$-divergences of the (degenerate) form $s''\ {}_P{\mathrel{\longleftarrow\!\!\!\Vert}}\ u' \rightarrow_\lambda t'$. Now, if some $p \in P$ is 'critical' here w.r.t. the root reduction step, we decompose the divergence into $s''\ {}_{P\setminus\{p\}}{\mathrel{\longleftarrow\!\!\!\Vert}}\ s'\ {}_p{\leftarrow} u' \rightarrow_\lambda t'$. For the case that the critical peak $s\ {}_p{\leftarrow} u \rightarrow_\lambda t$ corresponding to $s'\ {}_p{\leftarrow} u' \rightarrow_\lambda t'$ is joinable via $s \mathrel{\Vert\!\!\!\longrightarrow} t$ we get $s''\ {}_{P\setminus\{p\}}{\mathrel{\longleftarrow\!\!\!\Vert}}\ s' \mathrel{\Vert\!\!\!\longrightarrow} t'$ and can appeal to the induction hypothesis (as in the proof of Theorem 2.2.26) in order to obtain $s'' \mathrel{\Vert\!\!\!\longrightarrow} \circ \mathrel{\longleftarrow\!\!\!\Vert} t'$. Otherwise, i.e., if joinability is via $t \mathrel{\Vert\!\!\!\longrightarrow}_{\geq p} s$, we get $s''\ {}_{P\setminus\{p\}}{\mathrel{\longleftarrow\!\!\!\Vert}}\ s'\ {}_{\geq p}{\mathrel{\longleftarrow\!\!\!\Vert}}\ t'$ which obviously can be merged into one parallel step: $s'' \mathrel{\longleftarrow\!\!\!\Vert} t'$. Hence, in this case we directly obtain the desired joinability ($s'' \mathrel{\Vert\!\!\!\longrightarrow} \circ \mathrel{\longleftarrow\!\!\!\Vert} t'$) without using the induction hypothesis. ■

A simple example where this result but none of the previous ones applies is the TRS

$$\mathcal{R} = \left\{ \begin{array}{rcl} f(g(x)) & \rightarrow & f(h(x,x)) \\ g(a) & \rightarrow & g(g(a)) \\ h(a,a) & \rightarrow & g(g(a)) \end{array} \right.$$

which has only one critical peak $f(g(g(a)))\ {}_1{\leftarrow} f(g(a)) \rightarrow_\lambda f(h(a,a))$ that is joinable by $f(h(a,a)) \mathrel{\Vert\!\!\!\longrightarrow}_{\geq 1} f(g(g(a)))$.

For another interesting generalization of Theorem 2.2.26 we refer to [Toy88]. Further, more recent investigations by van Oostrom show how one can extend the range of such criteria by appropriately generalizing the notion of being *parallel* ([Oos94b; Oos95]). For another recent approach to confluence of possibly non-terminating, left-linear TRSs via *parallel critical pairs* we refer to [Gra95b].

2.2.2 Termination Criteria

Termination of (finite) TRSs is an undecidable property ([HL78]), even for one-rule systems ([Dau92]). But due to its fundamental importance for computation by rewriting many sufficient criteria, techniques and methods for proving termination have been developed (cf. [Der87] for a fairly comprehensive survey). We review here some of the most important concepts and results.

First we remark that termination of a TRS $\mathcal{R}^{\mathcal{F}}$ over $\mathcal{T}(\mathcal{F},\mathcal{V})$ is equivalent to termination of $\mathcal{R}^{\mathcal{F}}$ on all ground terms – provided that $\mathcal{T}(\mathcal{F})$ is non-empty as always assumed. Furthermore, $\mathcal{R}^{\mathcal{F}}$ is terminating if and only $\rightarrow^+$ is a well-founded partial ordering on $\mathcal{T}(\mathcal{F})$. Hence, a general method for proving termination of $\mathcal{R}^{\mathcal{F}}$ works as follows.[9]

[9] According to [Der87] this result — for finite $\mathcal{R}^{\mathcal{F}}$ — dates back to [MN70] and [Lan75].

Define a partial ordering $>$ on $\mathcal{T}(\mathcal{F})$ such that

$$(1) \quad > \text{ is well-founded, and}$$

$$(2) \quad \forall s,t \in \mathcal{T}(\mathcal{F}).\, s \rightarrow t \Longrightarrow s > t\ .$$

Then, obviously, $\mathcal{R}^{\mathcal{F}}$ is terminating. And conversely, if $\mathcal{R}^{\mathcal{F}}$ is terminating then $\rightarrow^+$ satisfies (1) and (2). The *well-founded mapping method* ([HO80]) suggests to take a well-founded partial ordering $>_D$ on some set D and some *termination function* $\tau : \mathcal{T}(\mathcal{F}) \rightarrow D$ for defining $>_D$ by

$$(3) \quad s > t \iff \tau(s) >_D \tau(t)\ .$$

This method is specialized to the *increasing interpretation* ([HO80], [Lan77]) or *monotone algebra* method ([Zan94]) by taking D to be an $\mathcal{F}$-algebra and τ to be the unique $\mathcal{F}$-homomorphism from $\mathcal{T}(\mathcal{F})$ to D. Then, (2) is guaranteed by closure of $>$ under context, i.e.

$$(4) \quad \forall s,t \in \mathcal{T}(\mathcal{F})\, \forall f \in \mathcal{F}.\, s > t \Longrightarrow f(\ldots,s,\ldots) > f(\ldots,t,\ldots)\ , \textit{and}$$

and closure (of (2), for the rules) under (ground) substitution, i.e.

$$(5) \quad \forall l \rightarrow r \in \mathcal{R}^{\mathcal{F}}.\, \forall \sigma\, , \sigma\ \mathcal{T}(\mathcal{F})\text{-ground substitution}.\, \sigma l > \sigma r\ .$$

Note that closure of $>$ under context, (4), can be weakened into

$$(4') \quad \forall s,t \in \mathcal{T}(\mathcal{F})\, \forall f \in \mathcal{F}.\, s \rightarrow t \wedge s > t \Longrightarrow f(\ldots,s,\ldots) > f(\ldots,t,\ldots)\ .$$

without affecting soundness (and completeness) of the method ([KL80]). Furthermore note that variables can easily be taken into account by defining for $s,t \in \mathcal{T}(\mathcal{F},\mathcal{V})$:

$$(3') \quad s > t \iff [\forall \alpha : \mathcal{V} \rightarrow D.\, (\alpha(s))_D >_D (\alpha(t))_D\,]$$

where the *term evaluation* $\alpha : \mathcal{T}(\mathcal{F},\mathcal{V}) \rightarrow D$ (in the algebra D) induced by a *variable assignment* $\alpha : \mathcal{V} \rightarrow D$ is inductively defined by

$$\alpha(f(t_1,\ldots,t_n)) = f_D(\alpha(t_1),\ldots,\alpha(t_n))\ .$$

In the terminology of [Zan94], a structure $\langle D, >_D \rangle$ with D an $\mathcal{F}$-algebra for which the underlying set is equipped with a well-founded ordering $>_D$ and where each algebra operation is strictly monotone in all arguments (w.r.t. $>_D$) is called a *well-founded monotone* $\mathcal{F}$-algebra. Then the above approach for characterizing termination of rewriting may be rephrased semantically as follows ([Zan94]): A TRS $\mathcal{R}^{\mathcal{F}}$ over $\mathcal{T}(\mathcal{F},\mathcal{V})$ is terminating if and only if there exists some non-empty well-founded monotone algebra satisfying $l > r$ for every rule $l \rightarrow r \in \mathcal{R}^{\mathcal{F}}$ (with $>$ as in (3′)).

For the special 'syntactical' case that the $\mathcal{F}$-algebra considered is the term algebra $\mathcal{T}(\mathcal{F},\mathcal{V})$, the above considerations motivate the following notions.

Definition 2.2.30 (rewrite ordering, reduction ordering, compatibility)

(1) A *rewrite ordering* is a partial ordering on $\mathcal{T}(\mathcal{F},\mathcal{V})$ which is a rewrite relation, i.e., which is closed under context and substitution (see Definition 2.2.8).

(2) A *reduction ordering* is a well-founded rewrite ordering.

(3) A TRS $\mathcal{R}$ is *compatible* with a partial ordering $>$ if there exists a rewrite ordering $\succ$ extending both $\rightarrow^+_{\mathcal{R}}$ and $>$, i.e., with $\rightarrow^+_{\mathcal{R}} \cup > \subseteq \succ$.[10]

Obviously, a TRS $\mathcal{R}$ is terminating if and only if $\rightarrow^+_{\mathcal{R}}$ is contained in some reduction ordering. If one wants to prove termination of a TRS by some recursively defined ordering on $\mathcal{T}(\mathcal{F},\mathcal{V})$ then closure under context and substitution are often easy to verify. Proving irreflexivity and transitivity often also turns out to be feasible, using some inductive reasoning and case analysis. But the most difficult task usually is to show well-foundedness which may be very hard to tackle directly. Fortunately, there is an important class of orderings for which well-foundedness is obtained for free, namely so-called *simplification orderings* which were first defined by Dershowitz ([Der79]). This result is based on the deep and powerful Tree Theorem of Kruskal ([Kru60]). Recently, simplification orderings and related notions of *simple termination* have attracted much attention since they enjoy interesting characterization and modularity properties. But there has also been considerable confusion about these notions as well as various different definitions in the literature. In particular, for the case of infinite signatures, things become rather subtle. The presentation here mainly follows the recent approach of Middeldorp and Zantema ([MZ94]) who succeeded to bring the definition of simplification orderings and simple termination in full accordance with Kruskal's Tree Theorem which is the basic motivation for the notion of simplification. This notion of simplification comprises two ingredients, namely:

(1) a term decreases by removing parts of it, and

(2) a term decreases by replacing a function symbol by a smaller one (w.r.t. some given *precedence*, i.e. partial ordering, on the signature).

If the signature is infinite, both of these ingredients are essential for the applicability of Kruksal's Tree Theorem which has not been taken fully into account in previous works. We shall also adopt the approach of [MZ94] to base the definition of simplification orderings on (strict) partial orderings instead of quasi-orderings[11] and consequently on *partial well-orderings* instead of *well-quasi-orderings* since this is less susceptive to mistakes and partially results in weaker proof obligations.[12]

Definition 2.2.31 (subterm property, homeomorphic embedding, self-embedding)
Let $\mathcal{T}(\mathcal{F},\mathcal{V})$ be given.

(1) A binary relation R on $\mathcal{T}(\mathcal{F},\mathcal{V})$ has the *subterm property* if $\rhd \subseteq R$ (cf. Definition 2.2.3(2)).

(2) The TRS $\mathcal{E}mb(\mathcal{F})$ consists of all *embedding* or *projection* rules

$$f(x_1,\ldots,x_n) \rightarrow x_i$$

[10]Note that our definition of compatibility is different from containment (in the sense: $l \succ r$ for all $l \rightarrow r \in \mathcal{R}$). In the literature, different definitions of compatibility are used, cf. e.g. [MZ94].

[11]However, here we shall not explicitly treat quasi-simplification orderings.

[12]But cf. also [Kru72] for advantages of wqo's.

with $f \in \mathcal{F}^n$ and $i \in \{1, \ldots, n\}$, where $x_1, \ldots, x_n$ are pairwise distinct variables. We abbreviate $\rightarrow^{+}_{\mathcal{E}mb(\mathcal{F})}$ by $\rhd_{emb}$ and $\leftarrow^{*}_{\mathcal{E}mb(\mathcal{F})}$ by $\unlhd_{emb}$. The relation $\unlhd_{emb}$ is called *(homeomorphic) embedding*.

(3) An infinite sequence $t_1, t_2, t_3, \ldots$ of terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is *self-embedding* if there exist indices i, j with $1 < i < j$ such that $t_i \unlhd_{emb} t_j$.

(4) A TRS $\mathcal{R}^{\mathcal{F}}$ is *self-embedding* if there exists an infinite derivation $t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow \ldots$ in $\mathcal{R}^{\mathcal{F}}$ with $t_i \unlhd_{emb} t_j$ for some i, j with $1 < i < j$.

The embedding relation and the subterm property are related as follows.

Lemma 2.2.32 A rewrite ordering $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ has the subterm property if and only if it contains the transitive closure of the reduction relation induced by the embedding rules, i.e. $\succ \supseteq \rhd_{emb}$.

For finite signatures the embedding relation enjoys the following fundamental property.

Theorem 2.2.33 (Kruskal's Tree Theorem – finite version, [Kru60])
Every infinite sequence of ground terms over some finite signature is self-embedding.

This motivates the definition of some properties of (rewrite) orderings and TRSs which are related to embedding.

Definition 2.2.34 (subterm compatibility, subterm compatible termination, simplification ordering, simple termination)

(1) A TRS $\mathcal{R}^{\mathcal{F}}$ is *subterm compatible* (or *simplifying*) if $\mathcal{R}$ is contained in some rewrite ordering with the subterm property. $\mathcal{R}^{\mathcal{F}}$ is *subterm compatibly terminating* if $\mathcal{R}$ is contained in some well-founded rewrite ordering (i.e. reduction ordering) with the subterm property.

(2) For a finite signature $\mathcal{F}$ a *simplification ordering* (on $\mathcal{T}(\mathcal{F}, \mathcal{V})$) is a rewrite ordering (on $\mathcal{T}(\mathcal{F}, \mathcal{V})$) with the subterm property. A TRS $\mathcal{R}^{\mathcal{F}}$ over some finite signature $\mathcal{F}$ is said to be *simply terminating* if $\mathcal{R}$ is contained in some simplification ordering (on $\mathcal{T}(\mathcal{F}, \mathcal{V})$).

The notion simple termination is justified by the following result which in essence is due to Dershowitz ([Der79]).

Theorem 2.2.35 (simple termination implies termination for finite signatures)
Any simply terminating TRS $\mathcal{R}^{\mathcal{F}}$ over some finite signature $\mathcal{F}$ is terminating.

Proof: Suppose $\mathcal{R}^{\mathcal{F}}$ is simply terminating, but not terminating. Hence $\mathcal{R}$ is contained in some simplification ordering $\succ$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$, and there exists a infinite reduction sequence $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \ldots$ involving only ground terms ($\in \mathcal{T}(\mathcal{F})$). Kruskal's Tree Theorem 2.2.33 now implies the existence of i, j with $1 < i < j$ such that $t_i \trianglelefteq_{emb} t_j$. By Lemma 2.2.32 and the fact that $\succ$ is a simplification ordering we get $t_j \succcurlyeq t_i$. However, since $\mathcal{R}$ is contained in $\succ$, $t_i \to_{\mathcal{R}}^{+} t_j$ implies $t_i \succ t_j$. But this is a contradiction with the fact that $\succ$ is a partial ordering. Hence $\mathcal{R}^{\mathcal{F}}$ must be terminating. ∎

Unfortunately, also simple termination of (finite) TRSs is undecidable, even for one-rule systems ([MG95]).

We note in particular, that simple termination and subterm compatible termination are equivalent notions for TRSs over finite signatures. More generally, the following equivalent characterizations are easy to prove.[13]

Lemma 2.2.36 (characterizations of simple termination for finite signatures, [KO90a], [KO90b], [Zan93], [Gra91], [Ohl92])
Let $\mathcal{R}^{\mathcal{F}}$ be a TRS over some finite signature $\mathcal{F}$. Then the following assertions are equivalent.

(1) $\mathcal{R}$ is simply terminating.

(2) $\mathcal{R}$ is subterm compatibly terminating.

(3) $\mathcal{R} \cup \mathcal{E}mb(\mathcal{F})$ is terminating.

(4) $\mathcal{R}$ is subterm compatible and terminating.

(5) $\mathcal{R}$ is subterm compatible.

(6) $\mathcal{R} \cup \mathcal{E}mb(\mathcal{F})$ is acyclic.

For infinite signatures it is well-known that a rewrite ordering with the subterm property need not be well-founded.

Example 2.2.37 (subterm compatibility does not imply termination)
Consider for instance the TRS $\mathcal{R}^{\mathcal{F}}$ consisting of infinitely many constants a_i and rewrite rules $a_i \to a_{i+1}$ for all $i \geq 1$. The rewrite ordering $\to_{\mathcal{R}}^{+}$ vacuously satisfies the subterm property, but $\mathcal{R}^{\mathcal{F}}$ is non-terminating:

$$a_1 \to_{\mathcal{R}} a_2 \to_{\mathcal{R}} a_3 \to_{\mathcal{R}} \ldots .$$

The reason is that Kruskal's Tree Theorem (its finite version above) is not applicable any more. But – as it is also well-known – applicability of the general version of Kruskal's Tree Theorem (see below) can be recovered by imposing an additional condition on the signature, namely to be *well-quasi-ordered* or, closely related, to be *well-partially ordered*. Syntactically this approach leads to a definition and characterization of the general version of simplification ordering and simple termination by means of an extended version of embedding.

[13] Kurihara and Ohuchi ([KO90a], [KO92]) were the first to make explicit some of these relations, by means of the embedding rules.

Definition 2.2.38 (extended embedding, [MZ94])
Let $\succ$ be a partial ordering on a signature $\mathcal{F}$. The TRS $\mathcal{E}mb(\mathcal{F},\succ)$ (over the signature $\mathcal{F}$) consists of all rewrite rules of $\mathcal{E}mb(\mathcal{F})$ together with all rewrite rules

$$f(x_1,\ldots,x_n) \to g(x_{i_1},\ldots,x_{i_m})$$

with $f \in \mathcal{F}^n$, $g \in \mathcal{F}^m$, $n \geq m \geq 0$, $f \succ g$, and $1 \leq i_1 < \ldots < i_m \leq n$ whenever $m \geq 1$. Here $x_1,\ldots,x_n$ are pairwise distinct variables. We abbreviate $\to^+_{\mathcal{E}mb(\mathcal{F},\succ)}$ by $\succ_{emb}$ and $\leftarrow^*_{\mathcal{E}mb(\mathcal{F},\succ)}$ by $\preccurlyeq_{emb}$. The latter relation is called *extended (homeomorphic) embedding*.[14]

Since $\mathcal{E}mb(\mathcal{F},\emptyset) = \mathcal{E}mb(\mathcal{F})$, extended (homeomorphic) embedding generalizes (homeomorphic) embedding.

Next we turn to the required property for signatures. We only recall here those (well-known) concepts and results from [MZ94] which are needed for the sake of understanding.[15]

Definition 2.2.39 (partial well-ordering)
A partial ordering $\succ$ on a set A is called a *partial well-ordering* (PWO for short) if every partial ordering extending $\succ$ (including $\succ$ itself) is well-founded.[16]

Every well-founded ordering on some finite set is a PWO. By definition every PWO is a well-founded ordering, but not vice versa in general. For instance, the empty relation on an infinite set is obviously a well-founded ordering but not a PWO. Clearly, every total well-founded ordering (also called *well-ordering*) is a PWO. Furthermore, PWOs enjoy many nice properties, e.g. they are preserved under intersection (on the same set) and under multiset extension. The following fundamental result due to Kruskal ([Kru60])[17] is the generalization of Theorem 2.2.33 to the case of arbitrary signatures.

Theorem 2.2.40 (Kruskal's Tree Theorem – general version, [Kru60])
If $\succ$ is a PWO on a signature $\mathcal{F}$ then $\succ_{emb}$ is a PWO on $\mathcal{T}(\mathcal{F})$.

If a precedence $\succ$ is a PWO on a signature $\mathcal{F}$ one may ask whether $\succ_{emb}$ can be restricted while retaining the property of being a PWO on $\mathcal{T}(\mathcal{F})$. In particular, one may ask whether all rewrite rules in $\mathcal{E}mb(\mathcal{F},\succ)$ are really necessary. In case there is a uniform bound on the arities of function symbols in $\mathcal{F}$, it is indeed possible to reduce the set $\mathcal{E}mb(\mathcal{F},\succ)$ (see [MZ94] for more details). And in the finite signature case the rules of $\mathcal{E}mb(\mathcal{F})$ are clearly sufficient since the empty relation is a PWO on any finite set.

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[14] In [MZ94] this extended version is called *homeomorphic embedding* and the ordinary version *embedding*. In order to be consistent with most definitions in the literature we prefer *extended (homeomorphic) embedding* and *(homeomorphic) embedding*.

[15] See [Kru72] for an early survey of the rich and well-developed theory of *well-quasi-orderings* (and *partial well-orderings)*

[16] For some equivalent definitions in terms of *good* (and *bad*) sequences, *chains* or *antichains* see [MZ94].

[17] Cf. [NW63] for a beautiful non-constructive proof.

The revised version of the notions of simplification ordering and simple termination for the arbitrary signature case now reads as follows.

Definition 2.2.41 (simplification ordering, simple termination, [MZ94])

(1) A *simplification ordering* is a rewrite ordering on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that contains $\succ_{emb}$ for some PWO $\succ$ on $\mathcal{F}$.

(2) A TRS $\mathcal{R}^{\mathcal{F}}$ is *simply terminating* if $\mathcal{R}$ is contained in some simplification ordering on $\mathcal{T}(\mathcal{F}, \mathcal{V})$.[18]

For the case of finite signatures this definition coincides with Definition 2.2.34(2). In other words, a rewrite ordering on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ with $\mathcal{F}$ finite is a simplification ordering (according to Definition. 2.2.41) if and only if it has the subterm property. Moreover, simple termination does also imply termination for arbitrary signatures.

Theorem 2.2.42 (simple termination implies termination, [MZ94])
Any simply terminating TRS is terminating.

Finally, the very useful characterization of simple termination of $\mathcal{R}^{\mathcal{F}}$ by termination of $\mathcal{R} \cup \mathcal{E}mb(\mathcal{F})$ for the finite signature case (cf. Lemma 2.2.36) also extends to the case of arbitrary signatures.

Lemma 2.2.43 (characterization of simple termination, [MZ94; MZ95])
For any TRS $\mathcal{R}^{\mathcal{F}}$ the following statements are equivalent:

(1) $\mathcal{R}^{\mathcal{F}}$ is simply terminating.

(2) $\mathcal{R}^{\mathcal{F}} \cup \mathcal{E}mb(\mathcal{F}, \succ)$ is terminating for some PWO $\succ$ on $\mathcal{F}$.

(3) $\mathcal{R}^{\mathcal{F}} \cup \mathcal{E}mb(\mathcal{F}, \succ)$ is acyclic[19] for some PWO $\succ$ on $\mathcal{F}$.

Next we collect the relations between the introduced notions for the case of arbitrary signatures.

Lemma 2.2.44 (subterm related notions of termination, general case)
For a TRS $\mathcal{R}^{\mathcal{F}}$ over an arbitrary signature $\mathcal{F}$ the following implications hold:

	$(1a)$	$\mathcal{R}$ is simply terminating.
$\Longleftrightarrow$	$(1b)$	$\mathcal{R} \cup \mathcal{E}mb(\mathcal{F}, \succ)$ is terminating.
$\Longleftrightarrow$	$(1c)$	$\mathcal{R} \cup \mathcal{E}mb(\mathcal{F}, \succ)$ is acyclic.
$\Longrightarrow$	(2)	$\mathcal{R}$ is subterm compatibly terminating.
$\Longleftrightarrow$	(3)	$\mathcal{R} \cup \mathcal{E}mb(\mathcal{F})$ is terminating.
$\Longrightarrow$	(4)	$\mathcal{R}$ is subterm compatible and terminating.
$\Longrightarrow$	(5)	$\mathcal{R}$ is subterm compatible.
$\Longleftrightarrow$	(6)	$\mathcal{R} \cup \mathcal{E}mb(\mathcal{F})$ is acyclic.

[18] In view of this revised definition of simplification ordering and simple termination, compared to previous terminology, Middeldorp & Zantema ([MZ95]) use the notion *pseudo-simple termination* instead of our subterm compatible termination.

[19] i.e., $\to^+_{\mathcal{R} \cup \mathcal{E}mb(\mathcal{F}, \succ)}$ is irreflexive

We note in particular that the proofs of the corresponding implications in Lemma 2.2.36 essentially carry over to the case of arbitrary signatures. Missing implications do not hold. The implication (5) $\Longrightarrow$ (4) is refuted by Example 2.2.37. Counterexamples to (2) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (2) are the following.

Example 2.2.45 ([MZ94])
Consider the TRS $\mathcal{R}^{\mathcal{F}}$ over the signature $\mathcal{F} = \{f_i, g_i \mid i \in \mathbb{N}\}$ with $\mathcal{R}$ consisting of all rules

$$f_i(g_j(x)) \to f_j(g_j(x))$$

where $i, j \in \mathbb{N}$ with $i < j$. This system is not simply terminating. To see this, let $\succ$ be any PWO on $\mathcal{F}$ and consider the infinite sequence $(f_i)_{i \geq 1}$. Due to the PWO property of $\succ$ we have $f_j \succ f_i$ for some $i < j$. Hence $\mathcal{E}mb(\mathcal{F}, \succ)$ contains the rule $f_j(x) \to f_i(x)$, yielding the infinite derivation

$$f_i(g_j(x)) \to f_j(g_j(x)) \to f_i(g_j(x)) \to \ldots$$

in the TRS $\mathcal{R} \cup \mathcal{E}mb(\mathcal{F}, \succ)$ ([MZ94]). But $\mathcal{R}^{\mathcal{F}}$ is subterm compatibly terminating since $\mathcal{R} \cup \mathcal{E}mb(\mathcal{F}) = \{f_i(g_j(x)) \to f_j(g_j(x)) \mid i, j \in \mathbb{N}, i < j\} \cup \{f_i(x) \to x, g_i(x) \to x\}$ is terminating as is easily proved by a lexicographic argument.[20]

Example 2.2.46 ([Ohl92])
Consider the TRS $\mathcal{R}^{\mathcal{F}}$ over the signature $\mathcal{F} = \{a, g, f_i \mid i \in \mathbb{N}\}$ with $\mathcal{R}$ consisting of all rules

$$f_j(a) \to f_{j+1}(g(a))$$

where $j \in \mathbb{N}$. Clearly, $\mathcal{R}^{\mathcal{F}}$ is terminating and also subterm compatible, since $\to^+_{\mathcal{R} \cup \mathcal{E}mb(\mathcal{F})}$ (which is irreflexive) is a rewrite ordering with the subterm property containing $\mathcal{R}$ (more precisely, $\to^+_{\mathcal{R} \cup \mathcal{E}mb(\mathcal{F})}$ is the smallest rewrite ordering with the subterm property containing $\mathcal{R}$). But $\mathcal{R}^{\mathcal{F}}$ is not subterm compatibly terminating because $\mathcal{R} \cup \mathcal{E}mb(\mathcal{F})$ is not terminating. For instance, we have the infinite derivation

$$f_1(a) \to f_2(g(a)) \to f_2(a) \to f_3(g(a)) \to f_3(a) \to \ldots$$

in $\mathcal{R} \cup \mathcal{E}mb(\mathcal{F})$. This means that any rewrite ordering with the subterm property containing $\to^+_{\mathcal{R}}$ is not well-founded.

Remark 2.2.47 (extensions of Lemma 2.2.44)
We note that the properties (2)-(6) in Lemma 2.2.44 are equivalent to each other not only in the case of finite signatures (cf. Lemma 2.2.36) but also for more general cases. The crucial point is to ensure applicability of Kruskal's Tree Theorem. This is guaranteed if for some given TRS $\mathcal{R}^{\mathcal{F}}$ every reduction sequence involves only finitely many function symbols. Simple ways of ensuring this property are to require finiteness of either $\mathcal{F}$ or $\mathcal{R}$. A more refined criterion is to require that $\mathcal{R}^{\mathcal{F}}$ *introduces only finitely many function symbols* ([Ohl92]), i.e., to require finiteness of $\bigcup_{l \to r \in \mathcal{R}} (Fun(r) \setminus Fun(l))$.

The most prominent and best known example of a well-founded rewrite ordering with the subterm property is the *recursive path ordering* of Dershowitz ([Der82]).

[20] In [MZ94] $\mathcal{R}^{\mathcal{F}}$ is shown to be even polynomially, hence totally terminating.

Definition 2.2.48 (recursive path ordering, [Der82])
Let $\succ$ be a partial ordering (*precedence*) on some (arbitrary) signature $\mathcal{F}$. The *(induced) recursive path ordering* (RPO for short) $>_{rpo}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined recursively as follows:[21]

$s = f(s_1, \ldots, s_n) >_{rpo} g(t_1, \ldots, t_m) = t$ if
- $f = g$ (and $m = n$) and $[s_1, \ldots, s_n] >_{rpo}^{mul} [t_1, \ldots, t_n]$, or
- $f \succ g$ and $[s] >_{rpo}^{mul} [t_1, \ldots, t_m]$, or
- $s_i \gtrsim_{rpo} t$ for some $i \in \{1, \ldots, n\}$.[22]

Theorem 2.2.49 ([Der82])
Any recursive path ordering is a rewrite ordering with the subterm property, and it is well-founded if and only if the precedence is well-founded.

The well-foundedness property of a recursive path ordering (over a well-founded precedence) is proved in [Der82] by applying Kruskal's Tree Theorem 2.2.33, exploiting the *incrementality* of the RPO-construction (i.e., extending the precedence extends the induced RPO) and the well-known fact that every well-founded partial ordering can be extended to a total well-founded partial ordering. Since $>_{rpo}$ extends $\succ_{emb}$, for any precedence $\succ$ on the signature, $>_{rpo}$ is a simplification ordering (according to Definition 2.2.41) whenever the precedence is a PWO. In particular, if the signature is finite then every RPO is a simplification ordering. In the case of arbitrary signatures every RPO is included in some simplification ordering. Hence, every TRS whose rewrite relation is included in some RPO over a well-founded precedence is simply terminating ([MZ94]).

Extensions and generalizations of RPO-like orderings have been investigated in [KL80] and subsequently by many authors. Many other variations of precedence based syntactical ordering have been developed, too, see [Ste94; Ste95b] for a comprehensive survey on simplification orderings (over finite signatures). Particularly interesting from a practical point of view are orderings or ordering schemes which allow for a flexible integration and combination of both syntactical and semantical termination arguments, for instance the *general path ordering* of Dershowitz & Hoot ([DH95])[23] Interesting transformation based techniques for proving termination are explored among others in [BD86], [BL90], [FZ95] [Zan94; Zan95], [Ste95a].

2.3 Conditional Term Rewriting Systems

In this Section we shall introduce conditional term rewriting systems. First some basic terminology is recalled, different ways of assigning operational semantics as well as aspects of expressive power are discussed. Moreover, additional complications which are specific for conditional rewriting are stressed. The focus of the presentation is on

[21] Variables are interpreted as constants which are unrelated w.r.t. the precedence.
[22] Here $\gtrsim_{rpo}$ denotes the associated quasi-ordering obtained by identifying permutatively equivalent terms, i.e. terms which are equal up to (recursive) permutation of arguments.
[23] Cf. also [Ges94] for an improved version.

confluence criteria for conditional rewriting, with or without termination (see Sections 2.3.1 and 2.3.2, respectively).

Definition 2.3.1 (conditional term rewriting system)
A *conditional term rewriting system* (CTRS for short) is a pair $(\mathcal{F}, \mathcal{R})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R}$ of *conditional rewrite rules* of the form

$$l \to r \Longleftarrow s_1 = t_1, \ldots, s_n = t_n$$

with $l, r,\ s_1, \ldots, s_n\ ,\ t_1, \ldots, t_n\ \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. The equations $s_1 = t_1, \ldots, s_n = t_n$ are the *conditions*[24] of the rule. As for unconditional TRSs (cf. Definition 2.2.6) we require $l \notin \mathcal{V}$ and $Var(r) \subseteq Var(l)$, i.e. no variable left-hand sides and no extra variables on right-hand sides. Extra variables in conditions are allowed if not stated otherwise. Rules without conditions (i.e., $n = 0$) will be written as $l \to r$. Instead of $(\mathcal{F}, \mathcal{R})$ we also write $\mathcal{R}^{\mathcal{F}}$ or simply $\mathcal{R}$ if $\mathcal{F}$ is clear from the context or irrelevant.

Remark 2.3.2 The CTRS

$$\left\{ \begin{array}{rcl} 0 \leq y \to true & & \\ s(x) \leq 0 \to false & & \\ s(x) \leq s(y) \to x \leq y & & \\ x \leq y \to true & \Longleftarrow & x \leq z = true, z \leq y = true \end{array} \right.$$

defining $\leq$ on natural numbers and expressing transitivity of $\leq$ has as extra variable z in the condition of the last rule which is ok. From a programming point of view examples like (cf. [DOS88b])

$$\left\{ \begin{array}{rcl} fib(0) \to \langle 0, s(0) \rangle & & \\ fib(s(x)) \to \langle z, y + z \rangle & \Longleftarrow & fib(x) = \langle y, z \rangle \end{array} \right.$$

involving extra variables on right-hand sides, too, are also interesting, but will not be considered here due to severe technical complications. Restricted classes of such systems are treated for instance in [ALS94] and [SMI95].

Note that, due to the implicit universal quantification of conditional rewrite rules (equations), when applying such a rule $l \to r \Longleftarrow s_1 = t_1, \ldots, s_n = t_n$ extra variables in the right hand side r and in the conditions $s_i = t_i$ may be viewed to be existentially quantified. This means that checking such a rule for applicability involves (besides recursive evaluation of the conditions) a search for appropriate instantiations of the extra variables.

Depending on the interpretation of the equality sign in the conditions of rewrite rules different rewrite relations may be associated with a given CTRS.

Definition 2.3.3 (types of CTRSs, [BK86], [DOS88a], [Klo92])

(1) In a *join* CTRS $\mathcal{R}$ equality in conditions is interpreted as joinability. Formally: $s \to t$ if there exists a rewrite rule $l \to r \Longleftarrow s_1 = t_1, \ldots, s_n = t_n \in \mathcal{R}$, a substitution σ and a context $C[]$ such that $s = C[\sigma l]$, $t = C[\sigma r]$ and $\sigma s_i \downarrow \sigma t_i$ for

[24] which are conjunctively connected.

all $i \in \{1, \ldots, n\}$. For rewrite rules of a join CTRS we shall henceforth use the notation

$$l \to r \Longleftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n \,.$$

(2) A *normal CTRS* $\mathcal{R}$ is a join CTRS with the additional property that for every rule $l \to r \Longleftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ all right-hand sides t_i of the conditions are ground terms which are irreducible w.r.t. $R_u = \{l \to r \,|\, l \to r \Longleftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n \in \mathcal{R}\}$. Due to the latter property, rules of a normal CTRS are denoted by

$$l \to r \Longleftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n \,.$$

(3) *Semi-equational* CTRSs are obtained by interpreting equality in conditions as convertibility, i.e. as $\leftrightarrow^*$. The corresponding notation for rules is

$$l \to r \Longleftarrow s_1 \leftrightarrow^* t_1, \ldots, s_n \leftrightarrow^* t_n \,.$$

(4) A *generalized* CTRS has rules of the form

$$l \to r \Longleftarrow P_1, \ldots, P_n$$

where the conditions P_i are formulated in some general mathematical framework, e.g. in some first-order language.

Remark 2.3.4 (a slightly more general notion of normality)
Note that for finite CTRSs normality according to the above definition can easily be decided. A slightly more general, yet well-defined (but undecidable) notion of normality is obtained by requiring that the right hand sides of conditions are ground terms which are irreducible w.r.t. the reduction relation induced by join semantics.

Since the conditions in the rewrite rules of join, normal and semi-equational CTRSs are positive, the rewrite relation $\to_{\mathcal{R}}$ in these cases is well-defined, notwithstanding the circularity in its definition. In the case of generalized CTRSs well-definedness of the rewrite relation has to be ensured explicitly. CTRSs with positive *and* negative (equational) conditions have been studied in [Kap88] and more recently in [WG94] from an algebraic specification point of view.

By associating with a given CTRS $\mathcal{R}^{\mathcal{F}}$ the ARS $\langle \mathcal{T}(\mathcal{F}, \mathcal{V}), \to_{\mathcal{R}} \rangle$ all notions defined in Section 2.2 carry over to CTRSs. For the sake of readability we shall use in the following compact notations for conditional rules and conjunctions of conditions. When writing $l \to r \Longleftarrow P$ for some conditional rewrite rule then P stands for the conjunction of all conditions. Similarly, $P \downarrow$ means joinability of all conditions in P, and $\sigma(P)$ (or σP) means that all conditions in P are instantiated by σ.

For establishing properties of CTRSs the following inductive definition of the reduction relation $\to_{\mathcal{R}}$ is fundamental.

Definition 2.3.5 (inductive definition of $\to_{\mathcal{R}}$, depth)
The reduction relation corresponding to a given (join, normal or semi-equational)

CTRS $\mathcal{R}$ can also be inductively defined as follows ($\square$ denotes $\downarrow$, $\to^*$ or $\leftrightarrow^*$, respectively):

$$\begin{aligned}\mathcal{R}_0 &= \emptyset\,,\\ \mathcal{R}_{i+1} &= \{\sigma l \to \sigma r \mid l \to r \Longleftarrow P \in \mathcal{R}, \sigma u \square_{\mathcal{R}_i} \sigma v \text{ for all } u \square v \text{ in } P\}\,.\end{aligned}$$ [25]

Note that $\mathcal{R}_i \subseteq \mathcal{R}_{i+1}$, for all $i \geq 0$. Furthermore we have $s \to_{\mathcal{R}} t$ if and only if $s \to_{\mathcal{R}_i}$ for some $i \geq 0$, hence $\to_{\mathcal{R}} = \bigcup_{i \geq 0} \to_{\mathcal{R}_i}$.

If $s \to_{\mathcal{R}} t$ then the *depth* of $s \to_{\mathcal{R}} t$ is defined to be the minimal n with $s \to_{\mathcal{R}_n} t$. For $s \to_{\mathcal{R}}^* t$ and $s \downarrow_{\mathcal{R}} t$ depths are defined analogously. More precisely, if $s \to_{\mathcal{R}}^* t$ then the *depth* of $s \to_{\mathcal{R}}^* t$ is defined to be the minimal n with $s \to_{\mathcal{R}_n}^* t$. The *depth* of $s \downarrow_{\mathcal{R}} t$ is the minimal n with $s \downarrow_{\mathcal{R}_n} t$. If the depth of $s \to_{\mathcal{R}}^* t$ is less than or equal to n we denote this by $s \xrightarrow{n}{}_{\mathcal{R}}^{*} t$.

Remark 2.3.6 (encoding of conditions via an equality predicate)
Note that instead of a CTRS $\mathcal{R}$ one may somehow equivalently consider the extended system $\mathcal{R}' := \mathcal{R} \uplus \{eq(x,x) \to true\}$. More precisely, taking – within a many-sorted framework – $\mathcal{R}' := \mathcal{R} \uplus \{eq(x,x) \to true\}$, with eq a fresh binary function symbol and $true$ a fresh constant of a new sort (with x a variable of the 'old' sort), it is easily shown that $\mathcal{R}'$ is a conservative extension of $\mathcal{R}$ in the following sense: for all 'old' terms s, t we have:[26]

$$\begin{aligned}
& s \xrightarrow{n}_{\mathcal{R}} t \iff s \xrightarrow{n}_{\mathcal{R}'} t,\\
& s \to_{\mathcal{R}} t \iff s \to_{\mathcal{R}'} t,\\
& s \square_{\mathcal{R}}^{n} t \iff eq(s,t) \square_{\mathcal{R}'}^{n} true \text{ (for } n \geq 1\text{)},^{27}\\
& eq(s,t) \xrightarrow{n}{}_{\mathcal{R}'}^{*} eq(u,v) \iff s \xrightarrow{n}{}_{\mathcal{R}}^{*} u \wedge t \xrightarrow{n}{}_{\mathcal{R}}^{*} v,\\
& eq(s,t) \xrightarrow{n}{}_{\mathcal{R}'}^{*} true \iff \exists w : eq(s,t) \xrightarrow{n}{}_{\mathcal{R}}^{*} eq(w,w) \to_{\mathcal{R}'} true \text{ (for } n \geq 1\text{)},
\end{aligned}$$

for $\square$ denoting $\downarrow$ or $\leftrightarrow^*$, respectively. From these properties it is straightforward to infer that properties like termination, confluence, local confluence and joinability of critical pairs are not affected by considering $\mathcal{R}'$ instead of $\mathcal{R}$, or vice versa. Note in particular, that for join CTRSs the equivalence $s \downarrow_{\mathcal{R}}^{n} t \iff eq(s,t) \downarrow_{\mathcal{R}'}^{n} true$ (for $n \geq 1$) means: $s \downarrow_{\mathcal{R}}^{n} t \iff eq(s,t) \xrightarrow{n}{}_{\mathcal{R}'}^{*} true$, since $true$ is irreducible. This (depth preserving!) encoding of joinability into reducibility by means of an equality predicate is particularly useful for proof-technical reasons as we shall see later on (in the proof of Theorem 3.6.1). Furthermore, observe that the above encoding allows to transform a given (join) CTRS into a non-left-linear normal one which behaves essentially equivalently. The non-left-linearity of the transformed system – due to the rule $eq(x,x) \to true$ – can in general not be avoided.

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[25]Note in particular that all unconditional rules of $\mathcal{R}$ are contained in $\mathcal{R}_1$ (because the empty conditions are vacuously satisfied) as well as all conditional rules with trivial conditions only, i.e. conditions of the form $s \square s$. In fact, rules of the latter class can be considered to be essentially unconditional.

[26]Note that $s \square_{\mathcal{R}}^{n} t$ is to denote that the depth of $s \square_{\mathcal{R}} t$ is at most n.

[27]In order to obtain the equivalence $s \square_{\mathcal{R}}^{n} t \iff eq(s,t) \square_{\mathcal{R}'}^{n} true$ for $n = 0$, too, one would have to include the rule $eq(x,x) \to true$ into $\mathcal{R}_0$ instead of $\mathcal{R}_1$ in Definition 2.3.5 as it is sometimes done in the literature.

Remark 2.3.7 (expressive power)
Concerning the logical strength of semi-equational and join CTRSs compared to the purely equational interpretation the following relations hold (here we denote by $=_{\mathcal{R}}$ the congruence induced by $\mathcal{R}$ when considered as a set of equations). The semi-equational and purely equational interpretation of some given CTRS $\mathcal{R}$ yield the same congruences: $\leftrightarrow^*_{\mathcal{R}} = =_{\mathcal{R}}$. But for join systems $\leftrightarrow^*_{\mathcal{R}} = =_{\mathcal{R}}$ need not hold, i.e., $\leftrightarrow^*_{\mathcal{R}} \supseteq =_{\mathcal{R}}$ may be violated. However, if $\mathcal{R}$ is confluent as a join system then all three congruences coincide and $\mathcal{R}$ is also confluent as a semi-equational CTRS ([Kap84]). Hence, when considering the most important type of CTRSs, namely join CTRSs, we don't lose expressive power provided confluence is given.

Definition 2.3.8 (conditional critical pairs)
Let $\mathcal{R}$ be a join CTRS, and let $l_1 \to r_1 \Longleftarrow P_1$, $l_2 \to r_2 \Longleftarrow P_2$ be two rewrite rules of $\mathcal{R}$ which have no variables in common. Let $p \in \mathcal{F}Pos(l_1)$ such that l_1/p and l_2 are unifiable with mgu σ. Then $\langle(\sigma l_1)[p \leftarrow \sigma r_2] = \sigma r_1\rangle \Longleftarrow \sigma(P_1), \sigma(P_2)$ is said to be a *(conditional) critical pair* of $\mathcal{R}$, determined by *overlapping* $l_2 \to r_2 \Longleftarrow P_2$ into $l_1 \to R_1 \Longleftarrow P_1$ at position p. If the two rules are renamed versions of the same rule of $\mathcal{R}$, we do not consider an overlap at root position. A (conditional) critical pair $\langle s = t\rangle \Longleftarrow P$ is said to be *joinable* if $\sigma s \downarrow \sigma t$ for every substitution σ with $(\sigma P)\downarrow$. A substitution σ which satisfies the conditions, i.e. for which $(\sigma P)\downarrow$ holds, is said to be *feasible*. Otherwise, σ is *infeasible*. Analogously, a (conditional) critical pair is said to be *feasible (infeasible)* if there exists some (no) feasible substitution for it. The set of all (conditional) critical pairs between rules of $\mathcal{R}$ is denoted by CP($\mathcal{R}$). Joinability of (all) critical pairs of $\mathcal{R}$ is abbreviated by JCP($\mathcal{R}$).

Note that testing joinability of conditional critical pairs is in general much more difficult than in the unconditional case since one has to consider all substitutions which satisfy the correspondingly instantiated conditions.

Definition 2.3.9 (non-overlapping, overlay CTRSs)
(cf. [BK86], [Klo92]) Let $\mathcal{R}$ be a CTRS and let $\mathcal{R}_u$ be its unconditional version, i.e., $\mathcal{R}_u = \{l \to r \mid l \to r \Longleftarrow P \in \mathcal{R}\}$. Then $\mathcal{R}$ is said to be *non-overlapping (NO)* / *weakly non-overlapping (WNO)* / *orthogonal* (ORTH) / *weakly orthogonal* (WORTH) / a *(conditional) overlay system* (OS) if $\mathcal{R}_u$ is non-overlapping / weakly non-overlapping / orthogonal / weakly orthogonal / an (unconditional) overlay system.

Conditional rewriting is inherently much more complicated than unconditional rewriting. Intuitively, the main reason is that for applying some rule $l \to r \Longleftarrow P$ the appropriately instantiated conditions must be verified recursively using again the reduction relation. This may lead to a non-terminating evaluation process for the conditions, even for terminating systems. In fact, the rewrite relation (and reducibility) may be undecidable even for (finite) complete (join) CTRSs without extra variables in the conditions ([Kap84]). A sufficient (but rather restrictive) condition for the decidability of irreducibility in normal CTRSs which does not imply termination is given in [BK86]. In the general case of join CTRSs one has to impose a stronger condition than termination for ensuring decidability of the basic notions.

Definition 2.3.10 (decreasing, reductive, simplifying)
Let $\mathcal{R}$ be a (join / normal / semi-equational) CTRS.[28] Then $\mathcal{R}$ is *decreasing* ([DOS88a]) if there exists a partial ordering $>$ satisfying

(1) $>$ is well-founded.

(2) $\rightarrow_{\mathcal{R}} \subseteq >$.

(3) If $l \rightarrow r \Longleftarrow s_1 = t_1, \ldots, s_n = t_n$ is a rule in $\mathcal{R}$ and σ is a substitution then $\sigma l > \sigma s_i$ and $\sigma l > \sigma t_i$ for $i = 1, \ldots, n$, and

(4) $>$ has the subterm property, i.e. $C[s] > s$ for every term s and every non-empty context $C[]$.

$\mathcal{R}$ is *reductive* ([JW86]) if there exists a partial ordering $>$ satisfying (1)-(3) and

(5) $>$ is closed under contexts.

$\mathcal{R}$ is *simplifying* ([Kap87]) if there exists a rewrite ordering $>$ with the subterm property such that

(3') If $l \rightarrow r \Longleftarrow s_1 = t_1, \ldots, s_n = t_n$ is a rule in $\mathcal{R}$ then $l > r$ and $l > s_i$, $l > t_i$ for $i = 1, \ldots, n$

holds (note that in this case, $>$ satisfies (2)-(5) and is closed under substitutions).

It is easy to see that every reductive system $\mathcal{R}$ is decreasing: If $>$ is a partial ordering satisfying the reductivity conditions (w.r.t. $\mathcal{R}$), then the extended ordering $(> \cup \trianglerighteq)^+$ satisfies the decreasingness conditions, in particular it is well-founded, too. Moreover, any decreasing system is clearly terminating. If $\mathcal{R}^{\mathcal{F}}$ is simplifying, with some rewrite ordering $>$ having the subterm property, such that $>$ is well-founded, then it is also reductive, hence decreasing. As mentioned earlier this is for instance the case if $\mathcal{R}$ or $\mathcal{F}$ is finite. Note that a system may be decreasing but not reductive.

Example 2.3.11 (decreasing $\not\Rightarrow$ reductive, [DOS88a])
The (join or semi-equational) CTRS

$$\left\{ \begin{array}{rcl} b \rightarrow c & & \\ f(b) \rightarrow f(a) & & \\ a \rightarrow c & \Longleftarrow & b = c \end{array} \right.$$

is easily shown to be decreasing, but it is not reductive since the third rule enforces $a > b$, hence $f(a) > f(b)$ by closure under context which contradicts the orientation of the second rule.

Note that a decreasing semi-equational CTRS is also decreasing as a join system but not vice versa in general.

[28] We write here $l \rightarrow r \Longleftarrow s_1 = t_1, \ldots, s_n = t_n$ for denoting rules of a join, normal or semi-equational CTRS.

Example 2.3.12 (decreasingness is type-dependent)
Consider the CTRS

$$\mathcal{R} = \begin{cases} a \to a & \Longleftarrow \quad c = d \\ b \to c & \\ b \to d & . \end{cases}$$

As join CTRS $\mathcal{R}$ is clearly decreasing due to unsatisfiability of $c \downarrow d$ but as semi-equational CTRS it is obviously non-terminating, hence cannot be decreasing.

Decreasingness exactly captures the finiteness of recursive evaluation of terms. For finite decreasing join CTRSs (such that the decreasing ordering $>$ is decidable) all the basic notions are decidable, e.g. reducibility and joinability (cf. [Kap87], [JW86], [DOS88a]).[29] Observe that this does not apply to semi-equational systems because there the recursive evaluation of conditions $s_i \leftrightarrow^* t_i$ may require inverse reduction steps which does not yield the desired decrease w.r.t. the corresponding ordering.

The problems of verifying confluence and termination of CTRSs are in general much more difficult than in the unconditional case. Concerning termination, it is quite natural to try to extend the existing machinery to the conditional case. If one can prove termination of the unconditional version $\mathcal{R}_u = \{l \to r \,|\, l \to r \Longleftarrow P \in \mathcal{R}\}$ of some given CTRS $\mathcal{R}$ then $\mathcal{R}$ is obviously terminating, too. In general, however, one has to take into account some semantic knowledge for proving termination of conditional rewriting. Namely, for some rewrite rule $l \to r \Longleftarrow P$ of $\mathcal{R}$ one must have $\sigma l > \sigma r$ (with $>$ some well-founded rewrite ordering) only for those substitutions which satisfy the instantiated condition σP. Natural candidates for such an approach are the *semantic path odering* of [KL80] and the *general path ordering* of [DH95], cf. also [Bev93]. But, from a practical and automation point of view, this approach is quite challenging and has not been much investigated yet. In fact, this is a field of research which is closely related to automating termination proofs of recursively defined algorithms within a functional framework (cf. [BM79], [Wal94b], [Gie95]).

2.3.1 Confluence without Termination

Testing confluence of CTRSs is in general much more difficult than for unconditional TRSs. For instance, the Critical Pair Lemma (cf. Lemma 2.2.17) does not hold for join CTRSs any more. And orthogonal join CTRSs need not be (locally) confluent (cf. Theorem 2.2.27).

Example 2.3.13 (orthogonal join CTRSs need not be confluent, [BK86])
The join CTRS

$$\mathcal{R} = \begin{cases} f(x) \to a & \Longleftarrow \quad f(x) \downarrow x \\ \quad b \to f(b) & \end{cases}$$

is non-overlapping and left-linear, hence orthogonal, but non-confluent and not even locally confluent. We have e.g. $f(a) \leftarrow f(f(b)) \to a$, but not $f(a) \downarrow a$, since both a

[29] For slightly generalized versions of decreasingness cf. [WG94].

and $f(a)$ are irreducible.

The crucial point here is that, unlike the unconditional case, variable overlaps may be critical: Let $\sigma l \rightarrow \sigma r$ due to the satisfied conditions $\sigma s_i \downarrow \sigma t_i$ for some rule $l \rightarrow r \Longleftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$. Now, if $\sigma x \rightarrow \sigma' x$ for some $x \in \mathcal{V}(l)$, then we obviously have $\sigma' l \ {}^{+}\!\!\leftarrow \sigma l \rightarrow \sigma r \rightarrow^* \sigma' r$, but not necessarily $\sigma' l \rightarrow \sigma' r$ since $\sigma' s_i \downarrow \sigma' t_i$ need not hold any more.

By requiring additionally normality this problem can be avoided.

Theorem 2.3.14 (confluence by orthogonality plus normality, [BK86], [Klo92])
Any weakly orthogonal, normal CTRS is confluent.

In fact, as remarked in [GM88], the proof in [BK86] shows *level-confluence*. More precisely, it even shows *shallow-confluence*.

Definition 2.3.15 (shallow-confluence, level-confluence)
Let $\mathcal{R}$ be a CTRS.

(1) $\mathcal{R}$ is *level-confluent* ([GM88]) if for all $n \geq 0$ and all terms s, t, u with $s \xrightarrow{n} t$, $s \xrightarrow{n} u$ there exists a term v such that $t \xrightarrow{n}^* v$ and $u \xrightarrow{n}^* v$, i.e., if $\mathcal{R}_n$ (cf. Definition 2.3.5) is confluent for all $n \geq 0$. $\mathcal{R}$ is *level-complete* if it is level-confluent and terminating.

(2) $\mathcal{R}$ is *shallow-confluent* ([DOS88b]) if for all $m, n \geq 0$ and all terms s, t, u with $s \xrightarrow{m}^* t$, $s \xrightarrow{n}^* u$ there exists a term v such that $t \xrightarrow{n}^* v$ and $u \xrightarrow{m}^* v$. $\mathcal{R}$ is *shallow-complete* if it is shallow-confluent and terminating. A critical pair $\langle s = t \rangle \Longleftarrow P$ of $\mathcal{R}$, obtained from a critical overlap $s \leftarrow u \rightarrow t$, is *shallow-joinable* if, for each feasible substitution σ, i.e. with $(\sigma P)\downarrow$, with $\sigma u \xrightarrow{m} \sigma s$, $\sigma u \xrightarrow{n} \sigma t$ there exists a term v such that $\sigma s \xrightarrow{n}^* v$, $\sigma t \xrightarrow{m}^* v$.

Clearly, we have the implications: shallow-confluence $\Longrightarrow$ level-confluence $\Longrightarrow$ confluence, both of which are proper. Level-confluence is interesting since it guarantees the completeness of narrowing for (join) CTRSs with extra variables in the conditions allowed ([GM88]). Recently, Theorem 2.3.14 has been extended to certain orthogonal CTRSs allowing extra-variables in right-hand sides, too (cf. [SMI95]).

Whereas the Critical Pair Lemma does not hold for join CTRSs in general, see Example 2.3.13 above, it does hold for semi-equational systems as is easily verified: Considering one-step divergences $s \leftarrow u \rightarrow t$, the variable overlap case is no problem if conditions are interpreted semi-equationally.

Definition 2.3.16 (closed predicate/CTRS, [Klo92])
Let $\mathcal{R}^{\mathcal{F}}$ be a CTRS with rewrite relation $\rightarrow$ and let P be an n-ary predicate on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Then P is said to be *closed with respect to* $\rightarrow$ if for all terms $t_i, t_i' \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $t_i \rightarrow^* t_i'$ $(i = 1, \ldots, n)$:

$$P(t_1, \ldots, t_n) \quad \Longrightarrow \quad P(t_1', \ldots, t_n') \,.$$

$\mathcal{R}$ is said to be *closed* if all conditions (appearing in some conditional rewrite rule of $\mathcal{R}$), viewed as predicates with the variables ranging over terms from $\mathcal{T}(\mathcal{F},\mathcal{V})$, are closed with respect to $\rightarrow$.

Theorem 2.3.17 (confluence criteria for closed CTRSs, [O'D77], [Klo92])

(1) Any generalized, weakly orthogonal, closed CTRS is confluent (cf. [O'D77], [Klo92]).

(2) Any weakly orthogonal, semi-equational CTRS is confluent.[30]

It seems open to what extent further confluence criteria which do not presume termination can be generalized from the unconditional to the conditional case (cf. e.g. Theorem 2.2.22). This should be clarified in future research.[31]

2.3.2 Confluence with Termination

Unfortunately, the Critical Pair Lemma does not even hold for terminating (join) CTRSs, cf. [DOS88b] for some illustrative counterexamples. A very simple counterexample (due to Aart Middeldorp) is the following.

Example 2.3.18 (SN $\wedge$ JCP $\not\Longrightarrow$ WCR)
Let

$$\mathcal{R} = \left\{ \begin{array}{lcl} h(x) \rightarrow k(b) & \Longleftarrow & h(x) \downarrow k(x) \\ k(a) \rightarrow h(a) & & \\ \quad a \rightarrow b & & \end{array} \right.$$

This system is easily shown to be terminating and the only critical peak (between the last two unconditional rules) $k(b) \leftarrow k(a) \rightarrow h(a)$ is joinable since $h(a) \rightarrow k(b)$ by the first rule (due to $h(a) \downarrow k(a)$). But we have $h(b) \leftarrow h(a) \rightarrow k(b)$ (again due to $h(a) \downarrow k(a)$) with both $h(b)$ and $k(b)$ irreducible.

We observe that in this example $\mathcal{R}$ is not shallow-joinable, since the step from $h(a)$ to $k(b)$ has depth 2. Furthermore $\mathcal{R}$ is left-linear, not normal, no overlay system and not decreasing. Normality (more precisely, its slightly more general version, cf. Remark 2.3.4) can easily be obtained by modifying the first rule appropriately.

Example 2.3.19 The join CTRS

$$\mathcal{R} = \left\{ \begin{array}{lcl} h(x) \rightarrow k(b) & \Longleftarrow & k(x) \downarrow h(b) \\ k(a) \rightarrow h(a) & & \\ \quad a \rightarrow b & & \end{array} \right.$$

is terminating, left-linear, normal (in the sense of Remark 2.3.4), has exactly one joinable, but not shallow-joinable, critical pair and is not shallow-joinable and not locally confluent (for the same reason as above).

[30] This is a corollary of (1).

[31] Some very recent progress in this direction is reported in [Wir95], within the more general setting of *positive/negative CTRSs*.

Dershowitz, Okada and Sivakumar succeeded to prove the following confluence criteria for terminating CTRSs. Note that in all three cases none of the conditions can be dropped as exemplified by counterexamples in [DOS88b].

Theorem 2.3.20 (shallow-confluence criterion, [DOS88b])
A terminating, left-linear, normal CTRS is (shallow-)confluent, if all its critical pairs are shallow-joinable.

Theorem 2.3.21 (overlay criterion for confluence, [DOS88b])
A terminating overlay (join) CTRS is confluent, if all its critical pairs are joinable.

Theorem 2.3.22 (confluence criterion for decreasing CTRSs, [DOS88b])
A decreasing (join) CTRS is confluent, if all its critical pairs are joinable.

Moreover, if a semi-equational CTRS is decreasing and has joinable critical pairs, then it is confluent, and in this case the corresponding join CTRS is (decreasing and) confluent, too ([DOS88a]. If a terminating semi-equational CTRS is confluent, then it need not be confluent as a join CTRS (cf. e.g. Examples 2.3.18, 2.3.19 above). We remark that Theorem 2.3.22 slightly generalizes the corresponding results for simplifying ([Kap87]) and reductive ([JW86]) join CTRSs. In Theorem 2.3.20 it is not possible to weaken shallow-joinability of critical pairs to level-joinability as demonstrated in [DO90].

Considering Theorem 2.3.21, we would like to mention that some generalizations are still possible (but are not treated here in order to keep this thesis reasonable in size). One may indeed allow non-overlay critical pairs, but then one needs a stronger condition than joinability of critical pairs, namely *quasi-overlay joinability* of (all) *shared parallel critical pairs (peaks)*, cf. [WG94], [GW96], [Wir95].

2.4 Combined Systems and Modularity Behaviour

In this section we give a brief introduction into the combination setting. Different kinds of combinations of systems are discussed as well as basic problems arising. For the special case of disjoint unions we then introduce in Section 2.4.2 the necessary terminology and basic theory. This is finally extended also to combinations of constructor sharing and of composable systems.

2.4.1 Introduction

Since all interesting properties of TRSs like confluence and termination are undecidable, the need for sufficient criteria, methods and proof techniques for verifying such properties is evident. In particular, it would be very desirable to have a well-developed structure theory for (term) rewriting which would allow for a *modular* analysis of combined systems (via a *divide and conquer* approach) and — from a dual point of view

— for a *modular* construction of complex systems with some desired properties (via general inheritance mechanisms). Unfortunately, under the combination of arbitrary TRSs all interesting properties get lost. For instance, combining the terminating one-rule TRSs $a \to b$ and $b \to a$ leads to non-termination, and combining the confluent one-rule TRSs $a \to b$ and $a \to c$ yields a non-confluent system.

Various structural restrictions are conceivable for specifying certain classes of combinations:

(a) structural restrictions concerning the signatures of the involved TRSs and the overall shape of their rules;

(b) syntactic and semantic restrictions concerning the required form and properties of rules and rule systems;

(c) combinations of (a) and (b).

The simplest type of restriction according to (a) above is the concept of *disjoint union* where the signatures of the component TRSs are required to be disjoint (hence also the corresponding rule sets).

Definition 2.4.1 (union, disjoint union)
Let $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ be two TRSs. Their *union* or *combination* $\mathcal{R}^{\mathcal{F}}$ is defined by $\mathcal{R}^{\mathcal{F}} := \mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_2^{\mathcal{F}_2} := (\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$. We say that $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ are *disjoint* if $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$. In that case $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$ (or simply $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$) denotes their (disjoint) union.

Definition 2.4.2 (modularity of TRS properties)
Let P be a property of TRSs.

- We say that P is *modular* (for arbitrary TRSs) if, for all TRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$, $\mathcal{R}^{\mathcal{F}}$ with $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_2^{\mathcal{F}_2}$: $P(\mathcal{R}_1^{\mathcal{F}_1}) \wedge P(\mathcal{R}_2^{\mathcal{F}_2}) \iff P(\mathcal{R}^{\mathcal{F}})$.
- We say that P is *modular for disjoint TRSs* (or *modular for disjoint unions of TRSs*) if, for all TRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ and $\mathcal{R}^{\mathcal{F}}$ with $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$: $P(\mathcal{R}_1^{\mathcal{F}_1}) \wedge P(\mathcal{R}_2^{\mathcal{F}_2}) \iff P(\mathcal{R}^{\mathcal{F}})$.

As already mentioned all interesting semantic properties like termination and confluence are not modular for arbitrary TRSs. However, many syntactic properties like left- (right- or full) linearity or being non-erasing are indeed modular (for arbitrary TRSs). This will be (sometimes tacitly) exploited.

Often, in particular for disjoint unions, one implication of the modularity of some property P, namely $P(\mathcal{R}^{\mathcal{F}}) \implies P(\mathcal{R}_1^{\mathcal{F}_1}) \wedge P(\mathcal{R}_2^{\mathcal{F}_2})$, is trivial (which is sometimes tacitly exploited, too). In such cases it suffices to concentrate on the difficult part $P(\mathcal{R}_1^{\mathcal{F}_1}) \wedge P(\mathcal{R}_2^{\mathcal{F}_2}) \Longrightarrow P(\mathcal{R}^{\mathcal{F}})$, i.e., to show preservation of P under the combination.

Some basic observations concerning modularity (for disjoint TRSs) are the following:

- If $P \Longrightarrow Q$ and P is modular, then Q need not be modular. To see this, take for instance confluence for P and ground confluence for Q. Then P is modular by Toyama's Theorem 4.1.2 and Q is not modular, since it is well-known that ground confluence is not even preserved under signature extensions.

- If $P \Longrightarrow Q$ and Q is modular, then P need not be modular. For instance, one can take termination (SN) for P, weak termination (WN) for Q and observe the non-modularity of SN (Example 5.1.1) and the modularity of WN (Theorem 5.2.2).

- If P is modular, then $\neg\, P$ need not be modular. For instance, taking the modular CR for P, we clearly have $\neg\mathrm{CR}(\mathcal{R}_1^{\mathcal{F}_1}) \wedge \neg\mathrm{CR}(\mathcal{R}_2^{\mathcal{F}_2}) \Longrightarrow \neg\mathrm{CR}(\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$ but not $\neg\mathrm{CR}(\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}) \Longrightarrow \neg\mathrm{CR}(\mathcal{R}_1^{\mathcal{F}_1}) \wedge \neg\mathrm{CR}(\mathcal{R}_2^{\mathcal{F}_2})$.

Furthermore we observe that the more interesting part of the modularity of some property P usually entails a two-step approach, in the following sense: For proving modularity of P, first prove

(1) $P(\mathcal{R}^{\mathcal{F}}) \Longrightarrow P(\mathcal{R}^{\mathcal{F}'})$ for any $\mathcal{F}' \supseteq \mathcal{F}$ (preservation of P under signature extension), and then

(2) $P({\mathcal{R}_1}^{\mathcal{F}}) \wedge P({\mathcal{R}_2}^{\mathcal{F}}) \Longrightarrow P(\mathcal{R}^{\mathcal{F}})$ where $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_2^{\mathcal{F}_2}$ (preservation of P under combination over the same signature).

For TRSs part (1) is often trivial or easy (and thus sometimes even omitted or ignored). However, (1) may be non-trivial, as we will see, for various properties of CTRSs.

As to the interaction possibilities of rewriting in a combined system $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_2^{\mathcal{F}_2}$, it is clear that TRSs satisfy the basic decomposition property (for terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$)

$$s \to_{\mathcal{R}} t \Longrightarrow s \to_{\mathcal{R}_1} t \vee s \to_{\mathcal{R}_2} t$$

and consequently $\to_{\mathcal{R}} = \to_{\mathcal{R}_1} \cup \to_{\mathcal{R}_2}$. This property is, however, violated for CTRSs as observed by Middeldorp ([Mid90]), since a reduction step $s \to_{\mathcal{R}} t$ in the (disjoint or non-disjoint) union $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_2^{\mathcal{F}_2}$, let's say using an $\mathcal{R}_1$-rule, may need $\mathcal{R}_2$-rules for satisfying the corresponding conditions. In fact, this is one of the additional complications arising when analyzing combinations of CTRSs.

Secondly, we observe that interaction of rewriting in a combined system is enabled by shared function symbols as well as by the (always) shared variables. Hence, even in a disjoint union interaction is possible due to the presence of variables. More precisely, it turns out that the main problems are caused by *collapsing* rules, i.e., rules with a variable as right hand side. For non-disjoint unions, additionally those rules are problematic which have a shared function symbol on top of their right hand side. Applying such rules may essentially modify the 'homogeneous' parts of 'mixed' terms thereby enabling the application of rules which were previously not applicable. Further problems of interaction (which are not specific for the combination setting here) are due to non-left-linear rules.

Relaxing the disjointness requirement for disjoint unions is possible in various ways. For instance, one may insist on the disjointness of the sets of defined symbols but allow shared constructors (cf. Definition 2.2.12), yielding combinations of *constructor sharing* systems. A further weakening is to allow even shared defined symbols provided the respective defining rules occur in both systems, yielding combinations of *composable* systems. Asymmetric *hierarchical* combinations are obtained by requiring that one system (the *base system*) does not *depend* — in a sense to be made precise — on the other one (the *extension*), but possibly vice versa. Hierarchical combinations as well as some other types of combinations that have been investigated in the literature will be briefly discussed in Chapter 6. Here we restrict ourselves to constructor sharing and composable systems.

Definition 2.4.3 (constructor sharing / composable TRSs, cf. [KO92], [MT91; MT93], [Ohl94c])
For $\mathcal{R}_i^{\mathcal{F}_i}$ be TRSs with sets $\mathcal{D}_i$ of defined symbols and $\mathcal{C}_i$ of constructors, respectively, for $i = 1, 2$. Let $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_2^{\mathcal{F}_2}$. Then the set $\mathcal{F}^s$ of *shared function symbols* is given by $\mathcal{F}_1 \cap \mathcal{F}_2$, the set $\mathcal{R}^s$ of *shared rules* by $\mathcal{R}_1 \cap \mathcal{R}_2$. The sets $\mathcal{F}_i^{ns}$ and $\mathcal{R}_i^{ns}$ of *non-shared* $\mathcal{F}_i$-symbols and *non-shared* $\mathcal{R}_i$-rules[32] (for $i = 1, 2$) are defined by $\mathcal{F}_i^{ns} = \mathcal{F}_i \setminus \mathcal{F}^s$ and $\mathcal{R}_i^{ns} = \{l \to r \in \mathcal{R}_i \mid root(l) \in \mathcal{F}_i^{ns}\}$.

- $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ are said to be *constructor sharing* if $\mathcal{F}_1 \cap \mathcal{D}_2 = \emptyset = \mathcal{F}_2 \cap \mathcal{D}_1$, i.e., if they share at most constructors (and no rules).[33]

- $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ are said to be *composable* if $\mathcal{C}_1 \cap \mathcal{D}_2 = \emptyset = \mathcal{C}_2 \cap \mathcal{D}_1$ and $\{l \to r \in \mathcal{R} \mid root(l) \in \mathcal{D}_1 \cap \mathcal{D}_2\} = \mathcal{R}_1 \cap \mathcal{R}_2$, i.e., if they share (some) constructors and the defining rules for all shared defined symbols. [34]

The notion of modularity is extended to constructor sharing and composable TRSs as follows.

Definition 2.4.4 (modularity: constructor sharing / composable TRSs)
Let P be a property of TRSs.

- We say that P is *modular for constructor sharing TRSs* (or *modular for constructor sharing unions*) if, for all TRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$, $\mathcal{R}^{\mathcal{F}}$ with $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_2^{\mathcal{F}_2}$ such that $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ are constructor sharing: $P(\mathcal{R}_1^{\mathcal{F}_1}) \wedge P(\mathcal{R}_2^{\mathcal{F}_2}) \iff P(\mathcal{R}^{\mathcal{F}})$.

- We say that P is *modular for composable TRSs* (or *modular for composable unions*) if, for all TRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$, $\mathcal{R}^{\mathcal{F}}$ with $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_2^{\mathcal{F}_2}$ such that $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ are composable: $P(\mathcal{R}_1^{\mathcal{F}_1}) \wedge P(\mathcal{R}_2^{\mathcal{F}_2}) \iff P(\mathcal{R}^{\mathcal{F}})$.

[32] i.e., $\mathcal{R}_i$-rules defining the non-shared (defined) function symbols from $\mathcal{F}_1$

[33] In other words: $\mathcal{F}^s \subseteq \mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{R}^s = \emptyset$, or, equivalently: $\mathcal{R}_i = \mathcal{R}_i^{ns}$ for $i = 1, 2$ (and consequently: $\mathcal{R} = {\mathcal{R}_1}^{ns} \uplus {\mathcal{R}_2}^{ns}$).

[34] In other words: $\mathcal{R}_i = \mathcal{R}_i^{ns} \uplus \mathcal{R}_s$ for $i = 1, 2$ (and consequently: $\mathcal{R} = {\mathcal{R}_1}^{ns} \uplus {\mathcal{R}_2}^{ns} \uplus \mathcal{R}^s$).

Clearly, disjoint TRSs are constructor sharing, and constructor sharing TRSs are composable, but not necessarily the other way round. Moreover, all these types of combinations and modularity definitions extend in the obvious way to conditional TRSs (CTRSs). Furthermore, the extension to corresponding combinations of more than two systems is also straightforward.

As we will see, many modularity results do only hold under certain conditions. When stating such results later on, we will use the following convention for the sake of readability (which applies in an analogous manner to the cases of constructor sharing and of composable systems). Assertions of the form

Some property P is modular for disjoint (C)TRSs satisfying Q

have to be interpreted as

$(P \wedge Q)$ is modular for disjoint (C)TRSs

i.e., implicitly the modularity of the conjunction of P and Q is meant. This convention allows to emphasize the important part of corresponding modularity results (in fact, often Q will be obviously modular).

2.4.2 Basic Terminology

Disjoint Unions

Here we introduce the basic terminology, notions, notations and facts needed for dealing adequately with disjoint unions of TRSs following mainly [Toy87b], [Mid90].

Let us assume subsequently that $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ are disjoint TRSs with $\mathcal{R}^{\mathcal{F}}$ denoting their disjoint union $\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$. Furthermore we shall use the abbreviating notations $\mathcal{T} = \mathcal{T}(\mathcal{F}, \mathcal{V})$ and $\mathcal{T}_i = \mathcal{T}(\mathcal{F}_i, \mathcal{V})$ for $i = 1, 2$. Instead of $\rightarrow_{\mathcal{R}}$ we shall also simply use $\rightarrow$.

Many definitions, notations and case distinctions are symmetric w.r.t. the two systems. The non-explicit case is therefore often indicated in parentheses (or omitted). First of all, in order to achieve better readability we introduce the mostly used chromatic terminology.

Definition 2.4.5 (chromatic terminology)
Function symbols from $\mathcal{F}_1$ ($\mathcal{F}_2$) are called *black* (*white*). Variables are *transparent*, i.e., have no colour. A term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is called *black* (*white*) if $s \in \mathcal{T}_1$ ($s \in \mathcal{T}_2$). We say that s is *top black* (*top white, top transparent*) if $root(s) \in \mathcal{F}_1$ ($root(s) \in \mathcal{F}_2$, $root(s) \in \mathcal{V}$). Terms in $\mathcal{T}_1 \cup \mathcal{T}_2$ are called *homogeneous*, terms from $\mathcal{T} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$ *mixed*.

Obviously every term $s \in \mathcal{T}$ has unique representations

$$s = \begin{cases} C^b\langle s_1, \ldots, s_l\rangle \text{ with } C^b\langle, \ldots, \rangle \in \mathcal{T}(\mathcal{F}_1, \mathcal{V}),\ root(s_i) \in \mathcal{F}_2 \\ C^w\langle t_1, \ldots, t_m\rangle \text{ with } C^w\langle, \ldots, \rangle \in \mathcal{T}(\mathcal{F}_2, \mathcal{V}),\ root(t_j) \in \mathcal{F}_1 \\ C^t\langle u_1, \ldots, u_n\rangle \text{ with } C^t\langle, \ldots, \rangle \in \mathcal{T}(\mathcal{F}_1 \cap \mathcal{F}_2, \mathcal{V}),\ root(u_k) \in \mathcal{F}_1 \cup \mathcal{F}_2 \end{cases}$$

which will subsequently be indicated by writing[35]

$$s = \begin{cases} C^b \langle\!\langle\, s_1, \dots, s_l \rangle\!\rangle \\ C^w \langle\!\langle\, t_1, \dots, t_m \rangle\!\rangle \\ C^t \langle\!\langle\, u_1, \dots, u_n \rangle\!\rangle \end{cases}$$

According to our conventions for denoting contexts (cf. Definition 2.2.4) we shall also use the notations

$$s = \begin{cases} C^b[\![s_1, \dots, s_l]\!] \\ C^w[\![t_1, \dots, t_m]\!] \end{cases}$$

in the case of strict contexts. Thus, any any mixed term s must be either of the form $s = C^b[\![s_1, \dots, s_l]\!]$ (if it is top black) or else of the form $s = C^w[\![t_1, \dots, t_m]\!]$ (if it is top white). In this case we shall often also drop the superscripts (for black and white) and simply write e.g. $s = C[\![s_1, \dots, s_l]\!]$.

Definition 2.4.6 (alien / special subterm, rank, top)

(1) A top black (top white) subterm s/p of $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, i.e., with $root(s/p) \in \mathcal{F}_1$ ($root(s) \in \mathcal{F}_2$), is called a black (white) *alien* or *special subterm* of s if the immediate superterm of s/p in s (if existent) is top white (top black). If $s = C^b[\![s_1, \dots, s_n]\!]$ ($s = C^w[\![s_1, \dots, s_n]\!]$) the s_i's, i.e., the maximal white (black) aliens of s, are called the white (black) *principal aliens* or *principal subterms* of s. For $s = C[\![s_1, \dots, s_n]\!]$ (meaning to stand for both cases) the s_i's are simply called *principal aliens* or *principal subterms*.

(2) The *rank* of a term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined by

$$rank(s) = \begin{cases} 0 & \text{if } \ s \in \mathcal{V} \\ 1 & \text{if } \ s \in (\mathcal{T}_1 \cup \mathcal{T}_2) \setminus \mathcal{V} \\ 1 + \max\{rank(s_i) \mid 1 \leq i \leq n\} & \text{if } \ s = C[\![s_1, \dots, s_n]\!] \end{cases}$$

The *rank* of a (possibly infinite) derivation

$$D: \ s_1 \to s_2 \dots$$

is defined by $rank(D) = \min\{rank(s_i) \mid s_i \text{ occurs in } D\}$.

(3) The *topmost homogeneous part* $top(s)$ of a term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined by

$$top(s) = \begin{cases} s & \text{if } \ s \in \mathcal{V} \\ C[, \dots,] & \text{if } \ s = C[\![s_1, \dots, s_n]\!] \end{cases}$$

Definition 2.4.7 (inner / outer reduction, destructive steps)

(1) If $s \to t$ by applying some rule in one of the principal aliens of s, we write $s \xrightarrow{i} t$, otherwise $s \xrightarrow{o} t$. The relations $\xrightarrow{i}$ and $\xrightarrow{o}$ are called *inner* and *outer* reduction, respectively.

[35] Actually, the last case is degenerate here since for disjoint unions a top transparent term must be a variable. However, for constructor sharing and composable systems these notations can be conveniently extended.

(2) A rewrite step $s \to t$ is *destructive at level 1* if the root symbols of s and t have different colours, i.e., if either s is top black and t top white or top transparent, or s is top white and t top black or top transparent. The rewrite step $s \to t$ is *destructive at level* $n+1$ if $s = C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \xrightarrow{i} C[s_1, \ldots, t_j, \ldots, s_n]$ with $s_j \to t_j$ destructive at level n. A step $s \to t$ is *destructive* if it is destructive at some level $n \geq 1$.

Clearly, if a step $s \to t$ is destructive then the applied rule[36] must be collapsing.

Lemma 2.4.8 (form of outer and inner steps)

(1) If $s \xrightarrow{o} t$ then $s = C\,\{\!\{\, s_1, \ldots, s_n \}\!\}$ and $t = C'\,\langle\!\langle\, s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle$ for some contexts $C\{, \ldots, \}$ and $C'\langle, \ldots, \rangle$, terms $s_1, \ldots, s_n$, and indices $i_1, \ldots, i_m \in \{1, \ldots, n\}$. Moreover, if $s \xrightarrow{o} t$ is not destructive at level 1, then t has the form $t = C'\,\{\!\{\, s_{i_1}, \ldots, s_{i_m} \}\!\}$. If additionally $rank(t) > 1$ then $t = C'[\![s_{i_1}, \ldots, s_{i_m}]\!]$.

(2) If $s \xrightarrow{i} t$ then $s = C[\![s_1, \ldots, s_j, \ldots, s_n]\!]$ and $t = C[s_1, \ldots, t_j, \ldots, s_n]$ for some context $C[, \ldots,]$, terms $s_1, \ldots, s_n, t_j$ and $j \in \{1, \ldots, n\}$ with $s_j \to t_j$. Moreover, if $s \xrightarrow{i} t$ is not destructive at level 2 then t has the form $t = C[\![s_1, \ldots, t_j, \ldots, s_n]\!]$.

An important basic fact is that reduction in the disjoint union is rank decreasing. This enables proofs by induction on the rank of terms.

Lemma 2.4.9 (reduction in the disjoint union is rank decreasing)
If $s \to^* t$ then $rank(s) \geq rank(t)$.

Definition 2.4.10 (consistent replacement)
Let $s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathcal{T}$.

(1) We write $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$ if $t_i = t_j$ whenever $s_i = s_j$, for all $1 \leq i < j \leq n$ (*consistent replacement*).

(2) We write $\langle s_1, \ldots, s_n \rangle \infty \langle t_1, \ldots, t_n \rangle$ if both $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$ and $\langle t_1, \ldots, t_n \rangle \propto \langle s_1, \ldots, s_n \rangle$, i.e., if $s_i = s_j \iff t_i = t_j$, for all $1 \leq i < j \leq n$.

Lemma 2.4.11 (identifying / injective abstraction)
Let $s = C\,\{\!\{\, s_1, \ldots, s_n \}\!\} \xrightarrow{o}_{l \to r} C'\,\langle\!\langle\, s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle = t$. Then we have:

(1) $C\{t_1, \ldots, t_n\} \xrightarrow{o}_{l \to r} C'\langle t_{i_1}, \ldots, t_{i_m} \rangle$ for all terms $t_1, \ldots, t_n$ with $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$ (*consistent replacement*). Furthermore, if $l \to r$ is left-linear then the restriction $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$ can be omitted.

(2) In particular: $top(s) = C\{, \ldots, \} \xrightarrow{o}_{l \to r} C'\langle, \ldots, \rangle = top(t)$ (*identifying abstraction*[37]).

[36] which need not be uniquely determined

[37] This means (consistent) replacement with maximal identification, i.e., all principal aliens are replaced by some (the same!) fresh variable (or constant $\Box$).

(3) And furthermore: $C\{x_1, \dots, x_n\} \xrightarrow{o}_{l \to r} C'\langle x_{i_1}, \dots, x_{i_m}\rangle$ for all fresh variables $x_1, \dots, x_n$ with $\langle s_1, \dots, s_n\rangle \infty \langle x_1, \dots, x_n\rangle$ (*injective abstraction*[38]). In this case we additionally get: If $s \xrightarrow{o}_{l \to r} t$ is an innermost reduction step, then the abstracted step $C\{x_1, \dots, x_n\} \xrightarrow{o}_{l \to r} C'\langle x_{i_1}, \dots, x_{i_m}\rangle$ is innermost, too. (A subtle point here is that 'freshness' of the x_i's is to be interpreted w.r.t. to all variables appearing in s (and t) because otherwise unnecessary identifications between abstracted principal aliens and variables occurring in s but not inside its principal aliens might become possible!)[39]

Definition 2.4.12 (colour changing reduction, preserved / inner preserved terms, collapsing reduction)

(1) A reduction (sequence) $s \to^+ t$ with s top black (top white) is said to be *colour changing* if t is top white or top transparent (top black or top transparent).

(2) A term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is called *preserved* if no $(\to)$-derivation issuing from s contains a destructive step (at some level ≥ 1). We say that s is *inner preserved* if all its principal aliens are preserved.

(3) The *collapsing* (or *layer coalescing*) reduction relation $\to_c$ (on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ w.r.t. $\to$) is defined as follows: $s \to_c t$ if $s = C[s_1]$, $t = C[t_1]$, s_1 is an alien of s, $s_1 \to^+ t_1$, and $s_1 \to^+ t_1$ is colour changing.

Lemma 2.4.13 (relating colour changing reduction, preservation and collapsing reduction)

(1) If $s \to_c t$ then $s \to^* t$.

(2) s is preserved if and only if it is $\to_c$-irreducible.

(3) s is preserved if and only if none of its aliens allows a colour changing reduction (sequence).

(4) $\mathrm{WN}(s, \to_c)$ implies that s has a preserved reduct (w.r.t. $\to$).

Definition 2.4.14 (extended notions for substitutions)
Let σ and τ be substitutions (over $\mathcal{T}(\mathcal{F}, \mathcal{V})$).

(1) We write $\sigma \propto \tau$ if $\sigma x = \sigma y$ implies $\tau x = \tau y$ for all $x, y \in \mathcal{V}$.

(2) By $\sigma \to^* \tau$ we mean $\sigma x \to^* \tau x$ for all $x \in \mathcal{V}$.

(3) σ is said to be *irreducible* or *in normal form* (w.r.t. $\to$) if σx is irreducible (w.r.t. $\to$) for all $x \in Dom(\sigma)$.

(4) σ is called *black* (*white*) if σx is black (white) for all $x \in Dom(\sigma)$. It is *top black* (*top white*) if σx is top black (top white) for all $x \in Dom(\sigma)$.

[38]This means consistent replacement with minimal identification, i.e., distinct principal aliens are replaced by distinct fresh variables.

[39]We remark that a technically more elegant version of injective abstraction is possible (cf. [FJ95]).

Lemma 2.4.15 (decomposition of substitutions)
Every substitution σ (over $\mathcal{T}(\mathcal{F},\mathcal{V})$) can be decomposed into $\sigma_2 \circ \sigma_1$ such that σ_1 is black, σ_2 top white, and $\sigma_2 \propto \epsilon$ (recall that ϵ denotes the empty substitution).

Definition 2.4.16 (auxiliary relations for conditional case, [Mid93b, Definition 3.2])
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs with $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$. Then the rewrite relation $\rightarrow_1$ (on $\mathcal{T}(\mathcal{F},\mathcal{V})$) is defined as follows: $s \rightarrow_1 t$ if there exists a rule $l \rightarrow r \Longleftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}_1$, a context $C[]$ and a substitution σ such that $s = C[\sigma(l)]$, $t = C[\sigma(r)]$ and $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ for $i = 1, \ldots, n$, where $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ means that s_i and t_i are joinable using only $\stackrel{o}{\rightarrow}_1$-reduction steps. The relation $\rightarrow_2$ is defined analogously. The union of $\rightarrow_1$ and $\rightarrow_2$ is denoted by $\rightarrow_{1,2}$.

Non-Disjoint Unions: Constructor Sharing and Composable Systems

For constructor sharing and for composable systems the basic notations, definitions and facts essentially carry over, with the following adaptations and extensions. Here we mainly follow Ohlebusch ([Ohl94a]), with some slight differences, however. We assume subsequently that $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ are two composable systems (note that this subsumes the case of constructor sharing systems) with $\mathcal{R}^{\mathcal{F}}$ denoting their union. Again we shall use the abbreviating notations $\mathcal{T} = \mathcal{T}(\mathcal{F},\mathcal{V})$ and $\mathcal{T}_i = \mathcal{T}(\mathcal{F}_i,\mathcal{V})$ for $i = 1, 2$. Instead of $\rightarrow_{\mathcal{R}}$ we shall also simply use $\rightarrow$.

First of all, the chromatic terminology extends (more precisely, is refined) by considering only the non-shared function symbols from $\mathcal{F}_1$ ($\mathcal{F}_2$), i.e., the elements of $\mathcal{F}_1^{ns}$ ($\mathcal{F}_2^{ns}$), to be black (white) and all shared function symbols, i.e., those from $\mathcal{F}^s$, as well as the variables to be transparent. Then the definitions of black (white), top black (top white), homogeneous and mixed extend in the obvious way. The same holds for the unique context notations which now have the form

$$s = \begin{cases} C^b \langle\!\langle s_1, \ldots, s_l \rangle\!\rangle \text{ with } C^b\langle, \ldots, \rangle \in \mathcal{T}_1,\ root(s_i) \in {\mathcal{F}_2}^{ns} \\ C^w \langle\!\langle t_1, \ldots, t_m \rangle\!\rangle \text{ with } C^w\langle, \ldots, \rangle \in \mathcal{T}_2,\ root(t_j) \in {\mathcal{F}_1}^{ns} \\ C^t \langle\!\langle u_1, \ldots, u_n \rangle\!\rangle \text{ with } C^t\langle, \ldots, \rangle \in \mathcal{T}(\mathcal{F}^s,\mathcal{V}),\ root(u_k) \in {\mathcal{F}_1}^{ns} \cup {\mathcal{F}_2}^{ns} \end{cases}$$

The s_i's are the *white principal subterms* (or *white principle aliens*) of s, the t_j's the *black principal subterms* (or *black principle aliens*) of s.

In particular, we observe that a mixed term must now have one of the following forms:

(1) $s = C^b[\![s_1, \ldots, s_l]\!]$ if s is top black (then the s_i's are all white principal aliens, and s is the only black principal alien, or else

(2) $s = C^w[\![t_1, \ldots, t_m]\!]$ if s is top white (then the t_j's are all black principal aliens, and s is the only white principal alien), or else

(3) $s = C^t[\![u_1, \ldots, u_n]\!]$ if s is top transparent (then the u_k's are white or black principal aliens of s).

The important notion of *rank* also naturally extends as follows (such that in particular transparent parts of a term do not contribute to its rank).

$$rank(s) = \begin{cases} 0 & \text{if } s \in \mathcal{T}(\mathcal{F}^s, \mathcal{V}) \\ 1 & \text{if } s \in (\mathcal{T}_1 \cup \mathcal{T}_2) \setminus \mathcal{T}(\mathcal{F}^s, \mathcal{V}) \\ 1 + \max\{rank(s_i) \mid 1 \leq i \leq n\} & \text{if } s = C[\![s_1, \ldots, s_n]\!] \end{cases}$$

As before, the *rank* of a derivation D is the minimal rank of all terms occurring in D. Moreover, the definition of the topmost homogeneous part is refined into a black and white version:

$$top^b(s) = C^b\langle, \ldots, \rangle$$

and

$$top^w(s) = C^w\langle, \ldots, \rangle$$

using the unique representations form above.

The notion of *inner* and *outer* reduction is extended (and refined) as follows:

(1) Let s be top black. If $s = C^b[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to C^b[s_1, \ldots, t_j, \ldots, s_n] = t$ and $s_j \to t_j$ then $s \xrightarrow{i} t$ (as before), otherwise we write $s \xrightarrow{o}_b t$ (the applied rule must be a black or transparent one, i.e., from $\mathcal{R}_1$).

(2) Let s be top white. If $s = C^w[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to C^w[s_1, \ldots, t_j, \ldots, s_n] = t$ and $s_j \to t_j$ then $s \xrightarrow{i} t$ (as before), otherwise we write $s \xrightarrow{o}_w t$ (the applied rule must be a white or transparent one, i.e., from $\mathcal{R}_2$).

(3) Let s be top transparent. If $s = C^t[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to C^t[s_1, \ldots, t_j, \ldots, s_n] = t$ and $s_j \xrightarrow{i} t_j$ then $s \xrightarrow{i} t$. If $s = C^t[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to C^t[s_1, \ldots, t_j, \ldots, s_n] = t$ and $s_j \xrightarrow{o}_b t_j$ (with s_j top black), then $s \xrightarrow{o}_b t$. If $s = C^t[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to C^t[s_1, \ldots, t_j, \ldots, s_n] = t$ and $s_j \xrightarrow{o}_w t_j$ (with s_j top white), then $s \xrightarrow{o}_w t$. If $s \to t$ by applying some transparent rule $l \to r$, i.e., with $l \to r \in \mathcal{R}^s$, at a position in the topmost transparent layer $C^t\,\{\!\{\,, \ldots, \}\!\}$ of $s = C^t\,\{\!\{\, s_1, \ldots, s_n\}\!\}$ then $s \xrightarrow{t} t$.

The relations $\xrightarrow{i}$, $\xrightarrow{o}_b$, $\xrightarrow{o}_w$, $\xrightarrow{o} = \xrightarrow{o}_b \cup \xrightarrow{o}_w$, and $\xrightarrow{t}$ are called *inner*, *black outer*, *white outer*, *outer* and *transparent* reduction, respectively. Further abbreviations used are $\xrightarrow{t,o}_b = \xrightarrow{t} \cup \xrightarrow{o}_b$, $\xrightarrow{t,o}_w = \xrightarrow{t} \cup \xrightarrow{o}_w$, and $\xrightarrow{t,o} = \xrightarrow{t} \cup \xrightarrow{o}$. Notice that every reduction step must either be inner or black outer or white outer or transparent.

Destructive steps (at levels $n \geq 1$) $s \to t$ for s top black or top white are defined as before. For the additional case that s is top transparent, and $s \to t$ we define: $s \to t$ is *destructive at level 0* if t is top black or top white (i.e., non-top-transparent), and it is *destructive at level* $m \geq 1$ if $s = C^t[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to C^t[s_1, \ldots, t_j, \ldots, s_n] = t$ with $s_j \to t_j$ destructive at level m.

Again it is easy to show that reduction in the union $\mathcal{R}^{\mathcal{F}}$ is rank decreasing.

The notion of *colour changing* reduction as well as that of *collapsing* (or *layer coalescing*) reduction is as before.[40] The same holds for the property of terms to be *preserved*.

[40] This is a slight but crucial difference to Ohlebusch ([Ohl94a]) who also defines destructive steps at

Inner preservation is refined as follows: A term s is *black (white) preserved* if all its black (white) principal aliens are preserved. With these adaptations Lemma 2.4.13 immediately extends to the more general setting. Furthermore the remaining notations and properties for the disjoint union case extend in the obvious way.

However, we need some more definitions in order to deal appropriately with the new 'layer coalescing' effect that rules with transparent right hand side root symbol can cause.

Definition 2.4.17 (constructor lifting / shared function symbol lifting / shared symbol lifting / layer preserving rules / systems)
We say that a rule $l \to r \in \mathcal{R}$ is

- *constructor lifting* if $root(r)$ is a shared constructor,
- *shared function symbol lifting* if $root(l) \in {\mathcal{F}_1}^{ns} \uplus {\mathcal{F}_2}^{ns}$ and $root(r) \in \mathcal{F}^s$ (i.e., $if root(l)$ is a non-shared function symbol of either $\mathcal{R}_1$ or $\mathcal{R}_2$, and $root(r)$ is a transparent (shared) function symbol),
- *shared symbol lifting* if $root(l) \in {\mathcal{F}_1}^{ns} \uplus {\mathcal{F}_2}^{ns}$ and $root(r) \in \mathcal{F}^s \uplus \mathcal{V}$ (i.e., if $l \to r$ is a non-shared rule of either $\mathcal{R}_1$ or $\mathcal{R}_2$ such that $root(r)$ is a transparent (shared function or variable) symbol).

We say that $\mathcal{R}_i$ (for $i = 1, 2$) is

- *constructor lifting* / *shared function symbol lifting* / *shared symbol lifting* if it has a constructor lifting / shared function symbol lifting / shared symbol lifting rule, respectively.
- *layer preserving* if it is not shared symbol lifting.

Hence, for the case of constructor sharing systems $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ the notions *constructor lifting* and *shared function symbol lifting* coincide (which is not the case if $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ are composable). Thus, two constructor sharing systems $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ are layer preserving if and only if they are non-collapsing and not constructor lifting. Furthermore, observe that two disjoint systems $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ are layer preserving if and only if they are non-collapsing. This means that layer preservation naturally extends the (important) non-collapsing property from the disjoint union case to the case of composable systems.

Finally, let us provide one more definition for denoting sets of normal forms: $\mathrm{NF}(\mathcal{R}^{\mathcal{F}}) := \{s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid s\ \mathcal{R}\text{-irreducible}\}$.

level 0 to be collapsing. Hence, also our notion of preservation slightly differs from the one of [Ohl94a].

Chapter 3

Relating Termination and Confluence Properties

Here we first review known results on termination and confluence properties of orthogonal TRSs. Then we show in Section 3.2 how to generalize almost all of these results to non-overlapping, but not necessarily left-linear systems. We give various counterexamples showing that the preconditions of obtained results cannot be dropped. Then, in Section 3.3 we relax the non-overlapping restriction by requiring the systems to be only overlaying and locally confluent. We show that the most important result for non-overlapping systems, namely the equivalence of innermost and general termination, does indeed also hold for this more general class of TRSs. In Section 3.4 we develop an alternate, incompatible approach for showing the equivalence of innermost and general termination thereby generalizing most of the results of Section 3.2. In Section 3.5 we exclusively deal with properties of innermost reduction and relate them to the corresponding properties of general reduction, which again leads to a couple of new results and to generalized versions of already known ones. Finally, in Section 3.6, we extend the previous analysis to conditional systems and show how to cope with the additional complications and problems arising there. In particular, we prove here a key lemma which expresses an interesting localized completeness property for conditional overlay systems without a full termination assumption.

The indeterminism of arbitrary rewriting considerably complicates termination proofs (and makes the confluence problem non-trivial). Essentially there are two kinds of indeterminism involved in computation by term rewriting, namely the choice of subterms as potential redexes and the choice of the rule(s) to be applied. Here we shall only be concerned with the first kind of restriction which amounts to rewriting under some (redex position selection) strategy. Clearly, the termination problem for rewriting under such a strategy may be considerably easier than for unrestricted rewriting.

Restricting rewriting derivations with such a strategy is quite common in many rewriting based computation models, e.g. in functional programming languages. For instance, a frequent restriction is innermost reduction, i.e. to require that every reduction step takes place at an innermost position of the term to be reduced. Innermost reduction corresponds closely to the functional evaluation mechanism employed in functional programming languages like LISP or ML. Other kinds of restrictions imposed on rewriting steps might also be conceivable according to the intended purpose, e.g. leftmost-outermost, parallel-outermost, top-down, bottom-up or other context-dependent strategies.

Of course, it may be the case that correspondingly restricted computations, e.g. innermost reduction sequences, always terminate but arbitrary computations (reduction sequences) do not necessarily terminate. As a very simple example illustrating this gap consider the TRS

$$\mathcal{R} = \left\{ \begin{array}{r} f(a) \to f(a) \\ a \to b \end{array} \right.$$

Here we have e.g. the infinite reduction sequence $f(a) \to f(a) \to f(a) \to \ldots$, which uses only non-innermost reduction steps. But of course, every innermost derivation in $\mathcal{R}$ (e.g. $f(a) \to f(b)$) is terminating.

In fact, for all conceivable strategies termination of rewriting under the strategy does in general not imply termination of unrestricted rewriting. Unfortunately, very little is known about termination of rewriting under such restrictions and its relation to general termination. However, there is one major exception, namely concerning the important and thoroughly investigated class of *orthogonal* TRSs, i.e., TRSs which are left-linear and non-overlapping (see [Klo92] for a survey of basic ideas, concepts and results about the theory of orthogonal TRSs). It is well-known that any orthogonal TRS is confluent notwithstanding the fact that it may be non-terminating (cf. Theorem 2.2.27).

In the following we shall study properties of rewriting under some fixed strategy. We focus here on sufficient criteria for the equivalence of termination of arbitrary rewriting and termination of restricted rewriting, in particular innermost rewriting. Confluence properties of restricted rewriting relations are touched, too, as by-products of the analysis (cf. also [Pla94], [Kri94a]). To this end we shall first collect and review some known results about termination properties of orthogonal TRSs. Then we shall investigate and develop various extensions and generalizations of these known results about orthogonal TRSs. This is done by weakening both the no-overlap and the left-linearity requirement but still guaranteeing local confluence.

The obtained results are very useful, from a theoretical, practical and conceptual point of view. They

- provide powerful abstract criteria for termination and confluence of rewriting in terms of restricted termination and confluence properties which considerably facilitates the burden of termination (and confluence) proofs for certain classes of TRSs
- are the basis for deriving a couple of interesting preservation results for properties of combined systems (cf. Chapter 5)
- can be generalized in a natural way to CTRSs, though this is proof-technically non-trivial (cf. Section 3.6).

3.1 Orthogonal Systems

Perhaps the most important and fundamental result concerning confluence of (possibly) non-terminating TRSs is the following: Any orthogonal TRS is confluent (cf. Theorem 2.2.27). This fundamental property is crucial for instance within the field of designing and implementing equational programming languages (cf. e.g. [O'D77], [O'D85]) and has initiated a couple of investigations about the class of orthogonal TRSs.

In particular, for orthogonal TRSs one also knows some sufficient criteria for termination (SN) which are formulated in terms of restricted termination properties (cf. e.g. [O'D77], [Klo92]). But as soon as the orthogonality requirement is weakened, either by allowing critical overlaps or by admitting non-left-linear rules, the main results (at least concerning confluence) do not hold any more, in particular the *parallel moves lemma* ([CF58]) which is the technical key lemma for inferring confluence for orthogonal TRSs.[1]

The most important known results about termination properties of orthogonal TRSs can be summarized as follows (recall that NE denotes the non-erasing property).

Theorem 3.1.1 (termination properties of orthogonal TRSs, cf. [Chu41], [Ros73], [O'D77], [Klo92])
Any orthogonal TRS $\mathcal{R}$ satisfies the following properties:

(1a) $\forall t : [\, \mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]$.

(1b) $\mathrm{WIN}(\mathcal{R}) \iff \mathrm{SIN}(\mathcal{R}) \iff \mathrm{SN}(\mathcal{R})$.

(2a) If $s \rightarrow_{p,\sigma,l\rightarrow r} t$ such that $\mathrm{SN}(t)$ and $\neg\mathrm{SN}(s)$ then s/p contains a proper subterm $s' = \sigma x$, for some $x \in Var(l)$, with $\neg\mathrm{SN}(s')$.[2]

(2b) If $s \xrightarrow[i]{} t$ and $\mathrm{SN}(t)$ then $\mathrm{SN}(s)$.

(3a) $\mathrm{NE}(\mathcal{R}) \implies [\forall t : [\, \mathrm{WN}(t) \iff \mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]\,]$.

(3b) $\mathrm{NE}(\mathcal{R}) \implies [\, \mathrm{WN}(\mathcal{R}) \iff \mathrm{WIN}(\mathcal{R}) \iff \mathrm{SIN}(\mathcal{R}) \iff \mathrm{SN}(\mathcal{R})\,]$.

Let us now investigate what happens if we drop the left-linearity requirement in Theorems 2.2.27 and 3.1.1. Examples 2.2.19, 2.2.20 show that if the left-linearity condition is dropped then confluence can get lost, i.e., non-overlapping TRSs need not be confluent. A closer look at these examples reveals that in both cases the TRS $\mathcal{R}$ is (necessarily non-terminating and) neither weakly innermost terminating (WIN) nor weakly terminating (WN). Weak termination is not crucial for the existence of such counterexamples as shown by the following example.

[1] directly, without making use of Theorem 2.2.26.

[2] This means that the non-terminating proper subterm $s' = \sigma x$ of s/p is *erased* in the step $s \rightarrow_{p,\sigma,l\rightarrow r} t$, and implies that $l \rightarrow r$ is an *erasing* rule, since $x \in Var(l) \setminus Var(r)$ due to $\mathrm{SN}(t)$.

Example 3.1.2 (Sivakumar'86 [DJ90])
Let $\mathcal{R}$ be the TRS given by

$$\mathcal{R} = \begin{cases} f(x,x) \to g(x) \\ f(x,g(x)) \to b \\ h(c,y) \to f(h(y,c),h(y,y)) \end{cases}$$

Clearly, $\mathcal{R}$ is non-overlapping but not confluent. We have e.g. the derivations

$$\begin{aligned} h(c,c) &\to f(h(c,c),h(c,c)) \to f(h(c,c),f(h(c,c),h(c,c))) \\ &\to f(h(c,c),g(h(c,c))) \to b \end{aligned}$$

and

$$h(c,c) \to f(h(c,c),h(c,c)) \to g(h(c,c)) \to^+ g(b)\,.$$

Here b and $g(b)$ are in normal from. Note that $\mathcal{R}$ is obviously not strongly normalizing and even not weakly innermost normalizing (consider e.g. the term $h(c,c)$), but weakly normalizing. Furthermore observe that $\mathcal{R}$ is not non-erasing (NE) due to the second rule.

These examples indicate that there might be some hope to generalize Theorem 3.1.1 to non-overlapping but possibly non-left-linear TRSs.

3.2 Non-Overlapping Systems

In the following we shall consider TRSs which are still non-overlapping but not necessarily left-linear. In fact, we will show that all normalization properties of Theorem 3.1.1 still hold for this more general class of TRSs. Throughout the following we assume that $\mathcal{R}$ is a non-overlapping TRS, i.e., NO($\mathcal{R}$) holds.

Let us start with an easy result about innermost reduction.

Lemma 3.2.1 (innermost reduction is uniformly confluent)
Innermost reduction in $\mathcal{R}$ is WCR^1, i.e., WCR$^1(\underset{i}{\longrightarrow})$ holds.

Proof: Let $s \underset{i}{\longrightarrow}_p t$ and $s \underset{i}{\longrightarrow}_q u$. If the innermost redex positions p, q of s are the same then the applied rule is unique due to NO($\mathcal{R}$) which implies $t = u$. Otherwise p and q are disjoint and v is uniquely defined by $s \underset{i}{\longrightarrow}_p t \underset{i}{\longrightarrow}_q v$ and $s \underset{i}{\longrightarrow}_q u \underset{i}{\longrightarrow}_p v$. ■

Combined with Theorem 2.1.9 this yields the following.

Corollary 3.2.2 (innermost reduction is confluent)
Innermost reduction in $\mathcal{R}$ is confluent, i.e. CR$(\underset{i}{\longrightarrow})$ holds.

Next we show that the existence of a terminating innermost derivation for some term t implies that any innermost derivation initiated by t is finite.

Lemma 3.2.3 (weak and strong innermost termination coincide)
$\forall t : [\,\mathrm{WIN}(t) \iff \mathrm{SIN}(t)\,]$.

Proof: By Lemma 3.2.1 we know that innermost reduction is WCR[1]. Hence, observing that WIN (SIN) is equivalent to "innermost reduction is WN (SN)", we can simply apply Lemma 2.1.27 which yields the desired result. ■

Furthermore, strong innermost normalization is equivalent to strong normalization.

Theorem 3.2.4 (innermost termination and termination coincide)
$\forall t : [\,\mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]$.

Proof: For a proof we refer to the proof of the more general Theorem 3.3.12 which uses a kind of parallel normalization technique. Note that a more direct proof by means of a parallel one-step reduction technique is also possible here (cf. Section 3.4, Theorem 3.4.11). ■

Combined with Lemma 3.2.3 this yields the following.

Corollary 3.2.5 $\forall t : [\,\mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]$.

The next result says that innermost reduction steps in non-overlapping TRSs cannot be critical in the sense that they may destroy the possibility of infinite derivations.

Lemma 3.2.6 (innermost steps cannot be critical)
If $s \xrightarrow[i]{} t$ and $\mathrm{SN}(t)$ then $\mathrm{SN}(s)$.

Proof: This is an immediate consequence of Corollary 3.2.5. ■

The properties mentioned in Corollary 3.2.5 are in general not equivalent to $\mathrm{WN}(t)$. To see this, consider the following

Example 3.2.7 (weak termination does not imply termination)
Consider the TRS

$$\mathcal{R} = \left\{ \begin{array}{r} a \to f(a) \\ f(x) \to b \end{array} \right.$$

which is clearly non-overlapping. We have $\mathrm{WN}(f(a))$ since for instance $f(a) \to b$ with b irreducible, but obviously not $\mathrm{SN}(f(a))$.

Note that in this example the innermost redex a which is not strongly normalizing disappears by applying the *erasing* rule $f(x) \to b$. In fact, this is crucial, and the absence of erasing rules turns out to be sufficient for the equivalence of all mentioned normalization properties as shown next.

Lemma 3.2.8 (critical steps must be erasing non-innermost)
If $s \to_{p,\sigma,l\to r} t$ such that $\mathrm{SN}(t)$ and $\neg\mathrm{SN}(s)$ then s/p contains a proper subterm $s' = \sigma(x)$, for some $x \in Var(l) \setminus Var(r)$,[3] with $\neg\mathrm{SN}(s')$.

[3] This means that $l \to r$ is an *erasing* rule.

Proof: Let $s = C[\sigma l]_p \to_{p,\sigma,l\to r} C[\sigma r]_p = t$ and SN(t), ¬SN(s). By Lemma 3.2.6 the step $s \to_{p,\sigma,l\to r} t$ must be non-innermost. NO($\mathcal{R}$) implies that σx is reducible for some $x \in V(l)$. And SN(t) implies in particular SN(σx) for all $x \in V(r)$. Now suppose that the rule $l \to r$ is non-erasing, i.e., $Var(l) = Var(r)$. Due to SN(t) this implies SN(σx) for all $x \in V(l)$. By innermost normalizing σ we thus obtain $s = C[\sigma l]_p \xrightarrow[i]{}^+ C[\sigma' l]_p \to C[\sigma' r]_p$ and $t = C[\sigma r]_p \xrightarrow[i]{}^* C[\sigma' r]_p$ with $\sigma' x$ a normal form, for all $x \in V(l) = V(r)$. Furthermore, the step $C[\sigma' l]_p \to_{p,\sigma',l\to r} C[\sigma' r]$ is innermost because of NO($\mathcal{R}$). Hence, from $s \xrightarrow[i]{}^* C[\sigma' r]_p$, $t \xrightarrow[i]{}^* C[\sigma' r]_p$ and SN(t) we obtain WIN(s) which, by Corollary 3.2.5, yields SN(s). But this is a contradiction to ¬SN(s). Hence, the rule $l \to r$ must be erasing, and there must exist some $x \in Var(l) \setminus Var(r)$ with ¬(SN(σx)). This concludes the proof. ∎

As direct consequence of this result we obtain the following.

Corollary 3.2.9 (critical steps are impossible for non-erasing TRSs)
Suppose NE($\mathcal{R}$). If $s \to t$ and SN(t) then SN(s).

Lemma 3.2.10
If NE($\mathcal{R}$) then : $\forall t : [\,\mathrm{WN}(t) \iff \mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]$.

Proof: Let t be a term with WN(t). This means there exists some normalizing derivation from t:

$$D:\ t =: t_0 \to t_1 \to t_2 \ldots \to t_n$$

with t_n in normal form, hence in particular SN(t_n). By a straightforward induction on the length of the derivation D we obtain $SN(t)$ from SN(t_n) using Lemma 3.2.9. Hence we are done. ∎

Finally let us summarize the results obtained for non-overlapping, but not necessarily left-linear TRSs.

Theorem 3.2.11 (termination properties of non-overlapping TRSs)
Any non-overlapping TRS $\mathcal{R}$ satisfies the following properties:

(1a) $\forall t : [\,\mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]$.

(1b) $\mathrm{WIN}(\mathcal{R}) \iff \mathrm{SIN}(\mathcal{R}) \iff \mathrm{SN}(\mathcal{R})$.

(2a) If $s \to_{p,\sigma,l\to r} t$ such that SN(t) and ¬SN(s) then s/p contains a proper subterm $s' = \sigma(x)$, for some $x \in Var(l) \setminus Var(r)$, with ¬SN($s'$).[4]

(2b) If $s \xrightarrow[i]{} t$ and SN(t) then SN(s).

(3a) $\mathrm{NE}(\mathcal{R}) \implies [\forall t : [\,\mathrm{WN}(t) \iff \mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]\,]$.

[4]This means that the non-terminating proper subterm $s' = \sigma(x)$ of s/p is *erased* in the step $s \to_{p,\sigma,l\to r} t$, and implies that $l \to r$ is an *erasing* rule.

(3b) $\mathrm{NE}(\mathcal{R}) \implies [\,\mathrm{WN}(\mathcal{R}) \iff \mathrm{WIN}(\mathcal{R}) \iff \mathrm{SIN}(\mathcal{R}) \iff \mathrm{SN}(\mathcal{R})\,]$.

(4a) $\forall t : [\,\mathrm{WIN}(t) \implies \mathrm{CR}(t)\,]$.

(4b) $\mathrm{WIN}(\mathcal{R}) \implies \mathrm{CR}(\mathcal{R})$.

(5a) $\mathrm{NE}(\mathcal{R}) \implies [\forall t : [\,\mathrm{WN}(t) \implies \mathrm{CR}(t)\,]\,]$.

(5b) $\mathrm{NE}(\mathcal{R}) \wedge \mathrm{WN}(\mathcal{R}) \implies \mathrm{CR}(\mathcal{R})$.

Proof: (1), (2a), (2b) and (3) correspond to Corollary 3.2.5, Lemma 3.2.8, Lemma 3.2.6 and Corollary 3.2.9. (4) follows from (1) and NO($\mathcal{R}$) (by Theorem 2.2.18), and (5) from Lemma 3.2.10 and NO($\mathcal{R}$) (again by Theorem 2.2.18). ■

3.3 Locally Confluent Overlay Systems

Considering Theorem 3.2.11(1) one may ask now whether instead of requiring NO($\mathcal{R}$) and WIN($\mathcal{R}$) it could also be sufficient to require WCR($\mathcal{R}$) and WIN($\mathcal{R}$) for guaranteeing SN($\mathcal{R}$) (and CR($\mathcal{R}$)). But this is not sufficient as witnessed by

Example 3.3.1 (WCR $\wedge$ WIN $\not\Rightarrow$ SN, CR)
For the TRS

$$\mathcal{R} = \begin{cases} b \to a \\ b \to c \\ c \to b \\ c \to d \end{cases}$$

we clearly have WCR($\mathcal{R}$) and WIN($\mathcal{R}$) but neither SN($\mathcal{R}$) nor CR($\mathcal{R}$).

Even the stronger requirement WCR($\mathcal{R}$) and SIN($\mathcal{R}$) is not sufficient for ensuring SN($\mathcal{R}$) (and CR($\mathcal{R}$)) as can be seen from

Example 3.3.2 (WCR $\wedge$ SIN $\not\Rightarrow$ SN, CR)
Consider the TRS

$$\mathcal{R} = \begin{cases} f(b) \to a & b \to e \\ f(b) \to f(c) & c \to e' \\ f(c) \to f(b) & f(e) \to a \\ f(c) \to d & f(e') \to d \end{cases}$$

Here it is easily verified that WCR($\mathcal{R}$) and SIN($\mathcal{R}$) hold but neither SN($\mathcal{R}$) nor CR($\mathcal{R}$).

Another possibly tempting conjecture might be to insist on SIN($\mathcal{R}$) and even require CR($\mathcal{R}$) instead of the weaker WCR($\mathcal{R}$) in order to infer SN($\mathcal{R}$). But this is also not true in general.

Example 3.3.3 (CR $\wedge$ SIN $\not\Rightarrow$ SN)
Consider the TRS

$$\mathcal{R} = \begin{cases} f(a) \to f(a) \\ \quad a \to b \end{cases}$$

for which we have CR($\mathcal{R}$) and SIN($\mathcal{R}$) but not SN($\mathcal{R}$).

But a common feature of the latter two counterexamples consists in the fact that for both systems some critical pairs were constructed by overlaps strictly below the root. This is crucial as will be shown. More precisely, we will prove that any strongly innermost normalizing, locally confluent overlay system is strongly normalizing (hence also confluent and complete). To this end, we make the following

General Assumption: Henceforth (i.e., throughout this section) we are dealing with a TRS $\mathcal{R}$ which is locally confluent, i.e., WCR($\mathcal{R}$) holds.

To simplify the presentation we introduce some easy definitions. Every term t can be (uniquely) written as $C\langle t_1, \ldots, t_n\rangle$ where $t_1, \ldots, t_n$ are the maximal complete subterms of t.[5]

Definition 3.3.4 (transformation by parallel normalization of all maximal complete subterms)
Let $\Phi(t) := C\langle t_1\!\downarrow, \ldots, t_n\!\downarrow\rangle$. Here $t_i\!\downarrow$ denotes the unique normal form of t_i.

Clearly we have $t \to^* \Phi(t)$.

If $s \to t$ by contracting a complete redex in s (i.e., $s \to_p t$ for some $p \in Pos(s)$, with s/p complete), we write $s \to_{\mathrm{c}} t$. If $s \to t$ by contracting a non-complete redex in s (i.e., $s \to_p t$ for some $p \in Pos(s)$, with s/p not complete), we write $s \to_{\mathrm{nc}} t$. Clearly, every reduction step can be written as $s \to_{\mathrm{c}} t$ or $s \to_{\mathrm{nc}} t$.[6] We observe that whenever $s \to_{\mathrm{c}} t$ by a root reduction step (with s complete) then this implies SN($s, \to_{\mathrm{c}}$) (which in this case is equivalent to SN($s, \to$). Furthermore, if $C[s] \to_{\mathrm{c}} C[t]$ (by contracting some complete redex in s) then $s \to_{\mathrm{c}} t$, and vice versa.

Lemma 3.3.5 ($\to_{\mathrm{c}}$ is terminating)
The relation $\to_{\mathrm{c}}$ is terminating, i.e., for all terms s we have SN($s, \to_{\mathrm{c}}$).

Proof: We proceed by induction on the term structure. If s is a variable we are obviously done. If s is a constant then it is either irreducible or else any reduction step $s \to_{\mathrm{c}} t$ is a root reduction step. In both cases s is complete implying SN($s, \to_{\mathrm{c}}$). For the induction step consider the case $s = f(t_1, \ldots, t_n)$ $(n \geq 1)$. Assume $D : s =: s_0 \to_{\mathrm{c}} s_1 \to_{\mathrm{c}} s_2 \to_{\mathrm{c}} \ldots$ is an infinite $\to_{\mathrm{c}}$-derivation. Since we have

[5] Recall that a term is complete if it is both confluent and terminating. Because of the general assumption WCR a term is complete if and only if it is terminating.

[6] Note however, that it is possible that s reduces to t both by contracting a complete or a non-complete redex in s. For instance, let $\mathcal{R} = \{a \to b, f(a) \to f(b), f(b) \to f(a)\}$. Then we have both $f(a) \to_{\mathrm{c}} f(b)$ and $f(a) \to_{\mathrm{nc}} f(b)$.Hence, in general we do not have $\to_{\mathrm{c}} \cap \to_{\mathrm{nc}} = \emptyset$. Nevertheless this could be enforced by slightly modifying the definition of $\to_{\mathrm{nc}}$ as follows: $\to_{\mathrm{nc}} = \to \setminus \to_{\mathrm{c}}$.

$\mathrm{SN}(t_i, \to_c)$ for all i by induction hypothesis, some (first) step in D must be a root reduction step $s_k \to_c s_{k+1}$. But then s_k is complete which contradicts the infinity of D. ∎

Lemma 3.3.6 (infinite derivations contain infinitely many $\to_{nc}$-steps)
Suppose $\neg\mathrm{SN}(\mathcal{R})$. Then every infinite derivation contains infinitely many $\to_{nc}$-steps (which are not $\to_c$-steps).

Proof: This is an immediate consequence of Lemma 3.3.5. ∎

Remark 3.3.7 (any non-terminating term even allows a minimal and constricting infinite derivation)
Actually, if a term s is non-terminating, then there even exists an infinite derivation (labelled by the positions of the contracted redexes)

$$s =: s_1 \to_{u_1} s_2 \to_{u_2} s_3 \to_{u_3} \dots$$

which is minimal in the following sense ([Gra92b]):

$$\forall k \geq 1\, \forall s'_{k+1}.\, s_k \to_{>u_k} s'_{k+1} \Longrightarrow \neg\mathrm{SN}(s'_{k+1})\,.$$

Note that this minimality property implies in particular

$$\forall k \geq 1\, \forall q \in Pos(s_k/u_k), q > \lambda.\, \mathrm{SN}(s_k/u_k q)\,,$$

i.e., all proper subterms of contracted redexes are terminating.

Moreover, as observed in [Pla93] and [DH95], minimal infinite derivations may even be chosen to be *constricting*. Here, an infinite (labelled) derivation as above is said to be *constricting*[7] if

$$v_1 \geq v_2 \geq v_3 \geq \dots\,,$$

where each v_k is the position of the minimal non-terminating superterm of s_k/u_k in s_k, formally:

$$v_k = \max\{v \in Pos(s_k) \mid \lambda \leq v \leq u_k, \neg\mathrm{SN}(s_k/v)\}\,.$$

A minimal, constricting infinite rewrite sequence for a non-terminating term s can be constructed as follows: Choose in $s_1 := s$ a non-terminating subterm $t_1 = s/p_1$ which is minimal in the sense that every proper subterm of it is terminating. Choose in s below p_1 a minimal subterm $s/p_1 q_1$ with the property that $s/p_1 q_1 = \sigma l$ for some rule $l \to r$ such that $s \to_{p_1 q_1, \sigma, l \to r} s[p_1 q_1 \leftarrow \sigma r] =: s'$ with $s/p_1[q_1 \leftarrow \sigma r] = s'/p_1$ still non-terminating.[8] If $q_1 > \lambda$ then repeat this process for s'. Constructing an infinite reduction issuing from s_1 in that way, eventually some non-terminating redex at position p_1 (of the current term s'_2 to be rewritten) must be contracted (with the

[7] Note that our definition of *constricting* differs from the one in [Pla93]. The problem with the latter one is that according to it any(!) infinite derivation would be constricting. However, the construction described in [Pla93] yields indeed an infinite derivation which is minimal and constricting in our sense.

[8] This choice still involves some indeterminism. For instance, one may proceed here according to a leftmost innermost strategy (as in [Pla93]), or according to any other strategy provided that the required minimality property is guaranteed.

contractum still non-terminating) since all proper subterms of s_1/p_1 are terminating. This results in a derivation of the form

$$s_1 \to^*_{>p_1} s'_2 \to_{p_1} s_2$$

such that s_2/p_1 is again non-terminating. Now we may repeat the whole process with s_2/p_1 instead of s_1. Doing this repeatedly one obtains an infinite derivation issuing from s_1 of the form

$$s_1 \to^*_{>p_1} s'_2 \to_{p_1} s_2 \to^*_{>p_1p_2} s'_3 \to_{p_1p_2} \dots s_n \to^*_{>p_1p_2\dots p_n} s'_{n+1} \to_{p_1p_2\dots p_n} s_{n+1} \dots$$

where in $s_i \to^*_{>p_1\dots p_i} s'_{i+1}$ only terminating redexes (at deepest possible positions) are contracted, and in $s'_{i+1} \to_{p_1\dots p_i} s_{i+1}$ the contracted redex $s'_{i+1}/p_1 \dots p_i$ (as well as its contractum $s_{i+1}/p_1 \dots p_i$) is non-terminating, and moreover, all proper subterms of $s'_{i+1}/p_1 \dots p_i$ are terminating and contracting any of these in $s'_{i+1}/p_1 \dots p_i$ would result in a terminating term.[9] Due to these properties the infinite derivation above is indeed minimal and constricting.

Lemma 3.3.8 (projection of $\to_c$-steps)
If $s \to_c t$ then $\Phi(s) \to^* \Phi(t)$.

Proof: Clearly, by definition of Φ and of $\to_c$, we have $t \to^* \Phi(s)$ by performing only reductions in complete subterms of t. Hence we get $\Phi(s) \to^* \Phi(t)$. ■

Observe that in general we do not have $\Phi(s) = \Phi(t)$ in Lemma 3.3.8 since the step $s \to_c t$ within some maximal complete subterm s/p of s may generate a new maximal complete subterm t/q in t with $q < p$. Take for instance $\mathcal{R}_1 = \{a \to b, f(a) \to f(a), f(b) \to c\}$ and consider the step $s = f(a) \to_c f(b) = t$ for which we have $\Phi(s) = f(b) \to c = \Phi(t)$. The analogous statement (of Lemma 3.3.8) for $\to_{nc}$ does not hold. Consider for instance the TRS $\mathcal{R}_2 = \{a \to b, f(a) \to g(a), g(x) \to f(x)\}$ and the step $s = f(a) \to_{nc} g(a) = t$. Here we have $\Phi(s) = f(b)$ and $\Phi(t) = g(b)$, but not $\Phi(s) \to^* \Phi(t)$. Note that $\mathcal{R}_2$ is not an overlay system. This is essential as will be shown next.

To this end we need the following crucial property of overlay systems.

Lemma 3.3.9 (crucial property of locally confluent overlay systems)
Suppose $\mathrm{OS}(\mathcal{R})$. If l is a left-hand side of a rule from $\mathcal{R}$ and σ a substitution such that σl is not complete then $\Phi(\sigma l) = (\Phi \circ \sigma)l$. Here $\Phi \circ \sigma$ denotes the composition of the substitution σ with the mapping Φ, i.e., the substitution defined by $(\Phi \circ \sigma)x = \Phi(\sigma x)$.

Proof: We have to show that normalization of all maximal complete subterms in σl can be achieved by normalizing all maximal complete subterms in the "substitution part σ of σl". If no subterm of σl is complete we clearly obtain $\Phi(\sigma l) = (\Phi \circ \sigma)l = \sigma l$

[9]Consequently, the constructed infinite derivation contains infinitely many such *essential* steps $s'_{i+1} \to_{p_1\dots p_i} s_{i+1}$ where contracting any proper (root-) reducible subterm of $s'_{i+1}/p_1 \dots p_i$ would necessarily destroy non-termination. Note that this property does also hold for infinite derivations which are minimal but not necessarily constricting! Actually, in proofs later on we shall only make use of the minimality property. Yet, the constricting property is appealing from an intuitive point of view and could additionally be assumed to ease comprehensibility.

by definition of Φ. Hence, we may assume that some subterm of σl is complete. Let $\sigma l = C[t_1, \ldots, t_n]$, $n \geq 1$, where the t_i's are the maximal complete subterms of σl, let's say with $\sigma l / p_i = t_i$. Note that, due to the assumption that σl is not complete, we have $\lambda < p_i$ for all p_i. Now, if p_i is below the position p of some variable x in l then we get $t_i \downarrow = (\Phi \circ \sigma)(l)/p_i$ since t_i is also a maximal complete subterm of σx. If p_i is a non-variable position of l then we have $t_i = \sigma(l/p_i)$. Since t_i is complete, for every variable x which occurs in l (strictly) below p_i, σx is also complete. Let $\sigma' x = (\Phi \circ \sigma)(x) = (\sigma x)\downarrow$ for these variables. By definition of Φ we get $t_i \downarrow = \sigma(l/p_i)\downarrow = \sigma'(l/p_i)\downarrow$. We still have to show $\sigma'(l/p_i)\downarrow = \sigma'(l/p_i)$. From $\mathrm{OS}(\mathcal{R})$, irreducibility of σ' and $\lambda < p_i$ we conclude that $\sigma'(l/p_i)$ must be irreducible (because otherwise there would exist a critical pair in $\mathcal{R}$ which is not an overlay). Hence we are done. ∎

Lemma 3.3.10 (non-empty projection of $\rightarrow_{\mathrm{nc}}$-steps)
Suppose $\mathrm{OS}(\mathcal{R})$. If $s \rightarrow_{\mathrm{nc}} t$ then $\Phi(s) \rightarrow^+ \Phi(t)$.

Proof: Suppose $s \rightarrow_{\mathrm{nc}} t$ by applying some rule $l \rightarrow r$ at position p with substitution σ, so $s/p = \sigma l$ and $t = s[p \leftarrow \sigma r]$. Because s/p is not complete, p is a position in $\Phi(s)$. Now, due to $\mathrm{OS}(\mathcal{R})$ we can apply Lemma 3.3.9 which yields $\Phi(s)/p = \tau l$ with substitution τ defined by $\tau x = \Phi(\sigma x)$ for all variables x. Hence we obtain $\Phi(s) \rightarrow \Phi(s)[p \leftarrow \tau r]$ again by the rule $l \rightarrow r$. Clearly we have $t \rightarrow^* \Phi(s)[p \leftarrow \tau r]$ by performing only reductions in complete subterms of t, and thus $\Phi(s)[p \leftarrow \tau r] \rightarrow^* \Phi(t)$. This implies $\Phi(s) \rightarrow \Phi(s)[p \leftarrow \tau r] \rightarrow^* \Phi(t)$, and hence $\Phi(s) \rightarrow^+ \Phi(t)$. ∎

Lemma 3.3.11 (Φ preserves non-termination for locally confluent overlay systems)
Suppose $\mathrm{OS}(\mathcal{R})$. If $\mathrm{SN}(\Phi(t))$ then $\mathrm{SN}(t)$.

Proof: Assume $\neg\mathrm{SN}(t)$, i.e., there exists an infinite derivation

$$D: \; t = t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \ldots$$

where every step is an $\rightarrow_{\mathrm{c}}$-step or an $\rightarrow_{\mathrm{nc}}$-step. By Lemma 3.3.6 there are infinitely many $\rightarrow_{\mathrm{nc}}$-steps in D. Hence, by applying Φ to D and using Lemmas 3.3.10 and 3.3.8 we obtain the infinite derivation

$$\Phi(D): \; \Phi(t) = \Phi(t_0) \rightarrow^* \Phi(t_1) \rightarrow^* \Phi(t_2) \rightarrow \ldots$$

which implies $\neg\mathrm{SN}(\Phi(t))$. ∎

Now we are prepared to prove the main result of this section.

Theorem 3.3.12 (innermost termination implies termination (and completeness) for locally confluent overlay systems)
Suppose $\mathrm{OS}(\mathcal{R})$. If $\mathrm{SIN}(t)$ then $\mathrm{SN}(t)$. (Taking into account the general assumption $\mathrm{WCR}(\mathcal{R})$, the global version of this result may be rephrased as follows: Any innermost terminating, locally confluent overlay system is terminating, hence complete.)

Proof: For a proof by contradiction, suppose that t admits an infinite derivation. Every infinite derivation starting from t must contain a non-innermost step, due to SIN(t). Now consider an infinite derivation D starting from t with the property that the first non-innermost step is essential: Selecting any innermost redex at that point would result in a term with the property SN. Let

$$D : t = t_0 \to t_1 \to \ldots \to t_n \to t_{n+1} \to \ldots$$

where $t_n \to t_{n+1}$ is the first non-innermost step. By assumption, contracting an innermost redex in t_n yields a strongly normalizing term. This implies that every innermost redex in t_n is complete. Since there is at least one innermost redex in t_n (the step $t_n \to t_{n+1}$ is non-innermost!), we conclude that $\mathrm{SN}(\Phi(t_n))$ holds. Since we also have $\neg\mathrm{SN}(t_n)$, this contradicts Lemma 3.3.11. Hence we are done. ∎

Note that we cannot weaken the precondition of Theorem 3.3.12 by omitting the global assumption WCR($\mathcal{R}$). To wit, consider the following example.

Example 3.3.13 (OS $\wedge$ SIN $\not\Rightarrow$ SN) Consider the TRS

$$\mathcal{R} = \begin{cases} f(a, b, x) \to f(x, x, x) \\ G(x, y) \to x \\ G(x, y) \to y \end{cases}$$

Obviously, $\mathcal{R}$ is an strongly innermost normalizing overlay system but it is not strongly normalizing as can be seen from the infinite (cyclic) derivation

$$\begin{array}{rcl} f(a, b, G(a, b)) & \to & f(G(a, b), G(a, b), G(a, b)) \\ & \to & f(a, G(a, b), G(a, b)) \\ & \to & f(a, b, G(a, b)) \\ & \to & \cdots . \end{array}$$

Moreover, omitting the condition OS($\mathcal{R}$) in Theorem 3.3.12 is not possible either, as demonstrated by Example 3.3.2.

According to Theorem 3.1.1(1a) (NO $\wedge$ LL $\wedge$ SIN) implies SN. Theorem 3.2.11(1) says that even (NO $\wedge$ SIN) implies SN. Moreover, from Theorem 3.3.12 we know that the weaker property (OS $\wedge$ WCR $\wedge$ SIN) implies SN, too. In view of these results another interesting conjecture would be the following:

$$(\mathrm{CR} \wedge \mathrm{LL} \wedge \mathrm{SIN}) \implies \mathrm{SN}.$$

But again this is not true in general.

Example 3.3.14 (Example 3.3.3 continued; CR $\wedge$ LL $\wedge$ SIN $\not\Rightarrow$ SN)
Let

$$\mathcal{R} = \begin{cases} f(a) \to f(a) \\ a \to b \end{cases}$$

Obviously, this system is confluent, left-linear and strongly innermost normalizing but is clearly not strongly normalizing.

Considering Theorems 3.2.11 and 3.3.12 one might be tempted to state the following conjectures by weakening the requirement NO to (OS $\wedge$ WCR).

$(C1)$ $\quad \mathrm{OS}(\mathcal{R}) \wedge \mathrm{WCR}(\mathcal{R}) \implies [\,\mathrm{WIN}(\mathcal{R}) \implies \mathrm{SIN}(\mathcal{R})\,]$,

$(C2)$ $\quad \mathrm{OS}(\mathcal{R}) \wedge \mathrm{WCR}(\mathcal{R}) \implies [\, s \xrightarrow[i]{} t \wedge \mathrm{SN}(t) \implies \mathrm{SN}(s)]$, and

$(C3)$ $\quad OS(\mathcal{R}) \wedge \mathrm{WCR}(\mathcal{R}) \wedge NE(\mathcal{R}) \implies [\,\mathrm{WN}(\mathcal{R}) \implies \mathrm{SN}(\mathcal{R})\,]$.

But $(C1)$, $(C2)$ and $(C3)$ are all refuted by the following very simple counterexample.

Example 3.3.15 Let

$$\mathcal{R} = \begin{cases} a \to a \\ a \to b \end{cases}$$

Clearly, $\mathcal{R}$ is a non-erasing overlay system where the only critical pair is a joinable overlay. Moreover, every term has a normal form that can be computed by innermost reduction, but obviously $\mathcal{R}$ is not strongly innermost normalizing and hence not strongly normalizing, too. Furthermore we have $a \xrightarrow[i]{} b$ with $\mathrm{SN}(b)$, but not $\mathrm{SN}(a)$.

3.4 Extensions

Next we shall show that it is possible to prove some interesting generalizations of the results in the previous two sections. Note that we do not presuppose any global assumption (like local confluence) any more!

The first crucial observation is that Theorem 3.2.4 (NO $\implies$ [SIN $\implies$ SN]), which is a special case of Theorem 3.3.12 (OS $\wedge$ JCP $\implies$ [SIN $\implies$ SN]), can be proved by means of a refined construction exploiting additional structural properties for the case of non-overlapping TRSs (as well as for more general classes of TRSs). More precisely, instead of parallel normalization of all maximal complete subterms we only perform a parallel one-step reduction at all terminating innermost redex positions. Two properties are essential for this construction to work.

Definition 3.4.1 (uniqueness of innermost reduction)
Let $\mathcal{R}$ be a TRS. We say that *innermost reduction (in $\mathcal{R}$) is unique* (at some fixed position), (denoted by UIR($\mathcal{R}$) or simply UIR), if

$$\forall s, t_1, t_2, p \in Pos(s) \,:\, s \xrightarrow[i]{}_p t_1 \wedge s \xrightarrow[i]{}_p t_2 \implies t_1 = t_2\,.$$

To state the second property, we first introduce a new definition and some compact notations.

Definition 3.4.2 (innermost-uncritical / innermost-critical reduction steps)

Let $\mathcal{R}$ be a TRS satisfying UIR. A reduction step $s \to t$ is called *innermost-uncritical*, denoted by $s \xrightarrow[iu]{} t$, if $s \to_{p,\sigma,l\to r} t$ (for some p, σ, $l \to r \in \mathcal{R}$) such that either s/p is an innermost redex (of s), or else all innermost redexes of s strictly below p correspond to variable overlaps w.r.t. l, i.e., for every innermost redex s/pp' of s, $p' > \lambda$, we have: $p' \geq q$ for some $q \in \mathcal{V}Pos(l)$ (if $s \xrightarrow[iu]{} t$ with some p, σ, $l \to r$ as above, we also write $s \xrightarrow[iu]{}_{p,\sigma,l\to r} t$). Otherwise, i.e., if $s \to t$ but not $s \xrightarrow[iu]{} t$, the step $s \to t$ is called *innermost-critical* and denoted by $s \xrightarrow[ic]{} t$.

Remark 3.4.3 By definition, $s \xrightarrow[ic]{} t$ means $s \to_{p,\sigma,l\to r} t$ (for some p, σ, $l \to r \in \mathcal{R}$) such that there exists an innermost redex $s/pp' = \tau l'$ (for some τ and $l' \to r' \in \mathcal{R}$) of s with $p' > \lambda$ and $p' \in \mathcal{F}Pos(l)$. Obviously, this entails the existence of a corresponding critical pair by overlapping $l' \to r'$ into $l \to r$. Moreover note that an innermost reduction step is by definition innermost-uncritical, and an innermost-critical step must be non-innermost, due to $\xrightarrow[i]{} \subseteq \xrightarrow[iu]{}$, $\to = \xrightarrow[iu]{} \uplus \xrightarrow[ic]{}$.

Now the two properties required subsequently read as follows.

Definition 3.4.4 (avoidance of innermost-critical steps)
Let $\mathcal{R}$ be a TRS. We say that *innermost-critical reduction steps can be avoided* (denoted by AICR($\mathcal{R}$) or simply AICR), if

$$\forall s, t \,:\, [s \to t \implies [s \xrightarrow[iu]{} t \vee \exists s' : s \xrightarrow[i]{} s' \to^* t]\,]\,.$$

This property guarantees that infinite reductions may be assumed to have a certain shape.

Lemma 3.4.5 (innermost-critical steps in infinite derivations can be avoided under AICR)
Let $\mathcal{R}$ be a TRS satisfying property AICR, and let t be a term. If t is non-terminating, then there exists an infinite derivation $D : t =: t_0 \to t_1 \to t_2 \ldots$ initiating from t which does only contain innermost-uncritical steps, i.e., for all $n \geq 0$ we have $t_n \xrightarrow[iu]{} t_{n+1}$.

Proof: Let t be non-terminating. We give a construction how to obtain an infinite derivation issuing from t with the desired property. Assume that we have already constructed an initial segment

$$t =: t_0 \to t_1 \to \ldots \to t_n$$

of length n of an infinite derivation, i.e., with t_n non-terminating such that every step $t_k \to t_{k+1}$, $0 \leq k < n$ is innermost-uncritical. Now, if there exists a non-terminating t_{n+1} such that $t_n \xrightarrow[iu]{} t_{n+1}$, we can simply extend the initial segment by one step of the desired form. Otherwise, any step $t_n \to t'_{n+1}$ with t'_{n+1} non-terminating must be innermost-critical (note that at least one such t'_{n+1} must exist, due to non-termination of t_n). By AICR we conclude that there exists some term t_{n+1} with $t_n \xrightarrow[i]{} t_{n+1} \to^* t'_{n+1}$. This means that we can properly extend the initial segment to

$$t =: t_0 \to t_1 \to \ldots \to t_n \to t_{n+1}$$

with t_{n+1} still non-terminating where all steps have the desired form (note that $\xrightarrow[i]{} \subseteq \xrightarrow[iu]{}$). By induction we finally conclude that there exists an infinite derivation issuing from t which contains only innermost-uncritical steps. ∎

In the previous section, due to the general assumption JCP ($\Longleftrightarrow$ WCR), a term was complete if and only if it was terminating. Here we do not require local confluence a priori. Therefore we need some new notions for reduction steps contracting terminating and non-terminating redexes, respectively. If $s \to t$ by contracting a terminating redex in s (i.e., $s \to_p t$ for some $p \in Pos(s)$, with $\mathrm{SN}(s/p)$), we write $s \to_{\mathrm{sn}} t$. If $s \to t$ by contracting a non-terminating redex in s (i.e., $s \to_p t$ for some $p \in Pos(s)$, with $\neg\mathrm{SN}(s/p)$), we write $s \to_{\neg\mathrm{sn}} t$. Obviously, every reduction step can be written as $s \to_{\mathrm{sn}} t$ or $s \to_{\neg\mathrm{sn}} t$. In other words, we have $\to\, = \,\to_{\mathrm{sn}} \cup \to_{\neg\mathrm{sn}}$[10]

The next easy result corresponds to Lemmas 3.3.5 and 3.3.6.

Lemma 3.4.6 Let $\mathcal{R}$ be a TRS. The relation $\to_{\mathrm{sn}}$ is terminating (on all terms). Moreover, any infinite $\to$-derivation must contain infinitely many $\to_{\neg\mathrm{sn}}$-steps which are not $\to_{\mathrm{sn}}$-steps.

Proof: Straightforward (as for Lemmas 3.3.5 and 3.3.6). ∎

Definition 3.4.7 (parallel one-step reduction of all terminating innermost redexes)
Let $\mathcal{R}$ be a TRS satisfying property UIR. Then the transformation function Ψ on terms is defined by a parallel one-step reduction of all terminating innermost redexes, i.e.:

$$\Psi(t) := C\langle t'_1, \dots, t'_n\rangle, \text{ if } t = C[\langle_1, \dots, t_n\rangle, \; t_k \underset{i}{\longrightarrow}_\lambda t'_k \; (1 \le k \le n),^{11}$$

where the t_k, $1 \le k \le n$, are all the terminating innermost redexes of t (equivalently: $t \mathrel{+\!\!\!\!\!\!\longrightarrow}_P \Psi(t)$ where P consists of the positions of all terminating innermost redexes of t).

Remark 3.4.8 Note that property UIR in the above definition ensures well-definedness of Ψ. However, a careful inspection of the definition of Ψ reveals that the slightly weakened condition

$$\forall s, t_1, t_2, p \in Pos(s) \; : \; s \underset{i}{\longrightarrow}_p t_1 \;\wedge\; s \underset{i}{\longrightarrow}_p t_2 \;\wedge\; \mathrm{SN}(s/p) \implies t_1 = t_2$$

would also suffice for guaranteeing well-definedness of Ψ.

Reduction steps in a given derivation D can be transformed into reduction steps of the transformed derivation $\Psi(D)$ as follows.

Lemma 3.4.9 Let $\mathcal{R}$ be a TRS satisfying property UIR, and let s, t be terms with $s \underset{iu}{\longrightarrow}_{p,\sigma,l\to r} t$. Then we have:

(1) $\mathrm{SN}(s/p) \implies \Psi(s) \to^* \Psi(t)$.

(2) $\neg\mathrm{SN}(s/p) \implies \Psi(s) \to^+ \Psi(t)$.

[10]However, as for $\to_{\mathrm{c}}$, $\to_{\mathrm{nc}}$, it is possible that s reduces to t both by contracting a terminating or a non-terminating redex in s. For $\mathcal{R} = \{a \to b, f(a) \to f(b), f(b) \to f(a)\}$, we have both $f(a) \to_{\mathrm{sn}} f(b)$ and $f(a) \to_{\neg\mathrm{sn}} f(b)$.Hence, in general we do not have $\to_{\mathrm{sn}} \cap \to_{\neg\mathrm{sn}} = \emptyset$. Nevertheless this could be enforced by slightly modifying the definition of $\to_{\neg\mathrm{sn}}$ as follows: $\to_{\neg\mathrm{sn}} = \to \setminus \to_{\mathrm{sn}}$.

Proof: Due to property UIR, the transformation Ψ is well-defined. Let P be defined by $s \mathrel{\Vert\!\!\!\!\rightarrow}_P \Psi(s)$.

(1) We distinguish two cases.

First assume that s/p is an innermost redex of s. Then we know by definition of Ψ, SN(s/p) and property UIR that $\Psi(s)/p = t/p$, and moreover, that $\Psi(s)$ is obtained from t via $t \mathrel{\Vert\!\!\!\!\rightarrow}_{P\setminus\{p\}} \Psi(s)$. Now, in order to obtain $\Psi(t)$ from $\Psi(s)$ one may have to contract additionally some newly introduced terminating innermost redexes in t. If $\sigma(r)$ is irreducible, new (terminating) innermost redexes can only be introduced above p in t, and only at positions disjoint from those of $P \setminus \{p\}$. If $\sigma(r)$ is reducible, all new innermost redexes in t must be below p (and all of these are indeed terminating, since the contracted redex s/p of s was terminating). Hence, in both cases, we obtain $\Psi(t)$ from $\Psi(s)$ by an additional parallel innermost reduction (at positions disjoint from those of P). Thus, we get $\Psi(s) \rightarrow^* \Psi(t)$ as desired.

In the other case s/p is a terminating non-innermost redex. Since the step $s \rightarrow_{p,\sigma,l\rightarrow r} t$ is non-innermost innermost-uncritical, every innermost redex of s (there exists at least one!) strictly below p, s/pq ($q > \lambda$), must correspond to a variable overlap, i.e., $q \geq u$, for some $u \in \mathcal{V}Pos(l)$. This implies (together with UIR) that $\Psi(s)/p$ is still an instance of l, let's say $\Psi(s)/p = \sigma'(l)$, with $\sigma \mathrel{\Vert\!\!\!\!\rightarrow} \sigma'$. Hence we get $s \xrightarrow[i]{}{}^{+}_{P} \Psi(s) \rightarrow_{p,\sigma',l\rightarrow r} \Psi(s)[p \leftarrow \sigma'(r)] =: t'$ and $s \rightarrow_{p,\sigma,l\rightarrow r} t \mathrel{\Vert\!\!\!\!\rightarrow}_Q t'$, where $Q = P \setminus \{q \in P \mid q \geq p\} \cup P'$ with P' consisting of the positions of those (terminating) innermost redexes of t which are descendents[12] of the terminating innermost redexes of s below p w.r.t. the step $s \rightarrow_{p,l\rightarrow r} t$. Again, in order to obtain $\Psi(t)$ from t', additionally some newly introduced terminating innermost redexes (above or below p) may have to be reduced. Thus we obtain $\Psi(s) \rightarrow t' \rightarrow^* \Psi(t)$, hence $\Psi(s) \rightarrow^+ \Psi(t)$ as desired.

(2) Again we distinguish two cases.

First assume that s/p is an innermost redex of s. Since s/p is non-terminating we have $p \notin P$ and $p|P$. Hence the two steps commute: $s \underset{i}{\mathrel{\Vert\!\!\!\!\rightarrow}}_P \Psi(s) \xrightarrow[i]{}_p t'$, $s \xrightarrow[i]{}_p t \underset{i}{\mathrel{\Vert\!\!\!\!\rightarrow}}_P t'$. Moreover, $\Psi(t)$ is obtained from t' by additionally contracting those terminating innermost redexes which are introduced (in t and also in t') by the step $s \xrightarrow[i]{}_p t$ (and $\Psi(s) \xrightarrow[i]{}_p t'$, respectively) above or below p. Again the positions of the latter introduced (terminating) innermost redexes are disjoint from those of $P \setminus \{p\}$. Hence we have $s \underset{i}{\mathrel{\Vert\!\!\!\!\rightarrow}}_P \Psi(s) \xrightarrow[i]{}_p t' \mathrel{\Vert\!\!\!\!\rightarrow} \Psi(t)$ and thus $\Psi(s) \rightarrow^+ \Psi(t)$.

In the other case s/p is a non-terminating, non-innermost redex. Hence, every innermost redex of s (there exists at least one!) strictly below p, s/pq ($q > \lambda$), must correspond to a variable overlap, i.e., $q \geq u$, for some $u \in \mathcal{V}Pos(l)$. This implies (together with UIR) that $\Psi(s)/p$ is still an instance of l, let's say $\Psi(s)/p = \sigma'(l)$, with $\sigma \mathrel{\Vert\!\!\!\!\rightarrow} \sigma'$. Hence we get $s \xrightarrow[i]{}{}^{+}_{P} \Psi(s) \rightarrow_{q,\sigma',l\rightarrow r} \Psi(s)[p \leftarrow \sigma'(r)] =: t'$ and $s \rightarrow_{q,\sigma,l\rightarrow r} t \mathrel{\Vert\!\!\!\!\rightarrow}_Q t'$, where $Q = P \setminus \{q \in P \mid q \geq p\} \cup P'$ with P'

[12]It should be intuitively clear what is meant here by *descendent*. In fact, a formally precise definition of this and related notions is not completely trivial (cf. e.g. [HL91]).

consisting of the positions of those (terminating) innermost redexes of t which are descendents of the terminating innermost redexes of s below p w.r.t. the step $s \rightarrow_{p,l\rightarrow r} t$. Again, in order to obtain $\Psi(t)$ from t', additionally some newly introduced terminating innermost redexes (above or below p) may have to be reduced. Thus we obtain $\Psi(s) \rightarrow t' \rightarrow^* \Psi(t)$, hence $\Psi(s) \rightarrow^+ \Psi(t)$ as desired.

■

Next we prove that under certain conditions the transformation Ψ preserves the possibility of infinite reductions.

Lemma 3.4.10 Let $\mathcal{R}$ be a TRS satisfying properties UIR and AICR. Then, for any term t we have: $\mathrm{SN}(\Psi(t)) \Longrightarrow \mathrm{SN}(t)$.

Proof: Assume $\neg\mathrm{SN}(t)$, i.e., there exists an infinite derivation D issuing from t. Due to property AICR and Lemma 3.4.5 we may assume that in

$$D : t =: t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \ldots$$

every step is innermost-uncritical. If $t_k \xrightarrow[iu]{}_{p,\sigma,l\rightarrow r} t_{k+1}$ with $\mathrm{SN}(t_k/p)$ (for some $p, \sigma, l \rightarrow r$), then, by Lemma 3.4.9(1), we know $\Psi(t_k) \rightarrow^* \Psi(t_{k+1})$. If $t_k \xrightarrow[iu]{}_{p,\sigma,l\rightarrow r} t_{k+1}$ with $\neg\mathrm{SN}(t_k/p)$ (for some $p, \sigma, l \rightarrow r$), we conclude by Lemma 3.4.9(2), that $\Psi(t_k) \rightarrow^+ \Psi(t_{k+1})$. Clearly, steps of the form are $t_k \rightarrow_p t_{k+1}$ with $\mathrm{SN}(t_k/p)$ are $\rightarrow_{\mathrm{sn}}$-steps, and steps of the form are $t_k \rightarrow_p t_{k+1}$ with $\neg\mathrm{SN}(t_k/p)$ are $\rightarrow_{\neg\mathrm{sn}}$-steps. By Lemma 3.4.6, D contains infinitely many $\rightarrow_{\neg\mathrm{sn}}$-steps of the latter form $t_k \xrightarrow[iu]{}_{p,\sigma,l\rightarrow r} t_{k+1}$ with $\neg\mathrm{SN}(t_k/p)$. We conclude that the transformed derivation

$$\Psi(D) : \Psi(t) = \Psi(t_0) \rightarrow^* \Psi(t_1) \rightarrow^* \Psi(t_2) \rightarrow^* \ldots$$

is infinite, too. Hence we are done. ■

Now we are in a position to establish the main result of this section.

Theorem 3.4.11 (abstract criterion for termination via innermost termination: UIR($\mathcal{R}$) $\wedge$ AICR($\mathcal{R}$) $\Longrightarrow$ [$\forall t$: SIN(t) $\Longrightarrow$ SN(t)])
Let $\mathcal{R}$ be a TRS satisfying properties UIR and AICR. Then, for any term t we have: $\mathrm{SIN}(t) \iff \mathrm{SN}(t)$.

Proof: For a proof by contradiction (of SIN $\Longrightarrow$ SN, the other direction is trivial), suppose that t admits an infinite derivation. Every infinite derivation starting from t must contain a non-innermost step, due to $\mathrm{SIN}(t)$. Now consider an infinite derivation D starting from t with the property that the first non-innermost step is essential: Contracting any innermost redex at that point would result in a term with the property SN. Let

$$D : t =: t_0 \rightarrow t_1 \rightarrow \ldots \rightarrow t_n \rightarrow t_{n+1} \rightarrow \ldots$$

where $t_n \rightarrow t_{n+1}$ is the first non-innermost step. Note that this step must be innermost-uncritical (i.e., $t_n \xrightarrow[iu]{} t_{n+1}$) because, otherwise, it could not be essential in the above sense, due to property AICR. By assumption, contracting an innermost redex in t_n yields a terminating term. This implies in particular that $\Psi(t_n)$, which is obtained from t_n by a (non-empty) parallel reduction step contracting all innermost redexes of

t_n (all of which must be terminating), is terminating. But then, by Lemma 3.4.10 we conclude that t_n is terminating, too. Hence, we have a contradiction (to the infinity of D). ■

This criterion for the sufficiency of the implication (SIN $\Longrightarrow$ SN) is still rather abstract in nature. Subsequently we shall investigate more concrete syntactic conditions satisfying the properties UIR, uniqueness of innermost reduction (at some fixed position), and AICR, the possibility to avoid (non-innermost) innermost-critical steps in infinite reductions. This will result in concrete critical pair conditions for the considered TRS $\mathcal{R}$.

First we introduce some more useful terminology for reasoning about critical pairs.

Definition 3.4.12 (left-to-right / overlay joinability, inside / outside critical peak, (strongly / weakly) left-to-right joinable critical pair / peak, (weakly) overlay joinable critical pair / peak)
Let $\mathcal{R}$ be a TRS.

(1) A critical peak $s\ {}_p\!\leftarrow u \rightarrow_\lambda t$ of $\mathcal{R}$ (with corresponding critical pair $\langle s = t\rangle \in \mathrm{CP}(\mathcal{R})$) is said to be *left-to-right joinable* if $s \rightarrow^* t$. It is *overlay joinable* or *outside joinable* if $p = \lambda$ and $s \downarrow t$. If $p > \lambda$ it is an *inside* critical peak, if $p = \lambda$ it is an *outside* critical peak ([Toy88]) or *critical overlay*.[13] If $p = \lambda$ and $s/p = s$ is an innermost redex, we speak of an *innermost critical overlay*.

(2) *$\mathcal{R}$ has left-to-right joinable critical peaks* (LRJCP) if all its critical peaks are left-to-right joinable. *$\mathcal{R}$ has strongly left-to-right joinable critical peaks* (SLRJCP) if all its critical peaks are left-to-right joinable and, moreover, all its outside critical peaks are even trivial. *$\mathcal{R}$ has weakly left-to-right joinable critical peaks* (WLRJCP) if all its inside critical peaks are left-to-right joinable and all its outside critical peaks are joinable.

(3) *$\mathcal{R}$ has overlay joinable critical peaks* (OJCP) if every critical peak of $\mathcal{R}$ is overlay joinable. *$\mathcal{R}$ has weakly overlay joinable critical peaks* (WOJCP) if all its inside critical peaks are trivial, and all its outside critical peaks are joinable.

Next we give a critical peak condition which is sufficient for the crucial properties UIR and AICR needed above.

Definition 3.4.13 (the critical peak condition CPC)
Let $\mathcal{R}$ be a TRS. We say that $\mathcal{R}$ *satisfies CPC* (CPC($\mathcal{R}$) or CPC for short) if the following holds: For every every critical peak $t_1\ {}_p\!\leftarrow s \rightarrow_\lambda t_2$ of $\mathcal{R}$ we have:[14]

(1) if $p = \lambda$ and both steps are innermost (i.e., s is an innermost redex), then $t_1 = t_2$, and

(2) if $p > \lambda$ and the inside step is innermost (i.e., $s \xrightarrow[i]{}_p t_1$), then $t_1 \rightarrow^* t_2$.

[13] Hence, a TRS is an overlay system (OS) if and only if all its critical peaks are outside ones.

[14] In other words, these requirements mean: every innermost critical overlay must be trivial, and every inside critical peak where the inside step is innermost must be left-to-right joinable.

Lemma 3.4.14 (CPC $\Longrightarrow$ UIR $\wedge$ AICR)
If a TRS satisfies the critical pair condition CPC then it satisfies both UIR and AICR (CPC $\Longrightarrow$ UIR $\wedge$ AICR).

Proof: Assume CPC. Then, proving UIR, uniqueness of innermost reduction, means to show: $\forall s, t_1, t_2, p \;:\; s \xrightarrow[i]{}_p t_1 \wedge s \xrightarrow[i]{}_p t_2 \implies t_1 = t_2$. To this end it is clearly sufficient to show: $\forall s, t_1, t_2 : s \xrightarrow[i]{}_\lambda t_1 \wedge s \xrightarrow[i]{}_\lambda t_2 \implies t_1 = t_2$. But this follows from CPC (1) above, since any such divergence where an innermost redex s is reduced at the root in two distinct ways (i.e., using different rules) must be an instance of a critical peak of the form in (1).

Proving AICR, the possibility of avoiding innermost-critical steps, amounts to showing

$$\forall s, t \;:\; [s \to t \implies [s \xrightarrow[iu]{} t \vee \exists s' : s \xrightarrow[i]{} s' \to^* t]\,]\,.$$

Now, w.l.o.g. we may assume $s \to t$, but not $s \xrightarrow[iu]{} t$, hence $s \xrightarrow[ic]{} t$. This means (cf. Remark 3.4.3) $s \to_{p,\sigma,l\to r} t$ (for some p, σ, $l \to r$) such that there exists an innermost redex in $s/p = \sigma(l)$ at some position $q \in \mathcal{F}Pos(l)$, $q > \lambda$, with $s/pq = \sigma(l)/q = \tau(l')$ for some rule $l' \to r' \in \mathcal{R}$. But this implies that the divergence

$$s/p[q \leftarrow \tau(r')]\; {}_q\!\!\leftarrow = s/p[q \leftarrow \tau(l')] = s/p\sigma(l) \to_\lambda \sigma(r) = t/p$$

is an instance of a critical peak of the form in (2) above. Therefore we get $s/p[q \leftarrow \tau(r')] \to^* \sigma(r)$ and hence

$$s \xrightarrow[i]{}_{pq} s[pq \leftarrow \tau(r')] \to^* s[p \leftarrow \sigma(r)] = t$$

as desired. ∎

Remark 3.4.15 (AICR $\not\Rightarrow$ CPC)
Note that the reverse implication in Lemma 3.4.14 does not hold. For example, the TRS

$$\mathcal{R} = \left\{ \begin{array}{c} f(a,a) \to f(a,a) \\ a \to b \\ f(b,x) \to f(x,x) \end{array} \right.$$

satisfies AICR($\mathcal{R}$) but not CPC($\mathcal{R}$). The point is that for the inside critical peak $f(a,b)\; {}_{>\lambda}\!\!\leftarrow f(a,a) \to_\lambda f(a,a)$ there exists no term t with $f(a,b) \xrightarrow[i]{} t \to^* f(a,a)$. However, the step $f(a,a) \to f(a,a)$ which is innermost-critical has two 'innermost-critical redexes'. And, indeed, choosing the left innermost-critical redex a corresponds to the other critical peak $f(b,a)\; {}_{>\lambda}\!\!\leftarrow f(a,a) \to_\lambda f(a,a)$ which is left-to-right-joinable, hence yielding $f(a,a) \xrightarrow[i]{} f(b,a) \to f(a,a)$.[15]

Combining Theorem 3.4.11 and Lemma 3.4.14 we obtain the following (local and global) results.

Theorem 3.4.16 (abstract critical pair criterion for SIN $\Longleftrightarrow$ SN)
Any innermost terminating (term in a) TRS $\mathcal{R}$ which satisfies the critical pair condition

[15] Due to this kind of indeterminism it seems hard to capture AICR exactly in terms of a reasonable (finitary) critical peak condition.

CPC is terminating.

In particular, we get the following concrete critical pair criteria, which extend the equivalence result (SIN $\Longleftrightarrow$ SN) for non-overlapping systems (cf. Theorem 3.2.11(1)) to more general classes of TRSs.

Theorem 3.4.17 (concrete critical pair criteria for SIN $\Longleftrightarrow$ SN)

(1) An innermost terminating (term in a) TRS $\mathcal{R}$ is terminating if $\mathcal{R}$ is weakly non-overlapping (i.e., WNO($\mathcal{R}$) $\Longrightarrow [\forall t : \mathrm{SIN}(t) \iff \mathrm{SN}(t)]$).

(2) Any innermost terminating (term in a) TRS $\mathcal{R}$ is terminating if $\mathcal{R}$ has strongly left-to-right joinable critical peaks (i.e., SLRJCP($\mathcal{R}$) $\Longrightarrow [\forall t : \mathrm{SIN}(t) \iff \mathrm{SN}(t)]$).

(3) Any innermost terminating (term in a) TRS $\mathcal{R}$ is terminating if $\mathcal{R}$ has left-to-right joinable critical peaks (i.e., LRJCP($\mathcal{R}$) $\Longrightarrow [\forall t : \mathrm{SIN}(t) \iff \mathrm{SN}(t)]$).

Proof: (1) is a special case of (2). And (2) holds by Theorem 3.4.16 and the fact that according to the definition of SLRJCP any violation of the critical pair condition CPC (in Lemma 3.4.14) is impossible. (3) is not a direct consequence of Theorem 3.4.16 and Lemma 3.4.14, but holds for similar reasons. Namely, LRJCP($\mathcal{R}$) still implies AICR($\mathcal{R}$) as is easily seen. However, UIR($\mathcal{R}$) need not hold any more. Nevertheless, a careful inspection of Definitions 3.4.2, 3.4.4, 3.4.7, Lemmas 3.4.9, 3.4.10 and Theorem 3.4.11 reveals that UIR($\mathcal{R}$) is only needed in the slightly weakened version of Remark 3.4.8, namely as:

$$\forall s, t_1, t_2, p \in Pos(s) \; : \; s \underset{i}{\longrightarrow}_p t_1 \wedge s \underset{i}{\longrightarrow}_p t_2 \wedge \mathrm{SN}(s/p) \implies t_1 = t_2 \, .$$

And indeed, an innermost divergence $t_1 \;{}_{\lambda}\!\!\leftarrow s \rightarrow_{\lambda} t_2$ with SN(s) which is non-trivial, i.e., $t_1 \neq t_2$, cannot be an instance of a non-trivial innermost critical overlay. Because then, by left-to-right-joinability of all innermost critical overlays (according to LRJCP($\mathcal{R}$)), we could conclude both $t_1 \rightarrow^* t_2$ and $t_2 \rightarrow^* t_1$, thus contradicting SN(s).[16] ■

In view of the latter two theorems and the situation in the preceding Sections 3.2 and 3.3, where we could infer termination from innermost termination for non-overlapping as well as for locally confluent systems, hence also confluence and completeness, the following question arises naturally in the current setting: Are the systems considered above also locally confluent, hence confluent and complete? Interestingly, this need not be the case in general for TRSs satisfying the critical pair condition CPC.

[16] In fact, any TRS $\mathcal{R}$ which has a non-trivial, left-to-right joinable critical overlay is necessarily non-terminating. For this reason, such systems are not so interesting from a termination point of view.

Example 3.4.18 (CPC $\not\Rightarrow$ WCR)
Consider the TRS

$$\mathcal{R} = \begin{cases} f(x) \to x \\ f(a) \to b \\ f(a) \to c \\ \quad a \to f(a) \end{cases}$$

It is easy to verify that $\mathcal{R}$ satisfies CPC (in particular, we note that CPC ignores the critical overlays between the first three rules since they comprise non-innermost divergences). However, $\mathcal{R}$ is not locally confluent, as shown by the non-joinable critical overlay $b \; {}_{\lambda}\!\!\leftarrow f(a) \to_{\lambda} c$.

We observe that in this example $\mathcal{R}$ is obviously not innermost terminating. And, fortunately, it turns out that SIN combined with CPC does indeed imply not only termination, but also confluence, hence completeness.

Theorem 3.4.19 (CPC $\wedge$ SIN $\Longrightarrow$ SN $\wedge$ CR)
Any innermost terminating TRS satisfying the critical pair condition CPC is complete.

Proof: Let $\mathcal{R}$ be a TRS with SIN($\mathcal{R}$) and CPC($\mathcal{R}$). By Theorem 3.4.16 we know that $\mathcal{R}$ must be terminating. Thus, it remains to show confluence. This can be done directly by induction: Using $\to^+$ as well-founded ordering, one proves

$$\forall s : [\, [\forall t : s \to^+ t \Longrightarrow \mathrm{CR}(t)] \implies \mathrm{CR}(s)\,]$$

by induction via a (not too difficult, but tedious) case analysis concerning the shape of the initial one-step divergence issuing from s in an arbitrary divergence $t_1 \;{}^{+}\!\!\leftarrow s \to^+ t_2$, and exploiting CPC($\mathcal{R}$). However, we shall not present this proof in detail but simply reduce it to a special case of some known result. More precisely, we apply the following *critical pair criterion* for confluence of terminating TRSs due to [KMN88]:

> If $\mathcal{R}$ is a terminating TRS such every critical pair which corresponds to a *prime* critical peak is joinable, then $\mathcal{R}$ is confluent.

Here, a critical peak $t_1 \;{}_{p}\!\!\leftarrow s \to_{\lambda} t_2$ of $\mathcal{R}$ is *prime* if it is not *composite*. And it is *composite* if the inner redex s/p has a proper reducible subterm. Indeed, it is straightforward to verify that CPC ignores only composite critical peaks, and the ones considered are clearly joinable according to CPC. Hence, applying this result we get confluence of $\mathcal{R}$ for free. ∎

Remark 3.4.20 In fact, the following stronger local version of Theorem 3.4.19 also holds: Any innermost terminating term TRS satisfying the critical pair condition CPC is complete. However, this is not a direct consequence of Theorems 3.4.19 and 3.4.16(via Lemma 2.2.17) since local confluence (below the considered term) is not guaranteed anymore. Yet, a direct proof analogous to the one in [KMN88] is possible.

Remark 3.4.21 Interestingly, it seems difficult to prove the implication (CPC $\wedge$ SIN $\Longrightarrow$ CR) directly, without making use of SN (which is allowed according to

Theorem 3.4.16). Any such direct proof probably would have to rely on some well-founded ordering induced by $\underset{i}{\longrightarrow}^+$. At least, all our natural attempts in this direction failed. This observation also pertains to direct proof attempts for the implication (JCP $\wedge$ OS $\wedge$ SIN $\Longrightarrow$ CR) without using SN, cf. Theorem 3.3.12.

Example 3.4.18 above showed that, given CPC, we cannot simply drop SIN when trying to infer CR or SN or both of them. In view of (the proof of) Theorem 3.4.19 one might be tempted to conjecture that the combination of SIN and the critical pair criterion of [KMN88], i.e., joinability of all prime critical peaks (KMN for short), suffices for guaranteeing CR and SN or at least one of them. However, this is also not the case.

Example 3.4.22 (SIN $\wedge$ KMN $\not\Rightarrow$ CR, SN)
Consider the TRS

$$\mathcal{R} = \begin{cases} f(b) \to f(c) \\ f(c) \to f(b) \\ \quad b \to a \\ \quad c \to d \end{cases}$$

This system is clearly innermost terminating and locally confluent, but non-terminating and non-confluent, and both critical peaks to be considered are obviously prime.

Let us mention that Theorem 3.4.17(1)–(2) captures only some special cases of Theorem 3.4.16. According to the latter result, for inferring termination from innermost termination it suffices to show that every innermost critical overlay is trivial, and every inside critical pair where the inside step is innermost is left-to-right joinable. In particular, non-trivial critical overlays may exist provided they comprise a non-innermost divergence.

Example 3.4.23 (CPC permits non-trivial critical overlays provided they are non-innermost)
Consider the TRS

$$\mathcal{R} = \begin{cases} f(a) \to b \\ f(a) \to f(c) \\ \quad a \to d \\ f(d) \to b \\ f(c) \to b \\ \quad d \to c \end{cases}$$

Here, it is easily verified that $\mathcal{R}$ is innermost terminating and satisfies CPC, hence must be complete by Theorem 3.4.16. But note that there exists a non-trivial critical overlay (between the first two rules) which is ignored by CPC since it is not a divergence by innermost reduction.

Before we shall finally investigate which other results of Section 3.2 for non-overlapping TRSs can be extended, too, to more general classes of systems, we should mention one interesting problem related to the above results which we have not been able to solve. As exhibited above, under uniqueness of innermost reduction (cf. Definition

3.4.4) the notion of left-to-right joinable critical pairs has turned out to be crucial for the implication (SIN $\Longrightarrow$ SN) to hold. Hence, in view of the fact that overlay joinability of all critical pairs (i.e., OS $\wedge$ JCP, or equivalently OJCP) also suffices for SIN $\Longrightarrow$ SN (cf. Theorem 3.3.12), it is quite natural to ask whether these two criteria can be combined. In other words, we have the following

Open Problem 1 Does termination follow from innermost termination if all critical overlays are joinable and all inside critical peaks are even left-to-right joinable (or, more concisely: WLRJCP $\wedge$ SIN $\Longrightarrow$ SN?)?

Unfortunately, we have neither been able to prove this criterion nor to find a counterexample. The careful reader may recognize that the problem lies with allowing both non-trivial innermost critical overlays and inside critical peaks. In fact, even for the special case that we require any inside critical peaks to be trivial, i.e., for the conjecture (WOJCP $\wedge$ SIN $\Longrightarrow$ SN), we could not provide a solution. Intuitively, the reason for our failure is due to the problem, that the two different proof techniques we applied somehow seem to be incompatible. More precisely, the latter proof technique in this section crucially relies on uniqueness of innermost reduction, which is destroyed by allowing non-trivial innermost critical overlays. And the other proof technique of Section 3.3 makes essential use of the (pure) overlay property which conflicts with allowing inside critical peaks.

Requiring less than left-to-right joinability for the inside critical peaks seems to be hopeless regarding the implication (SIN $\Longrightarrow$ SN), as witnessed by the very simple TRS

$$\mathcal{R} = \left\{ \begin{array}{c} f(a) \to f(a) \\ a \to b \end{array} \right.$$

of Example 3.3.14.

Up to now we have investigated how to obtain generalized sufficient criteria for the implication (SIN $\Longrightarrow$ SN). Next we shall study which other results of Theorem 3.2.11 for non-overlapping TRSs can be generalized, and how. A first easy result is the following.

Lemma 3.4.24 (UIR $\Longrightarrow$ WCR^1($\xrightarrow[i]{}$) $\Longrightarrow$ CR($\xrightarrow[i]{}$))
Uniqueness of innermost reduction implies uniform confluence (hence also confluence) of innermost reduction, i.e., UIR $\Longrightarrow$ WCR$^1(\xrightarrow[i]{})$ $\Longrightarrow$ CR$(\xrightarrow[i]{})$.

Proof: Straightforward by definition of UIR. ■

Furthermore, UIR also suffices for the equivalence of weak and strong innermost termination.

Lemma 3.4.25 (WIN and SIN coincide under UIR)
Let $\mathcal{R}$ be a TRS satisfying UIR. Then we have:
$\forall t : [\,\mathrm{WIN}(t) \iff \mathrm{SIN}(t)\,]$.

Proof: By applying Lemmas 3.4.24 and 2.1.27, analogous to the the proof of Lemma

3.2.3. ∎

Combining this Lemma with Theorem 3.4.11 we get the following.

Corollary 3.4.26 (WIN, SIN and SN coincide under UIR plus AICR)
Let $\mathcal{R}$ be a TRS satisfying both UIR and AICR. Then we have: $\forall t$: $[\,\mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]$.

Lemma 3.4.27 (critical steps cannot be innermost under UIR plus AICR)
Let $\mathcal{R}$ be a TRS satisfying both UIR and AICR. Then we have: If $s \xrightarrow[i]{} t$ and SN(t) then SN(s).

Proof: This is an immediate consequence of Corollary 3.4.26. ∎

Hence, in a TRS satisfying(UIR $\wedge$ AICR) an innermost reduction step $s \xrightarrow[i]{} t$ cannot be critical in the sense, that it destroys the possibility of infinite reductions, i.e., with $SN(t)$ but $\neg$SN(s). For non-overlapping systems we have seen that such critical steps must be erasing non-innermost, cf. Lemma 3.2.8. Interestingly, if we only require (UIR $\wedge$ AICR) instead of the stronger NO then this property does not hold any more.

Example 3.4.28 (critical steps need not be erasing under UIR plus AICR)
The TRS

$$\mathcal{R} = \left\{ \begin{array}{r} f(b) \to a \\ f(c) \to d \\ b \to c \\ c \to b \end{array} \right.$$

obviously satisfies UIR $\wedge$ AICR (and also CPC), but, for instance, the step $f(b) \to a$ is critical *and* non-erasing (note that the non-terminating proper subterm b of $f(b)$ is not matched by a variable of the applied rule).

This means that the equivalence WN(t) $\iff$ SN(t) does not hold in general for non-erasing TRSs satisfying (UIR $\wedge$ AICR). That is in contrast to the case of non-overlapping systems (cf. Corollary 3.2.9). The equivalence doesn't hold either for the more special class of TRSs with strongly left-to-right joinable critical pairs (SLRJCP), since the above counterexample clearly has the property SLRJCP. However, we get positive results for weakly non-overlapping (WNO) systems, i.e., we may allow trivial critical pairs, and also for a slightly more general class of TRSs.

Definition 3.4.29 (the critical peak condition CPC')
Let $\mathcal{R}$ be a TRS. We define the critical pair condition CPC' by slightly strengthening CPC into: For every every critical peak $t_1 \;{}_{p}\!\!\leftarrow s \to_\lambda t_2$ of $\mathcal{R}$ we have:[17]

(1) if $p = \lambda$ and both steps are innermost (i.e., s is an innermost redex), then $t_1 = t_2$, and

[17] In other words, every inside critical peak where the inside step is innermost must not only be left-to-right joinable, but even trivial.

(2) if $p > \lambda$ and the inside step is innermost (i.e., $s \underset{i}{\longrightarrow}_p t_1$), then $t_1 = t_2$.

We observe that the class of TRSs satisfying CPC' (properly) includes all TRSs satisfying CPC and in particular all weakly non-overlapping (WNO) systems (i.e., WNO $\Longrightarrow$ CPC' $\Longrightarrow$ CPC).

Lemma 3.4.30 (critical steps must be erasing non-innermost under CPC') Let $\mathcal{R}$ be a TRS satisfying CPC'. If $s \to_{p,\sigma,l\to r} t$ such that SN(t) and $\neg$SN(s) then s/p contains a proper subterm $s' = \sigma(x)$, for some $x \in Var(l) \setminus Var(r)$, with $\neg$SN(s').

Proof: Let $s = C[\sigma l]_p \to_{p,\sigma,l\to r} C[\sigma r]_p = t$ and SN(t), $\neg$SN(s). By Lemma 3.4.27 the step $s \to_{p,\sigma,l\to r} t$ must be non-innermost.

We claim that all innermost redexes of s/p are below variable positions of l. Otherwise, a proper subterm of σl at some non-variable position of l would be an innermost redex $\tau l'$, for some other rule $l' \to r' \in \mathcal{R}$. However, this would mean that there exists a corresponding inside critical peak between $l' \to r'$ and $l \to r$. According to CPC' this critical peak would have to be trivial. Hence we would get

$$s = C[\sigma l] = C[C'[\tau l']] \underset{i}{\longrightarrow} C[C'[\tau r']] = C[\sigma r] \,.$$

But this means that s reduces by an innermost step to t which, by Lemma 3.4.27 and the assumptions SN(t), $\neg$SN(s), is impossible. This finishes the proof of the claim.

Hence, all innermost redexes in s below p (there exists at least one!) are inside subterms which are matched by variables of l. Now, SN(t) implies in particular SN(σx) for all $x \in V(r)$. Now suppose that the rule $l \to r$ is non-erasing, i.e., $Var(l) = Var(r)$. This implies SN(σx) for all $x \in V(l)$. By innermost normalizing[18] σ we thus obtain $s = C[\sigma l]_p \underset{i}{\longrightarrow}^+ C[\sigma' l]_p \to C[\sigma' r]_p$ and $t = C[\sigma r]_p \underset{i}{\longrightarrow}^* C[\sigma' r]_p$ with σ irreducible, i.e., $\sigma' x$ a normal form, for all $x \in V(l) = V(r)$. Now we would like to conclude that the step $C[\sigma' l]_p \to_{p,\sigma',l\to r} C[\sigma' r]$ is innermost. However, $\sigma' l$ need not be an innermost redex in $C[\sigma' l]_p$. But we can reason as follows. If a proper subterm of $\sigma' l$ is reducible, then we consider such an innermost one. Again, the existence of such an innermost redex in $\sigma' l$ (and hence in $C[\sigma' l]_p$) corresponds to the existence of an inside critical pair (where the inner step is innermost) which, by the assumption CPC', must be trivial. Hence, we may argue, that in any case the step from $C[\sigma' l]_p$ to $C[\sigma' r]$ can be considered to be innermost. Summarizing, we have $s \underset{i}{\longrightarrow}^* C[\sigma' r]_p$, $t \underset{i}{\longrightarrow}^* C[\sigma' r]_p$ and SN(t). Thus we obtain WIN(s) which, by Corollary 3.4.26, yields SN(s). But this is a contradiction to $\neg$SN(s). Therefore, the rule $l \to r$ must be erasing, and there must exist some $x \in Var(l) \setminus Var(r)$ with $\neg$(SN(σx)). This accomplishes the proof. ■

As direct consequence of this result we obtain the following generalization of Corollary 3.2.9.

[18] Due to CPC' $\Longrightarrow$ CPC, $SN(\sigma x)$ and Theorem 3.4.19 we know that σx is complete. Thus, by an arbitrary reduction strategy we obtain a unique normal form $\sigma' x$.

Corollary 3.4.31 (critical steps are impossible for non-erasing TRSs satisfying CPC')
Let $\mathcal{R}$ be a non-erasing TRS satisfying CPC'. If $s \to t$ and SN(t) then SN(s).

Moreover, this entails also the corresponding generalization of Lemma 3.2.10.

Lemma 3.4.32 (all termination properties coincide for non-erasing TRSs with CPC')
Let $\mathcal{R}$ be a non-erasing TRS satisfying CPC'. Then we have:
$\forall t : [\,\mathrm{WN}(t) \iff \mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]$.

Proof: Analogous to the proof of Lemma 3.2.10, but using Corollary 3.4.31 instead of Corollary 3.2.9. ■

Finally let us summarize these additional results obtained which generalize those of Theorem 3.2.11 (for non-overlapping TRSs).

Theorem 3.4.33 (termination and confluence properties of TRSs with unique innermost reduction)
Consider the following properties of a TRS $\mathcal{R}$:

(1a) $\forall t : [\,\mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]$.

(1b) $\mathrm{WIN}(\mathcal{R}) \iff \mathrm{SIN}(\mathcal{R}) \iff \mathrm{SN}(\mathcal{R})$.

(2a) If $s \to_{p,\sigma,l\to r} t$ such that SN(t) and $\neg$SN(s) then s/p contains a proper subterm $s' = \sigma(x)$, for some $x \in Var(l) \setminus Var(r)$, with $\neg$SN(s').[19]

(2b) If $s \xrightarrow[i]{} t$ and SN(t) then SN(s).

(3a) $\mathrm{NE}(\mathcal{R}) \implies [\forall t : [\,\mathrm{WN}(t) \iff \mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]]$.

(3b) $\mathrm{NE}(\mathcal{R}) \implies [\,\mathrm{WN}(\mathcal{R}) \iff \mathrm{WIN}(\mathcal{R}) \iff \mathrm{SIN}(\mathcal{R}) \iff \mathrm{SN}(\mathcal{R})\,]$.

(4a) $\forall t : [\,\mathrm{WIN}(t) \implies \mathrm{CR}(t)\,]$.

(4b) $\mathrm{WIN}(\mathcal{R}) \implies \mathrm{CR}(\mathcal{R})$.

(5a) $\mathrm{NE}(\mathcal{R}) \implies [\forall t : [\,\mathrm{WN}(t) \implies \mathrm{CR}(t)\,]]$.

(5b) $\mathrm{NE}(\mathcal{R}) \wedge \mathrm{WN}(\mathcal{R}) \implies \mathrm{CR}(\mathcal{R})$.

Now, if $\mathcal{R}$ enjoys the critical pair condition CPC', then it satisfies all the above termination and confluence properties (1a,b)-(5a,b). In particular, this is the case if $\mathcal{R}$ is weakly non-overlapping (WNO).
If CPC holds for $\mathcal{R}$, then $\mathcal{R}$ still satisfies the properties (1a,b), (2b) and (4a,b), whereas (2a), (3a,b) and (5a,b) do not hold any more.

[19] This means that the non-terminating proper subterm $s' = \sigma(x)$ of s/p is *erased* in the step $s \to_{p,\sigma,l\to r} t$, and implies that $l \to r$ is an *erasing* rule.

Proof: Let $\mathcal{R}$ satisfy CPC. Then (1a,b) is established by Lemma 3.4.26, (2b) is Lemma 3.4.27, and (4a,b) follows from (1) combined with Theorem 3.4.19. (2a), (3a,b) and (5a,b) are all refuted by Example 3.4.18 as is easily verified.[20]

If $\mathcal{R}$ even satisfies CPC', then the missing (2a), (3a,b) correspond to Lemmas 3.4.30, 3.4.32, and (5a,b) follows from (3a,b) and Theorem 3.4.19. ∎

3.5 Confluence of Innermost Reduction

Next we shall be concerned with criteria for innermost confluence and its relation to general confluence.

We observe that innermost confluence, i.e., CR($\underset{i}{\longrightarrow}$), follows from uniform confluence (WCR^1), subcommutativity (WCR$^{\leq 1}$) or strong confluence (SCR) of $\underset{i}{\longrightarrow}$, according to the general confluence criteria of Theorem 2.1.9. The following critical pair conditions will turn out to be sufficient for the properties WCR^1($\underset{i}{\longrightarrow}$) and WCR$^{\leq 1}$($\underset{i}{\longrightarrow}$), respectively.

Definition 3.5.1 (critical pair conditions for strong confluence properties of innermost reduction)
Let $\mathcal{R}$ be a TRS. Now consider the following properties of critical peaks

$$t_1 = (\sigma l_1)[p \leftarrow \sigma r_2]\ {}_{p,\sigma,l_2\to r_2}\!\!\leftarrow (\sigma l_1)[p \leftarrow \sigma l_2] = s = \sigma l_1 \to_{\lambda,\sigma,l_1\to r_1} \sigma r_1 = t_2$$

(1) If $p = \lambda$ and both steps are innermost, then we have

  (a) $t_1 = t_2$, or

  (b) $t_1 = \sigma r_2 \underset{i}{\longrightarrow}_{q,\tau,l\to r} t_3\ {}_{q',\tau',l'\to r'}\underset{i}{\longleftarrow} \sigma r_1 = t_2$ for some q, τ, $l \to r$, q', τ', $l' \to r'$, t_3, such that

    (b1) no proper non-variable subterm of $\sigma r_2/q$ unifies with a left-hand-side of $\mathcal{R}$, and

    (b2) no proper non-variable subterm of $\sigma r_1/q'$ unifies with a left-hand-side of $\mathcal{R}$.

(2) If $p = \lambda$ and both steps are innermost, then we have

  (a) $t_1 = t_2$, or

  (b) $t_1 = \sigma r_2 \underset{i}{\longrightarrow}_{q,\tau,l\to r} t_3\ {}_{q',\tau',l'\to r'}\underset{i}{\longleftarrow} \sigma r_1 = t_2$ for some q, τ, $l \to r$, q', τ', $l' \to r'$, t_3, such that

    (b1) no proper non-variable subterm of $\sigma r_2/q$ unifies with a left-hand-side of $\mathcal{R}$, and

[20] There, $\mathcal{R}$ is non-erasing, weakly normalizing, satisfies CPC, but is not strongly normalizing and also not (locally) confluent. Furthermore, we have $f(a) \to b$ with SN(b) and $\neg$SN($f(a)$) using the second rule.

(b2) no proper non-variable subterm of $\sigma r_1/q'$ unifies with a left-hand-side of $\mathcal{R}$, or

(c) $t_1 = \sigma r_2 \xrightarrow[i]{}_{q,\tau,l\to r} t_2$ for some q, τ, $l \to r$, such that

(c1) no proper non-variable subterm of $\sigma r_2/q$ unifies with a left-hand-side of $\mathcal{R}$, or

(d) $t_1 \ {}_{q',\tau',l'\to r'}\xleftarrow[i]{} \sigma r_1 = t_2$ for some q', τ', $l' \to r'$, such that

(d1) no proper non-variable subterm of $\sigma r_1/q'$ unifies with a left-hand-side of $\mathcal{R}$.

We say that $\mathcal{R}$ satisfies CPCI_k, $1 \leq k \leq 2$ (denoted by $\mathrm{CPCI}_k(\mathcal{R})$), if ($k$) holds for every every critical peak of $\mathcal{R}$.

Theorem 3.5.2 (strong confluence criteria for innermost reduction)
Let $\mathcal{R}$ be a TRS. Then we have:

(1) $\mathrm{CPCI}_1(\mathcal{R}) \Longrightarrow \mathrm{WCR}^1(\xrightarrow[i]{})$.

(2) $\mathrm{CPCI}_2(\mathcal{R}) \Longrightarrow \mathrm{WCR}^{\leq 1}(\xrightarrow[i]{})$.

Proof: We have to analyse local divergences of the form

$$v_1 \ {}_{p_2,l_2\to r_2}\xleftarrow[i]{} u \xrightarrow[i]{}_{p_1,l_1\to r_1} v_2 \, .$$

Since the steps are innermost, there are only two cases, namely, innermost reduction at the same position or at parallel positions. Now, the latter case is obviously trivial (for both implications (1) and (2)). Obviously, in the former case, i.e., for $p_1 = p_2$, it suffices to focus on the contracted subterm $s' := u/p_1$ of u, i.e., to consider the divergence

$$t_1' = v_1/p_1 \ {}_{\lambda,l_2\to r_2}\xleftarrow[i]{} s' \xrightarrow[i]{}_{\lambda,l_1\to r_1} v_2/p_1 = t_2' \, .$$

Now, if $t_1' = t_2'$, we are done (for both (1) and (2)). Hence, assume $t_1' \neq t_2'$. This implies that the above innermost divergence is an instance of a critical overlay (between $l_1 \to r_1$ and $l_2 \to r_2$)

$$t_1 = \sigma r_2 \ {}_{\lambda,l_2\to r_2}\xleftarrow[i]{} \sigma l_2 = s = \sigma l_1 \xrightarrow[i]{}_{\lambda,l_1\to r_2} \sigma r_1 = t_2 \, ,$$

which must be innermost, too. Hence we have $s' = \psi s$, $t_1' = \psi t_1$, $t_2' = \psi t_2$ for some substitution ψ. Moreover, σ, ψ and $\psi\sigma$ are irreducible.[21] In case (1) of the theorem we have by $\mathrm{CPCI}_1(\mathcal{R})$:

$$t_1 = \sigma r_2 \xrightarrow[i]{}_{q,\tau,l\to r} t_3 \ {}_{q',\tau',l'\to r'}\xleftarrow[i]{} \sigma r_1 = t_2$$

for some q, τ, $l \to r$, q', τ', $l' \to r'$, t_3. By instantiation we get

$$t_1' = \psi t_1 = \psi\sigma r_2 \to_{q,\psi\tau,l\to r} \psi t_3 \ {}_{q',\psi\tau',l'\to r'}\!\leftarrow \psi\sigma r_1 = \psi t_2 = t_2' \, .$$

If we can show that both steps here are innermost, then we are done (for case (1)).

Now, the step $\psi\sigma r_2 \to_{q,\psi\tau,l\to r} t_3$ can only be non-innermost if some proper subterm of $\psi\sigma r_2/q$ is reducible, let's say $(\psi\sigma r_2)/q\hat{q} = \pi l_3$, $\hat{q} > \lambda$. Then, from $\sigma r_2 \xrightarrow[i]{}_{q,\tau,l\to r} t_3$ and

[21] The latter holds because otherwise the step $s' = \psi s = \psi\sigma l_2 \xrightarrow[i]{}_{\lambda,\psi\sigma,l_2\to r_2} \psi\sigma r_2 = t_1'$ could not have been innermost.

irreducibility of σ we know $q \in \mathcal{F}Pos(r_2)$. Irreducibility of $\psi\sigma$ and $(\psi\sigma r_2) \rightarrow_{q\hat{q},\pi,l_3\rightarrow r_3} (\psi\sigma r_2)[q\hat{q} \leftarrow \pi r_3]$ yields $q\hat{q} \in \mathcal{F}Pos(r_2)$. This implies

$$(\psi\sigma r_2)/q\hat{q} = \psi(\sigma r_2/q\hat{q}) = (\psi\sigma)(r_2/q\hat{q}) = \pi l_3\,,$$

hence the proper non-variable subterm $\sigma r_2/q\hat{q}$ of $\sigma r_2/q$ (as well as the proper non-variable subterm $r_2/q\hat{q}$ of r_2/q) is unifiable with a left-hand side of $\mathcal{R}$.[22] But this contradicts the assumption (b1).

Analogously (using (b2)), one shows that the step $\psi t_3 \;{}_{q',\psi\tau',l'\rightarrow r'}\!\!\leftarrow \psi\sigma r_1 = \psi t_2$ is innermost, too. This finishes the proof of (1).

The remaining cases for (2) are completely analogous, using the corresponding assumptions (b1) and (b2), (c1), (d1) for the cases (b), (c) and (d), respectively. ∎

Remark 3.5.3 We note that for strong confluence of innermost reduction (SCR($\xrightarrow[i]{}$)) one can also define an analogous critical pair condition (which is finitary for finite TRSs). However, technically this is more complex. The crucial property to be guaranteed is the following. Any innermost critical overlay

$$t_1 = \sigma r_2 \;{}_{\lambda,l_2\rightarrow r_2}\!\xleftarrow[i]{}\; \sigma l_2 = s = \sigma l_1 \xrightarrow[i]{}_{\lambda,l_1\rightarrow r_1} \sigma r_1 = t_2$$

(with σ necessarily irreducible) must be strongly innermost joinable, i.e.,

$$t_1 \xrightarrow[i]{}{}^{*} t_3 \;{}^{=}\!\xleftarrow[i]{}\; t_2 \quad \text{and} \quad t_1 \xrightarrow[i]{}{}^{=} t_4 \;{}^{*}\!\xleftarrow[i]{}\; t_2$$

(for some terms t_3, t_4) and moreover, after instantiation with some irreducible substitution ψ (such that $\psi\sigma$ is irreducible, too), this strong innermost joinability should be preserved in the sense that the instantiated reduction steps are still innermost.

A careful inspection of Theorem 3.5.2 (and its proof) shows that the following result is an easy consequence.

Corollary 3.5.4 (strong confluence criteria for innermost reduction, weak version)
Let $\mathcal{R}$ be a TRS. Furthermore assume that no non-variable irreducible proper subterm of a right-hand side r (of $\mathcal{R}$) unifies with a left-hand side l (of $\mathcal{R}$), where w.l.o.g. l and r do not have common variables. Then we have:

(1) If for every non-trivial innermost critical overlay $t_1 \;{}_{\lambda}\!\xleftarrow[i]{}\; s \xrightarrow[i]{}_{\lambda} t_2$ we have $t_1 \xrightarrow[i]{} \circ \xleftarrow[i]{} t_2$, then innermost reduction is uniformly confluent (WCR^1($\xrightarrow[i]{}$)).

(2) If for every innermost critical overlay $t_1 \;{}_{\lambda}\!\xleftarrow[i]{}\; s \xrightarrow[i]{}_{\lambda} t_2$ we have $t_1 \xrightarrow[i]{}{}^{=} \circ \;{}^{=}\!\xleftarrow[i]{}\; t_2$, then innermost reduction is uniformly confluent (WCR$^{\leq 1}$($\xrightarrow[i]{}$)).

An obvious consequence of Theorem 3.5.2 is the following.

Corollary 3.5.5 (innermost confluence via strong innermost confluence)
Any TRS satisfying one of the critical pair conditions CPCI$_k$ ($1 \leq k \leq 2$) is innermost confluent.

[22] Note that w.l.o.g. we may assume here that $\sigma r_2/q\hat{q}$ and l_3 (as well as $r_2/q\hat{q}$ and l_3) do not share any variable.

Since the critical pair criteria CPCI_k $(1 \le k \le 2)$ can be effectively tested (for finite TRSs), this yields decidable sufficient criteria for innermost confluence of (finite) TRSs. In particular, we obtain the following result.

Theorem 3.5.6 (specialized criteria for innermost confluence)
If a TRS $\mathcal{R}$ has no non-trivial innermost critical overlays, then it is uniformly innermost confluent, hence innermost confluent.[23]

Proof: It is straightforward to verify that condition $\mathrm{CPCI}_1(\mathcal{R})$ is satisfied for such TRSs, hence innermost reduction is uniformly confluent ($\mathrm{WCR}^1(\xrightarrow[i]{})$) by Theorem 3.5.2(1). ■

Theorem 3.5.6 implies in particular that any weakly non-overlapping TRS is innermost confluent. However, overlay systems need not be innermost confluent, even if (they are confluent and) all innermost critical overlays are strongly innermost joinable.

Example 3.5.7
The TRS

$$\mathcal{R} = \begin{cases} f(x) \to g(k(x)) \\ f(x) \to a \\ g(x) \to a \\ k(a) \to k(k(a)) \end{cases}$$

evidently is an overlay system, and the two innermost critical overlays,

$$g(k(x)) \;{}_{\lambda}\xleftarrow[i]{} f(x) \xrightarrow[i]{}_{\lambda} a$$

and its mirrored version, are strongly innermost joinable via

$$g(k(x)) \xrightarrow[i]{} a\,.$$

However, the innermost divergence

$$g(k(a)) \;{}_{\lambda}\xleftarrow[i]{} f(a) \xrightarrow[i]{}_{\lambda} a$$

which is an instance of the above critical overlay (via the irreducible substitution $\psi = \{x \mapsto a\}$), is not innermost joinable any more. Indeed, the instantiated step $g(k(a)) \to a$ is not innermost any more, and the only innermost reductions from $g(k(a))$ are

$$g(k(a)) \xrightarrow[i]{} g(k(k(a))) \xrightarrow[i]{} g(k(k(k(a)))) \xrightarrow[i]{} \dots\,.$$

The point is (cf. CPCI_1) that the non-variable irreducible proper subterm $k(x)$ of the right-hand side $g(k(x))$ (instantiated with the irreducible identity substitution ϵ) is unifiable with a left-hand side, namely $k(a)$.

All the above criteria for innermost confluence (which are based on guaranteeing strong confluence of innermost reduction) do not need a termination assumption. Using Newman's Lemma (Theorem 2.1.20) one also easily gets the following.

[23] This generalizes the corresponding result of [Pla94] where all critical overlays are forbidden.

Theorem 3.5.8 (characterizing innermost confluence under innermost termination)
Let $\mathcal{R}$ be innermost terminating (SIN($\mathcal{R}$)). Then $\mathcal{R}$ is innermost confluent if and only if, for any innermost critical overlay

$$t_1 = \sigma r_2 \ {}_{\lambda,\sigma,l_2 \to r_2}\!\underset{i}{\longleftarrow} \sigma l_2 = s = \sigma l_1 \rightarrow_{\lambda,\sigma,l_1 \underset{i}{\to} r_1} \sigma r_1 = t_2$$

(with σ necessarily irreducible) and for any substitution ψ with $Dom(\psi) \subseteq Var(s)$, such that

$$\psi t_1 = \psi\sigma r_2 \ {}_{\lambda,\psi\sigma,l_2 \to r_2}\!\underset{i}{\longleftarrow} \psi\sigma l_2 = \psi s = \psi\sigma l_1 \rightarrow_{\lambda,\psi\sigma,l_1 \underset{i}{\to} r_1} \psi\sigma r_1 = \psi t_2$$

(i.e., the instantiated divergence is still innermost which implies in particular that ψ must be irreducible), we have that $\psi t_1 = \psi\sigma r_2$ and $\psi t_2 = \psi\sigma r_1$ are joinable by innermost reduction.[24]

Proof: It is straightforward to verify that the condition stated above is equivalent to local confluence of innermost reduction (WCR($\underset{i}{\longrightarrow}$)), hence, due to SIN($\mathcal{R}$) ($\iff$ SN($\underset{i}{\longrightarrow}$)), also to innermost confluence by Newman's Lemma. ■

However, this latter result has a serious drawback. Namely, (assuming innermost termination) its precondition cannot be effectively tested, since it is inherently infinitary.

Finally, let us briefly investigate the relation between confluence and innermost confluence.

In general, if for some abstract reduction relations $\rightarrow_1$, $\rightarrow_2$, we have $\rightarrow_1 \subseteq \rightarrow_2$ (or, $\rightarrow_1 \subseteq \rightarrow_2^*$), then the respective confluence properties need not be related, i.e., neither CR($\rightarrow_1$) $\Longrightarrow$ CR($\rightarrow_2$) nor CR($\rightarrow_2$) $\Longrightarrow$ CR($\rightarrow_1$) need to hold. This is also the case here, for $\underset{i}{\longrightarrow}$ and $\rightarrow$ (in TRSs), as observed e.g. in [Pla94]. For terminating TRSs, confluence clearly implies innermost (and outermost) confluence. Non-terminating systems can be confluent, but not innermost (outermost) confluent, as observed by Middeldorp (cf. [Pla94]). And, of course, a TRS may be innermost (outermost) confluent without being confluent. For instance, any non-confluent, non-overlapping TRS[25] is a counterexample to the implication CR($\underset{i}{\longrightarrow}$) $\Longrightarrow$ CR($\rightarrow$), because non-overlapping systems are innermost confluent (cf. Corollary 3.2.2).

First, we consider sufficient conditions for the implication CR($\rightarrow$) $\Longrightarrow$ CR($\underset{i}{\longrightarrow}$), or, more generally, for the implication CR($\rightarrow$) $\Longrightarrow$ CR($\rightarrow_s$), where – intuitively – $\mathcal{R}_s$ may be viewed as rewriting under some arbitrary strategy. To this end, the next general result about abstract reduction systems (ARSs) is useful.

Theorem 3.5.9 (criterion for confluence of restricted rewriting)
Let $\rightarrow$, $\rightarrow_s$ be abstract reduction relations (on the same set A) satisfying (i) WN($\rightarrow_s$), (ii) $\rightarrow_s \subseteq \rightarrow^*$ and (iii) NF($\rightarrow_s$) = NF($\rightarrow$).[26] Then we have: CR($\rightarrow$) $\Longrightarrow$ CR($\rightarrow_s$).

[24] This slightly generalizes the following result of [Kri94a] (which in turn slightly generalizes the corresponding result in [Pla94] where termination instead of innermost termination is required): *An innermost terminating TRS is innermost confluent if, for every critical pair $\langle s,t\rangle$ corresponding to a critical overlay and for every irreducible substitution ψ, ψs and ψt are joinable by innermost rewriting.*

[25] Hence, by Theorem 2.2.27 it cannot be left-linear.

[26] Actually, if $\rightarrow_s \subseteq \rightarrow^*$ holds, then NF($\rightarrow_s$) = NF($\rightarrow$) is equivalent to NF($\rightarrow_s$) $\subseteq$ NF($\rightarrow$).

Proof: Let $\to$, $\to_s$ be as above such that $\to$ is confluent. Consider u, v_1, v_2 such that $v_1 \; {}_s^*\!\!\leftarrow u \to_s^* v_2$. We have to show that there exists w such that $v_1 \to_s^* w \; {}_s^*\!\!\leftarrow v_2$. By CR($\to$) and (ii) we know that there exists some v_3 with $v_1 \to^* v_3 \; {}^*\!\!\leftarrow v_2$. Furthermore, (i) yields $v_1 \to_s^* v_4$, $v_2 \to_s^* v_5$, with v_4, v_5 normal forms w.r.t. $\to_s$ and, by (iii), w.r.t. $\to$. Now, (ii), CR($\to$) and $v_4 \in$ NF($\to$) imply $v_3 \to^* v_4$. Then, CR($\to$) combined with $v_2 \to^* v_3 \to^* v_4$, $v_2 \to^* v_5$ (by (ii)) and $v_5 \in$ NF($\to$) yield $v_4 = v_5$, hence choosing $w = v_4 = v_5$ we are done. ■

For TRSs, if $\to_s$ is rewriting under some arbitrary well-defined[27] (e.g., redex selection/exclusion) strategy, then the set of $\to$- and of $\to_s$-normal forms coincide. Hence Theorem 3.5.9 is applicable. In particular, for innermost rewriting we obtain the following.

Corollary 3.5.10 (confluence implies innermost confluence under weak innermost termination, [Kri94a])
A weakly innermost terminating TRS is innermost confluent if it is confluent, i.e., WN($\underset{i}{\longrightarrow}$) $\wedge$ CR($\to$) $\Longrightarrow$ CR($\underset{i}{\longrightarrow}$).

For the converse implication in Theorem 3.5.9, we have the following abstract result.

Theorem 3.5.11 (equivalence condition for confluence of arbitrary and of restricted rewriting)
Let $\to$, $\to_s$ be ARSs (on the same set A) satisfying (i) WN($\to_s$), (ii) $\to_s \subseteq \to^*$ and (iii) NF($\to_s$) = NF($\to$).Then,

$$(*) \qquad \to \; \subseteq \; \to_s^* \circ {}_s^*\!\!\leftarrow \; .$$

implies that confluence of $\to$ and of $\to_s$ are equivalent (i.e., CR($\to$) $\iff$ CR($\to_s$)).

Proof: Under the stated assumptions, the implication CR($\to$) $\Longrightarrow$ CR($\to_s$) holds by Theorem 3.5.9 (even without ($*$)). Conversely, assume ($*$) and CR($\to_s$). Then we get $\to^* \subseteq \leftrightarrow_s^*$ from ($*$), hence also ${}^*\!\!\leftarrow \circ \to^* \subseteq \leftrightarrow_s^*$. Confluence of $\to_s$ and (ii), $\to_s \subseteq \to^*$, yield $\leftrightarrow_s^* \subseteq \to_s^* \circ {}_s^*\!\!\leftarrow \; \subseteq \to^* \circ {}^*\!\!\leftarrow$. Hence we obtain ${}^*\!\!\leftarrow \circ \to^* \subseteq \to^* \circ {}^*\!\!\leftarrow$ as desired.[28] ■

For applying Theorem 3.5.11 under the stated preconditions via ($*$), one only has to consider an arbitrary $\to$-step which is not an $\to_s$-step (due to $\to_s \subseteq \to^*$).

Now let us consider the special case that — as in [Kri94a] — $\to$ and $\to_s$ are general and innermost reduction, respectively, in TRSs.

[27]Here, well-definedness is to include in particular that if a term t is $\to$-reducible then it is also $\to_s$-reducible.

[28]It should be mentioned that this result and the proof we give is essentially an abstracted version of the corresponding statements and reasoning in [Kri94a] for a special case. Namely, $\to$ and $\to_s$ in [Kri94a] are general and innermost reduction in TRSs and the proof there uses compatible refinements (cf. Lemma 2.1.11 and Theorem 2.1.12).

Lemma 3.5.12 (a sufficient condition for expressing non-innermost steps by innermost conversions)
Let $\mathcal{R}$ be a weakly innermost terminating TRS (WIN($\mathcal{R}$)) such that every inside critical peak of $\mathcal{R}$, where the inside step is innermost, is trivial.[29] Then $\mathcal{R}$ satisfies the instantiated version of $(*)$, namely:

$$\forall s,t:\ s \to t \Longrightarrow [\,\exists u:\ s \xrightarrow[i]{}{}^{*} u \wedge\ t \xrightarrow[i]{}{}^{*} u\,]\ .$$

Proof: Let $s \to t$. If $s = t$ or $s \xrightarrow[i]{} t$ we are obviously done. Otherwise, we have $s = C[\sigma l]_p \to_{p,\sigma,l\to r} C[\sigma r] = t$ (for some $C[]_p$, σ and $l \to r \in \mathcal{R}$) such that some proper subterm of $s/p = \sigma l$ is reducible. By WIN($\mathcal{R}$) we obtain $\sigma \xrightarrow[i]{}{}^{*} \sigma'$ by innermost normalizing σx into $\sigma' x$, for all $x \in Dom(\sigma) = Var(l)$. This yields

$$s = C[\sigma l]_p \xrightarrow[i]{}{}^{*} C[\sigma' l]_p \to_{p,\sigma',l\to r} C[\sigma' r]_p =: t'$$

and

$$t = C[\sigma r]_p \xrightarrow[i]{}{}^{*} C[\sigma' r]_p = t'\ .$$

Now it suffices to show

$$s'\ :=\ C[\sigma' l]_p \xrightarrow[i]{} C[\sigma' r]_p = t'\ .$$

If $\sigma' l$ is an innermost redex of s, we are done. Otherwise, we know by irreducibility of σ' that a proper subterm of $\sigma' l$ at some position $q \in \mathcal{F}Pos(l)$ is an innermost redex, let's say $\sigma' l/q = \tau l'$, for some rule $l' \to r' \in \mathcal{R}$ and some irreducible substitution τ. Hence, the divergence

$$(\sigma' l)[q \leftarrow \tau r']\ {}_{q,\tau,l'\to r'}\!\xleftarrow[i]{}\ \sigma' l \to_{\lambda,\sigma',l\to r} \sigma' r$$

is an instance of an inside critical peak of $\mathcal{R}$ (determined by overlapping $l' \to r'$ into $l \to r$ at position q) with innermost inside step. By assumption, the resulting critical pair is trivial which implies

$$(\sigma' l)[q \leftarrow \tau r'] = \sigma' r\ ,$$

hence also

$$s' = C[\sigma' l]_p \xrightarrow[i]{} C[\sigma' r]_p = t'$$

as desired. ∎

Combining this result with Theorem 3.5.11 we get the following.

Theorem 3.5.13 (equivalence condition for confluence and innermost confluence)
Let $\mathcal{R}$ be a weakly innermost terminating TRS (WIN($\mathcal{R}$)) such that every inside critical peak of $\mathcal{R}$, where the inside step is innermost, is trivial. Then, confluence and innermost confluence (of $\mathcal{R}$) are equivalent.

As a special case, we obtain that confluence and innermost confluence are equivalent

[29] The critical pair condition we require here is only one sufficient condition for inferring (*). It might be possible to still generalize or modify it appropriately. However, this seems to be non-trivial and needs a more refined investigation.

for weakly innermost terminating, weak overlay systems and, slightly more special, for weakly innermost terminating overlay systems. Actually, the latter result has already been shown in [Kri94a].[30]

3.6 Conditional Rewrite Systems

In the following we shall show that our results (in Sections 3.2 and 3.3) on restricted termination and confluence properties of unconditional TRSs can be generalized to the conditional case.[31] This generalization has to take into account the additional complications arising with CTRSs. In particular, we need a kind of local completeness property implying that variable overlaps are not critical under certain conditions. More precisely, we have the following result which is a generalized local version of Lemma 2 in (Dershowitz et al. [DOS88b])[32] which in turn is the main technical result for deriving Theorem 2.3.21, i.e., for inferring confluence of terminating CTRSs with overlay joinable critical peaks (cf. Theorem 4 in (Dershowitz et al. [DOS88b]), cf. also Theorem 6.2 in (Wirth & Gramlich [WG94]) which handles the more general case of positive / negative conditional rewrite systems). Note that extra variables (in conditions) are allowed here.

Theorem 3.6.1 (a localized structural confluence property for CTRSs with overlay joinable critical peaks)
Let $\mathcal{R}$ be a CTRS with OS($\mathcal{R}$) and JCP($\mathcal{R}$). Let s, t, u, v be terms with SN(s) and let Π be some set of mutually disjoint positions. Then we have the following implication:[33]

$$u = C[s]_\Pi \rightarrow^* v \wedge s \rightarrow^* t \quad \Longrightarrow \quad C[t]_\Pi \downarrow v\,.$$

Proof: A detailed proof (which is quite involved) is given in Appendix A. ■

Note that Theorem 3.6.1 is interesting by itself because it describes a non-trivial structural confluence property for (conditional) overlay systems with joinable critical pairs without a full termination assumption (it is applicable even in situations where the whole system need not terminate and it may have other potential applications than the ones mentioned below).

Lemma 3.6.2
Let $\mathcal{R}$ be a CTRS with OS($\mathcal{R}$) and JCP($\mathcal{R}$) and let u, v, w be terms with $v\ {}^*\!\leftarrow$

[30] In [Kri94a] also outermost confluence (i.e., confluence of outermost reduction) and the relation between confluence and outermost confluence are investigated.

[31] Again, we shall tacitly assume that all CTRSs considered are join CTRSs (which is the most important case in practice), except for cases where another kind of CTRSs is explicitly mentioned.

[32] In Lemma 2 of (Dershowitz et al. [DOS88b]) the proof (i.e. the induction ordering) makes use of the general termination assumption $SN(\mathcal{R})$ for the considered CTRS $\mathcal{R}$. Our proof of Theorem 3.6.1 has a similar structure but is based on a slightly different notion of *depth* and – more importantly – the induction ordering only needs the local termination assumption $SN(s)$.

[33] Recall that the notation $u = C[s]_\Pi$ means that Π is some set of parallel positions in u such that all subterms of u at these positions are equal to s.

$u \to^* w$. Then $v \downarrow w$ holds provided that v is obtained from u by performing only reductions in strongly normalizing (parallel) subterms of u, formally: $u = C[s_1, \ldots, s_n]$, $v = C[t_1, \ldots, t_n]$ for some context $C[, \ldots,]$, and $SN(s_i)$, $s_i \to^* t_i$ for $i = 1, \ldots, n$.

Proof: Straightforward by repeated application of Theorem 3.6.1. ■

One may ask now whether the assumption in the above results, that all critical peaks are joinable overlays, can be weakened to requiring only joinability. This is not the case, clearly, for CTRSs, since there exist terminating CTRSs with joinable critical pairs that are non-confluent, hence necessarily not even locally confluent (cf. e.g. Example 2.3.18). For unconditional TRSs such a counterexample involving a terminating TRS with joinable critical pairs cannot exist since for TRSs – in contrast to CTRSs – the Critical Pair Lemma 2.2.17 holds, i.e., the property WCR($\mathcal{R}$) is equivalent to JCP($\mathcal{R}$). By Newman's Lemma (Theorem 2.1.20) any such TRS must then be confluent. Hence, in the unconditional case the above question may be rephrased as follows: Does there exist a locally confluent (non-terminating) TRS which is not an overlay system violating the confluence property in Lemma 3.6.2 (and Theorem 3.6.1) above? This is indeed the case as shown by the following simple example.

Example 3.6.3 (Example 3.4.22 continued)
Consider the TRS

$$\mathcal{R} = \begin{cases} f(b) \to f(c) \\ f(c) \to f(b) \\ \quad b \to a \\ \quad c \to d \end{cases}$$

This system which is not an overlay system is clearly locally confluent, but non-confluent (and hence necessarily non-terminating). We have for instance $f(a) \leftarrow f(b) \to^* f(d)$ with both $f(a)$ and $f(d)$ irreducible, and moreover, in the step $f(b) \to f(a)$ the proper subterm b of $f(b)$ which is contracted is clearly terminating, i.e., satisfies $SN(b)$.

Thus, even for unconditional TRSs, Theorem 3.6.1 and Lemma 3.6.2 capture indeed a non-trivial structural confluence property of overlay systems!

But let us return now to the conditional case. Similar to Lemma 3.6.2 above we obtain from Theorem 3.6.1 in particular the following sufficient criterion for a variable overlap in CTRSs to be non-critical.

Lemma 3.6.4 (a sufficient condition for variable overlaps to be non-critical)
Let $\mathcal{R}$ be a CTRS with OS($\mathcal{R}$) and JCP($\mathcal{R}$), and let s, t be terms with $s = C[\sigma l]_p \to_{p,\sigma,l\to r\Leftarrow P} C[\sigma r] = t$. Furthermore assume SN($\sigma$), i.e., SN($\sigma x$) for all $x \in Dom(\sigma)$, and let σ' be given with $\sigma \to^* \sigma'$, i.e., $\sigma x \to^* \sigma' x$ for all $x \in Dom(\sigma)$. Then we have: $s = C[\sigma l]_p \to^* C[\sigma' l]_p \to_{p,\sigma',l\to r\Leftarrow P} C[\sigma' r]_p$ (due to $(\sigma' P)\downarrow$), and $t = C[\sigma r]_p \to^* C[\sigma' r]_p$.

Proof: Straightforward by verifying $(\sigma' P)\downarrow$ using the encoding of $s \downarrow_{\mathcal{R}} t$ into $eq(s,t) \rightarrow^*_{\mathcal{R}'} true$[34] and applying Lemma 3.6.2 together with the fact that *true* is irreducible. ∎

Choosing $C[]_\Pi$ to be the empty context (and accordingly $\Pi = \{\lambda\}$) in Theorem 3.6.1 we obtain as corollary the following local version of a confluence criterion.

Theorem 3.6.5 (local completeness criterion for conditional overlay systems) Let $\mathcal{R}$ be a CTRS with OS($\mathcal{R}$) and JCP($\mathcal{R}$), and let s be a term with SN(s). Then we have CR(s). In other words, for a conditional overlay system with joinable critical pairs, a term is terminating if and only if it is complete.

The termination assumption concerning s in this result is crucial as demonstrated by the following example.

Example 3.6.6 (Example 2.3.13 continued) Here

$$\mathcal{R} = \left\{ \begin{array}{rcl} f(x) \rightarrow a & \Longleftarrow & f(x) \downarrow x \\ b \rightarrow f(b) & & \end{array} \right.$$

clearly is an overlay system with joinable critical pairs (it is even orthogonal). Moreover we have $f(a) \leftarrow f(f(b)) \rightarrow a$ but not $a \downarrow f(a)$. Obviously, for the inner contracted redex $f(b)$ of $f(f(b))$, SN($f(b)$) does not hold due to the presence of the rule $b \rightarrow f(b)$ in $\mathcal{R}$ (note that SIN($f(b)$) doesn't hold either).

Remark 3.6.7 (local completeness is not a consequence of the global result of [DOS88b], Theorem 2.3.21)
Note that the local completeness property of Theorem 3.6.5 obviously implies the global one of Theorem 2.3.21. Hence, Theorem 2.3.21 is a direct consequence of Theorem 3.6.5. However, vice versa, for deriving Theorem 3.6.5 from Theorem 2.3.21 we cannot simply apply Newman's Lemma 2.1.20 since local confluence (below the terminating term considered) is not available as assumption, but only joinability of all critical pairs.

3.6.1 Non-Overlapping Conditional Systems

Now let us consider non-overlapping CTRSs. We shall show that all normalization properties of Theorem 3.1.1 also hold for non-overlapping CTRSs. The proofs of the corresponding results are very similar to those of the corresponding results for unconditional non-overlapping TRSs in Section 3.2. However, some additional problems arising with CTRSs have to be taken into account.

Throughout this subsection we assume that $\mathcal{R}$ is a non-overlapping CTRS, i.e. NO($\mathcal{R}$) holds.

Let us start with an easy result about innermost reduction.

[34] cf. Remark 2.3.6

Lemma 3.6.8 Innermost reduction in $\mathcal{R}$ is WCR[1], i.e. $\mathrm{WCR}^1(\xrightarrow[i]{})$ holds.

Proof: As for Lemma 3.2.1. ∎

Corollary 3.6.9 Innermost reduction in $\mathcal{R}$ is confluent, i.e. $\mathrm{CR}(\xrightarrow[i]{})$ holds.

The following result shows that for non-overlapping systems the existence of a terminating innermost derivation for some term t implies that any innermost derivation initiated by t is finite.

Lemma 3.6.10 $\forall t : [\,\mathrm{WIN}(t) \iff \mathrm{SIN}(t)\,]$.

Proof: As for Lemma 3.2.3. ∎

Furthermore, innermost termination is equivalent to termination.

Theorem 3.6.11 $\forall t : [\,\mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]$.

Proof: For a proof we refer to the more general Theorem 3.6.19. ∎

Combined with Lemma 3.6.10 this yields the following

Corollary 3.6.12 $\forall t : [\,\mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]$.

The next result says that innermost reduction steps in non-overlapping CTRSs cannot be critical in the sense that they may destroy the possibility of infinite derivations.

Lemma 3.6.13 If $s \xrightarrow[i]{} t$ and $\mathrm{SN}(t)$ then $\mathrm{SN}(s)$.

Proof: This is an immediate consequence of Corollary 3.6.12. ∎

Furthermore, as in the unconditional case, the non-erasing property is crucial for the equivalence of weak and strong termination of non-overlapping CTRSs.

Lemma 3.6.14 (critical steps must be erasing non-innermost)
If $s \rightarrow_{p,\sigma,l\rightarrow r\Leftarrow P} t$ such that $\mathrm{SN}(t)$ and $\neg\mathrm{SN}(s)$, then s/p contains a proper subterm $s' = \sigma(x)$, for some $x \in Var(l)$, with $\neg\mathrm{SN}(s')$, and moreover $l \rightarrow r$ is an *erasing* rule, more precisely: $x \in Var(l) \setminus Var(r)$.

Proof: Let $s = C[\sigma l]_p \rightarrow_{p,\sigma,l\rightarrow r} C[\sigma r]_p = t$ and $\mathrm{SN}(t)$, $\neg\mathrm{SN}(s)$. By Lemma 3.6.13 the step $s \rightarrow_{p,\sigma,l\rightarrow r} t$ must be non-innermost. $\mathrm{NO}(\mathcal{R})$ implies that σx is reducible for some $x \in V(l)$. And $\mathrm{SN}(t)$ implies in particular $\mathrm{SN}(\sigma x)$ for all $x \in V(r)$. Now suppose that the rule $l \rightarrow r$ is non-erasing, i.e., $Var(l) = Var(r)$. Due to $\mathrm{SN}(t)$ this implies $\mathrm{SN}(\sigma x)$ for all $x \in V(l)$. By innermost normalizing σ we thus obtain $s = C[\sigma l]_p \xrightarrow[i]{}{}^+ C[\sigma' l]_p \rightarrow_{p,\sigma',l\rightarrow r\Leftarrow P} C[\sigma' r]_p$ and $t = C[\sigma r]_p \xrightarrow[i]{}{}^* C[\sigma' r]_p$ with $\sigma' x$ a normal form, for all $x \in V(l) = V(r)$. Note in particular, that the step $C[\sigma' l]_p \rightarrow_{p,\sigma',l\rightarrow r\Leftarrow P} C[\sigma' r]_p$ is possible by Lemma 3.6.4 and innermost because of $\mathrm{NO}(\mathcal{R})$. Hence, from $s \xrightarrow[i]{}{}^* C[\sigma' r]_p$, $t \xrightarrow[i]{}{}^* C[\sigma' r]_p$ and $\mathrm{SN}(t)$ we obtain $\mathrm{WIN}(s)$ which, by Corollary 3.6.12, yields SN(s). But this is a contradiction to $\neg$SN(s). Hence, the rule $l \to r$ must be erasing, and there must exist some $x \in Var(l) \setminus Var(r)$ with $\neg(\mathrm{SN}(\sigma x))$. This concludes the proof. ■

As direct consequence of this result we obtain the following.

Corollary 3.6.15 (critical steps are impossible for non-erasing CTRSs)
Suppose NE($\mathcal{R}$). If $s \to t$ and SN(t) then SN(s).

Lemma 3.6.16
If $NE(\mathcal{R})$ then : $\forall t : [\,\mathrm{WN}(t) \iff \mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]$.

Proof: As for Lemma 3.2.10 using Lemma 3.6.15 instead of Lemma 3.2.9. ■

Finally let us summarize the results obtained for non-overlapping, but not necessarily left-linear CTRSs.

Theorem 3.6.17 Any non-overlapping CTRS $\mathcal{R}$ satisfies the following properties:

(1a) $\forall t : [\,\mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]$.

(1b) $\mathrm{WIN}(\mathcal{R}) \iff \mathrm{SIN}(\mathcal{R}) \iff \mathrm{SN}(\mathcal{R})$.

(2a) If $s \to_{p,\sigma,l\to r \Leftarrow P} t$ such that SN(t) and $\neg$SN(s) then s/p contains a proper subterm $s' = \sigma(x)$, for some $x \in Var(l) \setminus Var(r)$, with $\neg$SN(s').[35]

(2b) If $s \underset{i}{\longrightarrow} t$ and SN(t) then SN(s).

(3a) $\mathrm{NE}(\mathcal{R}) \implies [\forall t : [\,\mathrm{WN}(t) \iff \mathrm{WIN}(t) \iff \mathrm{SIN}(t) \iff \mathrm{SN}(t)\,]]$.

(3b) $\mathrm{NE}(\mathcal{R}) \implies [\,\mathrm{WN}(\mathcal{R}) \iff \mathrm{WIN}(\mathcal{R}) \iff \mathrm{SIN}(\mathcal{R}) \iff \mathrm{SN}(\mathcal{R})\,]$.

(4a) $\forall t : [\,\mathrm{WIN}(t) \implies \mathrm{CR}(t)\,]$.

(4b) $\mathrm{WIN}(\mathcal{R}) \implies \mathrm{CR}(\mathcal{R})$.

(5a) $\mathrm{NE}(\mathcal{R}) \implies [\forall t : [\,\mathrm{WN}(t) \implies \mathrm{CR}(t)\,]]$.

(5b) $\mathrm{NE}(\mathcal{R}) \wedge \mathrm{WN}(\mathcal{R}) \implies \mathrm{CR}(\mathcal{R})$.

Proof: (1)-(3) have been shown above.[36] To verify the confluence criteria (4) and (5) one has to combine (1) and (3), respectively, with Theorem 3.6.19 (below), observing that a non-overlapping CTRS is in particular an overlay system with joinable critical pairs. ■

[35] This means that the non-terminating proper subterm $s' = \sigma(x)$ of s/p is *erased* in the step $s \to_{p,\sigma,l\to r} t$, and implies that $l \to r$ is an *erasing* rule.

[36] Actually, the proofs thereof crucially rely on Theorem 3.6.19, which is independently proved below.

3.6.2 Conditional Overlay Systems with Joinable Critical Pairs

Now we show that Theorem 3.3.12 can be generalized from TRSs to CTRSs, i.e., we prove that any innermost terminating (term in a) conditional overlay system with joinable critical pairs is terminating, hence complete by Theorem 3.6.5. The proof is mainly analogous to the unconditional case, but again with some additional complications (arising from the problem that variable overlaps for CTRSs may be critical unlike the situation with TRSs). In order to ensure that variable overlaps are not critical in the proof construction, one needs in particular a stronger minimality property of an assumed infinite derivation which guarantees that whenever a non-terminating redex is contracted then all its proper subterms are terminating (cf. Remark 3.3.7).

Lemma 3.6.18 (properties of the transformation Φ)
Let $\mathcal{R}$ be a conditional overlay system with joinable critical pairs. Then the transformation Φ from Definition 3.3.4 is well-defined and the following properties hold:

(1) If $s = C[\sigma l]_p \rightarrow_{p,\sigma,l\rightarrow r \Leftarrow P} C[\sigma r]_p = t$ such that $\neg\mathrm{SN}(s/p)$ and $\mathrm{SN}(s/pq)$ for all $q \in Pos(s/p)$, $q > \lambda$, then $\Phi(s) = \Phi(s)[p \leftarrow (\Phi \circ \sigma)(l)] \xrightarrow[i]{}_{p,\Phi\circ\sigma,l\rightarrow r \Leftarrow P} \Phi(s)[p \leftarrow (\Phi \circ \sigma)(r)] \xrightarrow[i]{}^{*} \Phi(t)$.

(2a) If $s \rightarrow_p t$ with $\mathrm{SN}(s/p)$ then $\Phi(s) \xrightarrow[i]{}^{*} \Phi(t)$.

(2b) If $s \rightarrow_{>p} t$ with $\neg\mathrm{SN}(s/p)$, $\neg\mathrm{SN}(t/p)$ and $\mathrm{SN}(s/pq)$ for all $q \in Pos(s/p)$, $q > \lambda$, then $\Phi(s) = \Phi(t)$.

Proof: Under the assumptions of the Lemma we know by Theorem 3.6.5 that a term is terminating if and only if it complete. Hence the transformation Φ from Definition 3.3.4 is well-defined here. The proof of (1) is analogous to the proof of Lemma 3.3.10 exploiting in particular the property $\mathrm{OS}(\mathcal{R})$ and additionally Lemma 3.6.4 (as well as Theorem 3.6.5). (2a) is analogous to Lemma 3.3.8 and the refined version (2b) is straightforward by definition of Φ (and Theorem 3.6.5).[37] ∎

Theorem 3.6.19 (innermost termination implies termination and completeness for conditional overlay systems with joinable critical pairs)
For any CTRS $\mathcal{R}$ we have:

(a) $\mathrm{OS}(\mathcal{R}) \wedge \mathrm{JCP}(\mathcal{R}) \wedge \mathrm{SIN}(\mathcal{R}) \Longrightarrow \mathrm{SN}(\mathcal{R}) \wedge \mathrm{CR}(\mathcal{R})$, and

(b) $\mathrm{OS}(\mathcal{R}) \wedge \mathrm{JCP}(\mathcal{R}) \Longrightarrow [\forall s : [\mathrm{SIN}(s) \Longrightarrow \mathrm{SN}(s) \wedge \mathrm{CR}(s)]]$.

Proof: We prove the local version (b) which implies the global (a). Let $\mathcal{R}$ be a conditional overlay system with joinable critical pairs. Now assume s is innermost

[37] That the resulting derivations are innermost (or, more precisely, can be chosen to be innermost) is based on $\mathrm{OS}(\mathcal{R})$ and the fact that normalizing a complete term can be done by an arbitrary strategy, hence also by innermost reduction.

terminating but not terminating. Then according to Remark 3.3.7 there exists an infinite (constricting) derivation D issuing from s of the form

$$D: \; s = s_1 \rightarrow^*_{>q_1} s'_2 \rightarrow_{q_1} s_2 \rightarrow^*_{>q_2} s'_3 \rightarrow_{q_2} \ldots s_n \rightarrow^*_{>q_n} s'_{n+1} \rightarrow_{q_n} s_{n+1} \ldots$$

with $q_i \leq q_{i+1}$ for all i such that in $s_i \rightarrow^*_{>q_i} s'_{i+1}$ only terminating redexes are contracted, and in $s'_{i+1} \rightarrow_{q_i} s_{i+1}$ the contracted redex s'_{i+1}/q_i (as well as its contractum s_{i+1}/q_i) is non-terminating, and moreover, all proper subterms of s'_{i+1}/q_i are terminating and contracting any of these in s'_{i+1}/q_i would result in a terminating term. We observe in particular that there are infinitely many steps $s'_{i+1} \rightarrow_{q_i} s_{i+1}$ in D where a non-terminating redex, all of whose proper subterms are terminating, is contracted. Now, applying the transformation Φ to this derivation D, we know by Lemma 3.6.18 that ΦD has the form

$$\Phi D: \; \Phi s = \Phi s_1 = \Phi s'_2 \xrightarrow[i]{}^{+} \Phi s_2 = \Phi s'_3 \xrightarrow[i]{}^{+} \ldots \Phi s_n = \Phi s'_{n+1} \xrightarrow[i]{}^{+} \Phi s_{n+1} \ldots \; ,$$

hence is an infinite innermost derivation. Since we also have $s \xrightarrow[i]{}^{*} \Phi s$ this yields a contradiction to the assumption that s is innermost terminating. Thus we are done. ■

Finally, let us mention that most of the results developed for unconditional TRSs in Section 3.4 can also be extended to the conditional case. We shall not describe this in detail (here) but only sketch the crucial ideas which are necessary for this extension. First of all, Definitions 3.4.1 (UIR, uniqueness of innermost reduction), 3.4.2 (innermost-uncritical, innermost-critical steps), 3.4.4 (AICR, avoidance of innermost-critical reduction steps) and 3.4.7 (transformation by parallel one-step reduction of all terminating innermost redexes) are extended to CTRSs in the obvious way. Then the adapted versions of Lemmas 3.4.5 and 3.4.6 still hold. However, for the adapted version of the crucial Lemma 3.4.9, on which the subsequent results in Section 3.4 are based, we need an additional property, namely the following *parallel stability property* for the CTRSs considered:

> If $\sigma l \rightarrow_{\lambda,\sigma,l\rightarrow r \Leftarrow P} \sigma r$, $\sigma \rightarrow \sigma'$ and $\mathrm{SN}(\sigma)$, then also $\sigma' l \rightarrow_{\lambda,\sigma',l\rightarrow r \Leftarrow P} \sigma' r$ (and $\sigma r \not\!\!\!\twoheadrightarrow \sigma' r$).

The following interesting counterexample shows that left-to-right joinability of all critical peaks does not suffice for this property.

Example 3.6.20 The CTRS

$$\mathcal{R} = \left\{ \begin{array}{rcl} f(a,a) \rightarrow b & & \\ a \rightarrow b & & \\ f(b,x) \rightarrow b & \Longleftarrow & f(x,x) \downarrow b \\ f(x,b) \rightarrow b & \Longleftarrow & f(x,x) \downarrow b \end{array} \right.$$

is terminating (but not decreasing). It has two (symmetric) outside critical peaks (between the last two rules) without feasible instances, and the two inside critical peaks

$$f(b,a) \leftarrow f(a,a) \rightarrow b$$

and

$$f(a,b) \leftarrow f(a,a) \rightarrow b$$

both of which are left-to-right joinable ($f(a,b) \rightarrow b$, $f(b,a) \rightarrow b$). However, it is not (locally) confluent. For instance, we have

$$\sigma' l = f(b,b) \leftarrow f(b,a) = \sigma l \rightarrow \sigma r = b\,,$$

where the rule $l \rightarrow r \Longleftarrow P$ applied in the right step is $f(b,x) \rightarrow b \Longleftarrow f(x,x) \downarrow b$, with matching substitution $\sigma = \{x \mapsto a\}$, and where $\sigma' = \{x \mapsto b\}$, hence $\sigma \rightarrow \sigma'$. But the two reducts $f(b,b)$ and b are both irreducible. In particular, $\sigma' l \rightarrow_{\lambda,\sigma',l \rightarrow r \Longleftarrow P} \sigma' r$ does not hold any more. Intuitively, the reason for that is that recursively checking the condition $(\sigma' P)\downarrow$ leads to the *parallel critical peak*

$$f(b,b) \;{}_{>\lambda}\!\leftarrow\!\!\!\!\!\;\Vdash f(a,a) \rightarrow_{\lambda} b$$

which is not (left-to-right) joinable any more.

Fortunately, the above mentioned parallel stability property is satisfied for some interesting classes of CTRSs, namely for CTRSs with *left-to-right joinable shared parallel critical peaks* as well as for the even more general case of CTRSs with *quasi-overlay joinable critical peaks* (cf. [GW96]). These classes of CTRSs are a proper generalization of conditional overlay systems with joinable critical peaks (i.e., CTRSs with overlay joinable critical peaks). In [GW96] local and global completeness results for these classes of CTRSs are presented. With the help of these latter results it seems quite plausible that the main results of Section 3.4 can be extended (in adapted form) to CTRSs. But the details still have to be worked out.

Chapter 4

Modularity of Confluence Properties

In this chapter the modularity of confluence and related properties is dealt with. We provide an overview of known results and sketch the basic problems, ideas and proof techniques. The systematic and unifying presentation entails some slight improvements / simplifications of already known modularity results and proofs, respectively. Furthermore, we show by counterexamples that the properties *local confluence* and *joinability of all critical pairs*, which are well-known to be non-equivalent for conditional systems, are not even preserved under signature extensions. The presentation is structured as follows. First we consider confluence and local confluence in Section 4.1. Then related unique normal form properties are treated in Section 4.2. The case of non-disjoint unions is covered in Section 4.3, and corresponding modularity results for conditional rewrite systems are surveyed in Section 4.4

4.1 Confluence and Local Confluence

In this section we shall tacitly assume that $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ are disjoint TRSs with $\mathcal{R}^{\mathcal{F}}$ denoting their disjoint union $\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$.

The first property we consider is local confluence (WCR). Obviously, the sets of critical pairs in the disjoint union $\mathcal{R}$ and in the component systems $\mathcal{R}_1$, $\mathcal{R}_2$ are related as follows: $\mathrm{CP}(\mathcal{R}) = \mathrm{CP}(\mathcal{R}_1) \cup \mathrm{CP}(\mathcal{R}_2)$. Hence, by the Critical Pair Lemma 2.2.17 one easily obtains the following (almost trivial) well-known result.

Theorem 4.1.1 Local confluence is modular for disjoint unions of TRSs.

Much less obvious is the famous modularity result for confluence due to Toyama.

Theorem 4.1.2 (confluence is modular, [Toy87b])
Confluence is modular for disjoint unions of TRSs.

The original proof in [Toy87b] is quite involved. Simple proofs of the special cases that confluence is preserved under signature extensions and under the disjoint union of left-linear TRSs are given in [Mid90]. In the general case, the main problem is that destructive reduction steps (by applying collapsing rules) essentially modify the layer structure of mixed terms. An elegant and considerably simplified proof of this important result has later been given in [KMTV94].

Remark 4.1.3 (main ideas and proof structure of [KMTV94])
We sketch here the main ideas of the latter proof (for the non-trivial direction $\mathrm{CR}(\mathcal{R}_1^{\mathcal{F}_1}) \wedge \mathrm{CR}(\mathcal{R}_2^{\mathcal{F}_2}) \Longrightarrow \mathrm{CR}(\mathcal{R}^{\mathcal{F}})$) and its structure because it is the basis for various generalizations of Theorem 4.1.2.

(a) A first general idea (which is easily formulated in terms of abstract reduction relations, too) is to define and use *representatives* of sets of pairwise confluent terms: A set $\hat{S}$ of terms is said *to represent* a given set S of confluent terms, if (i) every term $s \in S$ has a unique reduct in $\hat{S}$, the *representative* of s, and (ii) joinable terms in S have the same representative in $\hat{S}$. If S is a set of confluent terms, joinability is an equivalence relation on S. If these equivalence classes are finite, for each of them a common (unique) reduct can be defined (its representative). Hence, every finite set of confluent terms can be represented.

(b) A second idea (which is specific for the combination setting) is to introduce the notion of *witnesses* which simplifies the main step of the proof. Intuitively, a witness of a term s is obtained by a synchronous reduction of all its principal aliens to preserved reducts. Formally, for $s = C \langle\!\langle s_1, \ldots, s_n \rangle\!\rangle$ a *witness* of s is an inner preserved term $t = C \langle\!\langle t_1, \ldots, t_n \rangle\!\rangle$ (i.e., with $t_1, \ldots, t_n$ preserved) such that $s_i \to^* t_i$ (for $1 \leq i \leq n$) and $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$. This definition implies in particular that every term of rank 1 is a (more precisely, the only) witness of itself. Note that the existence of witnesses depends on the existence of preserved reducts (for the principal aliens).

Now the proof structure is as follows:

(1) Every term (in $\mathcal{T}(\mathcal{F}, \mathcal{V})$) has a preserved reduct, hence every term has a witness (this is verified in [KMTV94] by showing weak (and even strong) termination of the *collapsing reduction relation* $\to_{\mathrm{c}}$).

(2) The outer reduction relations $\xrightarrow{o}_{\mathcal{R}_1}$ and $\xrightarrow{o}_{\mathcal{R}_2}$ are confluent (on $\mathcal{T}(\mathcal{F}, \mathcal{V})$) (this is straightforward by injective abstraction of all principal aliens).

(3) Preserved terms are confluent (by induction on the rank, using (2) and (a)).

(4) Inner preserved terms are confluent (by case analysis using (3) and (a)).[1]

(5) If $s \to t$ and all principal subterms of s are confluent, then $\dot{s} \downarrow \dot{t}$, for arbitrary witnesses $\dot{s}$ and $\dot{t}$ of s and t, respectively (by case analysis).

(6) The disjoint union $\mathcal{R}^{\mathcal{F}}$ is confluent. (By induction on *rank*(t) on shows confluence of t. In the induction step, for an arbitrary conversion of the form $t_1 \,{}^*\!\!\leftarrow t \to^* t_2$, one first reduces every term in this conversion to a witness. Since

[1] Actually, in [KMTV94], (4) (which is stronger than (3)) is directly proved by induction on the rank using case analysis, (2) and (a).

all principal subterms occurring in this conversion have rank less than $rank(t)$, they are confluent by induction hypothesis. Repeated application of (5) yields a conversion between the witnesses in which all terms are inner preserved. Then repeated application of (4) yields a common reduct of t_1 and t_2.)

Essentially, there are two main prerequisites for this proof (structure) to go through, namely, that reduction in the (disjoint) union is rank decreasing (which enables induction over the rank of terms), and that every term has a preserved reduct (which enables the projection of terms to witnesses, i.e., inner preserved reducts, hence makes the induction step go through by reducing it to confluence of inner preserved terms). In particular, the latter observation turns out to be crucial for generalizing the result to constructor sharing and to composable TRSs.

4.2 Unique Normal Form Properties

Next we summarize known modularity results (w.r.t. disjoint unions) for various normal form properties (cf. Definition 2.1.17) which are weakened versions of confluence.

Theorem 4.2.1 (UN is modular, [Mid90])
The property of having unique normal forms (UN) is modular for disjoint unions of TRSs.

Proof (idea): The central step in the proof (of the non-trivial direction $\mathrm{UN}(\mathcal{R}_1^{\mathcal{F}_1}) \wedge \mathrm{UN}(\mathcal{R}_2^{\mathcal{F}_2}) \Longrightarrow \mathrm{UN}(\mathcal{R}^{\mathcal{F}})$) of [Mid90] is a construction which shows that every TRS can be conservatively extended to a confluent TRS with the same set of normal forms. More precisely, the following is shown:

Every TRS $\mathcal{R'}^{\mathcal{F'}}$ with unique normal forms ($\mathrm{UN}(\mathcal{R'}^{\mathcal{F'}})$) can be extended to a TRS $\mathcal{R''}^{\mathcal{F''}}$ (i.e., $\mathcal{F'} \subseteq \mathcal{F''}$, $\mathcal{R'} \subseteq \mathcal{R''}$) such that:

(1) $\mathcal{R''}^{\mathcal{F''}}$ is confluent,

(2) $\forall s, t \in \mathcal{T}(\mathcal{F''}, \mathcal{V}).\, s \leftrightarrow^*_{\mathcal{R'}} t \iff s \leftrightarrow^*_{\mathcal{R''}} t$, and

(3) $\mathrm{NF}(\mathcal{R'}^{\mathcal{F'}}) = \mathrm{NF}(\mathcal{R''}^{\mathcal{F''}})$.

Using this interesting (construction and) fact it is not difficult to reduce the above implication $\mathrm{UN}(\mathcal{R}_1^{\mathcal{F}_1}) \wedge \mathrm{UN}(\mathcal{R}_2^{\mathcal{F}_2}) \Longrightarrow \mathrm{UN}(\mathcal{R}^{\mathcal{F}})$ to (the non-trivial direction of) Theorem 4.1.2. ■

Due to the implications CR $\Longrightarrow$ NF $\Longrightarrow$ UN (cf. Lemma 2.1.19) one might conjecture that the normal form property NF is modular, too. However, this is not the case.

Example 4.2.2 (counterexample to modularity of NF, [Mid90])
The disjoint TRSs $\mathcal{R}_1 = \{f(x,x) \to a\}$ and

$$\mathcal{R}_2 = \begin{cases} A \to B \\ A \to C \\ B \to B \\ C \to C \end{cases}$$

both have the property NF. However, in $\mathcal{R}_1 \oplus \mathcal{R}_2$ we have

$$f(B,C) \leftarrow f(A,C) \leftarrow f(A,A) \to a\,,$$

where a is a normal form and, obviously, $f(B,C)$ does not reduce to a. Hence, $\mathcal{R}_1 \oplus \mathcal{R}_2$ does not have property NF.

The non-left-linearity of one of the systems in the above counterexample is essential as is obvious from the next positive result due to Middeldorp.

Theorem 4.2.3 (NF is modular for left-linear TRSs, [Mid90])
The normal from property (NF) is modular for disjoint unions of left-linear TRSs.

In a way similar to Example 4.2.2 above, one can also construct a counterexample to the modularity of $\mathrm{UN}^{\to}$, the uniqueness of normal forms w.r.t. reduction.

Example 4.2.4 ($\mathrm{UN}^{\to}$ is not modular, [Mid90])
The disjoint TRSs $\mathcal{R}_1 = \{f(x,x) \to a\}$ and

$$\mathcal{R}_2 = \begin{cases} A \to B \\ A \to C \\ C \to C \\ D \to C \\ D \to E \end{cases}$$

both have the property $\mathrm{UN}^{\to}$. However, in $\mathcal{R}_1 \oplus \mathcal{R}_2$ the term $f(A,D)$ reduces to two different normal forms:

$$f(B,E) \leftarrow f(B,D) \leftarrow f(A,D) \to f(C,D) \to f(C,C) \to a\,.$$

Therefore, $\mathcal{R}_1 \oplus \mathcal{R}_2$ does not have the property $\mathrm{UN}^{\to}$.

Again, non-left-linearity in this example is essential. Actually, Middeldorp showed that $\mathrm{UN}^{\to}$ is modular for disjoint unions of left-linear TRSs without collapsing rules, and conjectured that the non-collapsing requirement can be omitted ([Mid90]). This conjecture was recently settled in the affirmative by Marchiori.

Theorem 4.2.5 ($\mathrm{UN}^{\to}$ is modular for left-linear TRSs, [Mar94])
Uniqueness of normal forms w.r.t. reduction ($\mathrm{UN}^{\to}$) is modular for disjoint unions of left-linear TRSs.

The proof technique employed here (as well as in the closely related papers [SSP94], [SMP95]) which is based on a very subtle interaction between properties of left-linear systems and of certain collapsing reduction (sequences) therein is interesting by itself and will be discussed in more detail later on (in Section 5.3.3).

Finally let us introduce two more important properties of TRSs which are weak versions of UN and $\mathrm{UN}^{\rightarrow}$, respectively, in the sense that instead of all terms in normal form only variables are considered.

Definition 4.2.6 (consistency properties)
A TRS $\mathcal{R}^{\mathcal{F}}$ is *consistent* (CON) if distinct variables are not convertible, i.e., if $\forall x, y \in \mathcal{V}.\, x \leftrightarrow^*_{\mathcal{R}^{\mathcal{F}}} y \Longrightarrow x = y)$.[2] $\mathcal{R}^{\mathcal{F}}$ is *consistent with respect to reduction* ($\mathrm{CON}^{\rightarrow}$) if no term reduces to two distinct variables i.e., if $\forall s \in \mathcal{T}(\mathcal{F}, \mathcal{V})\, \forall x, y \in \mathcal{V}.\, x \;{}^*\!\!\leftarrow s \rightarrow^* y \Longrightarrow x = y)$.[3]

The modularity behaviour of CON and $\mathrm{CON}^{\rightarrow}$ turns out to be analogous to that of UN and $\mathrm{UN}^{\rightarrow}$, respectively (cf. Theorems 4.2.1, 4.2.5 and Example 4.2.4 above).

Theorem 4.2.7 (CON is modular, [Sch89])
Consistency (CON) is modular for disjoint unions of TRSs (and of equation systems).[4]

Example 4.2.8 ($\mathrm{CON}^{\rightarrow}$ is not modular, [Mar94])
The disjoint TRSs

$$\mathcal{R}_1 = \begin{cases} f(x,x,y) \rightarrow y \\ f(x,y,y) \rightarrow x \end{cases}$$

and

$$\mathcal{R}_2 = \begin{cases} G(x) \rightarrow x \\ G(x) \rightarrow A \end{cases}$$

both have the property $\mathrm{CON}^{\rightarrow}$. However, in $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ we have

$$x \leftarrow f(x,A,A) \;{}^+\!\!\leftarrow f(G(x),G(y),G(z)) \rightarrow^+ f(A,A,z) \rightarrow z\,,$$

hence the disjoint union $\mathcal{R}^{\mathcal{F}}$ is not $\mathrm{CON}^{\rightarrow}$.

We observe that one of the systems in this example is not left-linear. This is indeed essential due to the following result.

Theorem 4.2.9 ($\mathrm{CON}^{\rightarrow}$ is modular for left-linear TRSs, [SMP95], [Mar94])
Consistency with respect to reduction ($\mathrm{CON}^{\rightarrow}$) is modular for disjoint unions of left-linear TRSs.

The relations between the various confluence (and normal form) properties considered as well as their modularity behaviour (for disjoint unions) are depicted graphically

[2] Actually, the property CON as defined here makes also sense in an equational setting where no restrictions concerning (the occurrences of) variables in equations / rewrite rules are imposed as it is common e.g. in unification theory ([Sch89], [BS94]). The same holds for UN.

[3] Consistency with respect to reduction ($\mathrm{CON}^{\rightarrow}$) is called *r-consistency* in [SMP95].

[4] In fact, the corresponding result in [Sch89] is more general since the construction there yields a conservative extension of two given disjoint equational theories.

$$\begin{tikzcd}
\mathrm{NF} \arrow[r, Rightarrow] & \boxed{\mathrm{UN}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{UN}^{\rightarrow} \arrow[d, Rightarrow] \\
\boxed{\mathrm{CR}} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \boxed{\mathrm{CON}} \arrow[r, Rightarrow] & \mathrm{CON}^{\rightarrow}
\end{tikzcd}$$

Figure 4.1: confluence properties and their modularity behaviour

in Figure 4.1 (where missing implications do not hold, and modular properties are enclosed in rectangles).

4.3 Non-Disjoint Unions

A first easy observation is that local confluence (WCR) or, equivalently, joinability of all critical pairs (JCP), is modular for composable TRSs,[5] due to the Critical Pair Lemma 2.2.17. This also holds for a couple of other properties related to (the form of joinability of) critical peaks / pairs.

Lemma 4.3.1 (modularity of critical peak properties for composable TRSs)
The following 'critical peak properties' are modular for composable TRSs:

(1) WCR(local confluence, or, equivalently, JCP, joinability of all critical pairs)

(2) OS (overlaying)

(3) NO (non-overlapping)

(4) WNO (weakly non-overlapping)

(5) UIR (uniqueness of innermost reduction)

(6) AICR (avoidance of innermost-critical steps)

(7) CPC (critical peak condition)

(8) SLRJCP (strongly left-to-right joinable critical peaks)

(9) LRJCP (left-to-right joinable critical peaks)

Proof: Straightforward by the respective definitions and the definition of composable TRSs. ∎

As observed by Kurihara & Ohuchi ([KO92]), the modularity of confluence (CR) is lost for constructor sharing (and hence also for composable) TRSs.

[5] as remarked e.g. in [Ohl94a]

Example 4.3.2 (confluence is not modular for constructor sharing TRSs) Consider the following partition of (a renamed version of) the TRS in Example 2.2.19 into

$$\mathcal{R}_1 = \left\{ \begin{array}{r} f(x,x) \to a \\ f(x,c(x)) \to b \end{array} \right.$$

and

$$\mathcal{R}_2 = \left\{ \ d \to c(d) \ \right\} .$$

which are constructor-sharing (c is a shared constructor). Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are obviously confluent, but their union $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \uplus \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$ is non-confluent. Indeed, in $\mathcal{R}^{\mathcal{F}}$ we have the divergence

$$a \ {}_{\mathcal{R}_1}\!\!\leftarrow f(d,d) \to_{\mathcal{R}_2} f(d,c(d)) \to_{\mathcal{R}_1} b$$

issuing from the term $f(d,d)$ of rank 2, with a, b in normal form. Intuitively, the reason for the non-confluence here is due to the fact that the topmost black layer of $f(d,d)$ (recall that symbols of $\mathcal{R}_1$ ($\mathcal{R}_2$) not occurring in $\mathcal{R}_2$ ($\mathcal{R}_1$) are considered to be black (white); hence f is black and d white) is essentially modified by the constructor lifting rule $d \to c(d)$ of $\mathcal{R}_2$ thus enabling the subsequent $\mathcal{R}_1$-step $f(d,c(d)) \to_{\mathcal{R}_1} b$ which was not possible before. In other words, the term $f(d,d)$ is not preserved and also not inner preserved, and even worse, the (two equal) principal top white aliens d do not have a preserved reduct at all, i.e., a reduct with a stable layer structure (cf. Definition 2.4.12). Indeed, the only reductions possible from d are of the form

$$d \to_{\mathcal{R}_2} c(d) \to_{\mathcal{R}_2} c(d) \to_{\mathcal{R}_2} c(c(d)) \ldots$$

where every step is layer coalescing (namely, destructive at level 1).

If the existence of preserved reducts in the union is guaranteed then — analogous to the situation for disjoint unions where this property is always guaranteed (cf. Section 4.1, Remark 4.1.3) — such a phenomenon (i.e., non-confluence of the union of two confluent constructor-sharing or composable TRS) cannot occur. Ohlebusch ([Ohl94b]) was the first to recognize this for constructor sharing TRSs and to succeed in adapting the proof structure of [KMTV94] to this case. The resulting generalization of Theorem 4.1.2 to constructor-sharing TRSs which in essence is due to [Ohl94b][6] reads as follows.

Theorem 4.3.3 (confluence of unions of constructor sharing TRSs) Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be two confluent constructor sharing TRSs. Their union $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \uplus \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$ is confluent provided every term (in $\mathcal{T}(\mathcal{F},\mathcal{V})$) has a preserved reduct).

[6] The corresponding result in [Ohl94b] is formulated more operationally in the sense that instead of the existence of preserved reducts in the union the layer coalescing (or collapsing) reduction relation $\to_{\mathrm{c}}$ is required to be weakly terminating. For confluent component TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ these two conditions are equivalent as is not too difficult to verify. However, in general a term (in the union) may have a preserved reduct even if the layer coalescing reduction $\to_{\mathrm{c}}$ is not weakly terminating. To wit, consider $\mathcal{R}_1 = \{f(x,x) \to a, f(x,c(x)) \to b\}$ as in Example 4.3.2 above and $\mathcal{R}_2 = \{d(e(a)) \to c(d(e(a))), e(a) \to a\}$. Then a and c are shared constructors. In the union every term has a preserved reduct (in fact, even a normal form), however we have e.g. $\neg\mathrm{WN}(d(e(a)), \to_{\mathrm{c}})$ since the only $\to_{\mathrm{c}}$-reductions issuing from $d(e(a))$ are: $d(e(a)) \to_{\mathrm{c}} c(d(e(a))) \to_{\mathrm{c}} c(c(d(e(a)))) \to_{\mathrm{c}} \ldots$. But note that $\mathcal{R}_2$ here is clearly non-confluent!

Proof: The proof is essentially the same as in [Ohl94b], following the proof structure of [KMTV94] as outlined in Remark 4.1.3 above. However, the assumption of weak termination of $\to_c$ is replaced by the requirement that every term (in the union) has a preserved reduct. (cf. also Theorem 4.3.4 below where the necessary adaptations for the composable case are sketched). ■

This preservation result for confluence also extends in a natural manner to composable TRSs as shown in [Ohl94a]. Our formulation is again slightly different from the one in [Ohl94a].[7]

Theorem 4.3.4 (confluence of unions of composable TRSs)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be two confluent composable TRSs. Their union $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$ is confluent provided every term (in $\mathcal{T}(\mathcal{F}, \mathcal{V})$) has a preserved reduct).

Proof (sketch): The proof is essentially the same as in [Ohl94a], with some slight modifications, and follows the proof structure of [KMTV94] as outlined in Remark 4.1.3 above. More precisely, the structure is as follows.

First the notions of preservation and witness are adapted (cf. Section 2.4.2). A term (in $\mathcal{T}(\mathcal{F}, \mathcal{V})$) is preserved if no ($\mathcal{R}^{\mathcal{F}}$-)derivation issuing from it contains a step which is destructive at level $m \geq 1$ (this implies in particular that all transparent terms, i.e., those from $\mathcal{T}(\mathcal{F}^s, \mathcal{V})$, are preserved).[8] A term is white (black) preserved if all its white (black) principal aliens are preserved (black / white preservation replaces inner preservation of [KMTV94]). For $s = C^b \langle\langle s_1, \ldots, s_n \rangle\rangle$, a white witness of s is a white preserved term $t = C^b \langle\langle t_1, \ldots, t_n \rangle\rangle$ such that $s_i \to^* t_i$ (for $1 \leq i \leq n$) and $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$. The definition of black witnesses is symmetric. Note that this implies in particular that any transparent term is a (more precisely, the only witness) of itself.

Now the adapted proof structure reads as follows:

(1) By assumption we know that every term (in $\mathcal{T}(\mathcal{F}, \mathcal{V})$) has a preserved reduct, hence every term has a black (white) witness.

(2) The reduction relations $\xrightarrow{t}$, $\xrightarrow{t,o}_b$ and $\xrightarrow{t,o}_w$ are confluent (on $\mathcal{T}(\mathcal{F}, \mathcal{V})$).

(3) Preserved terms are confluent.

(4) White (black) preserved terms are confluent.

[7] Actually, we only require the existence of preserved reducts instead of the more operational condition in [Ohl94a] that the layer coalescing reduction relation $\to_c$ is be weakly terminating. Furthermore, we note that our definition of $\to_c$ (cf. Definition 2.4.12) slightly differs from the one of [Ohl94a] in the sense that we do not consider a reduction step which is destructive at level 0, i.e., which reduces a top transparent term to a top black or top white one, as a layer coalescing step, i.e., as $\to_c$-step (since it does not lead to the problematic case of a coalescence of two originally distinct layers). Clearly, weak termination of $\to_c$ implies the existence of preserved reducts (for all terms in the union). Vice versa, it seems quite plausible that — for confluent composable TRSs — the existence of preserved reducts also implies weak termination of $\to_c$. However, we have not investigated this in detail.

[8] This slightly differs from [Ohl94a].

(5) If $s \to t$ and all white (black) principal subterms of s are confluent, then $\dot{s} \downarrow \dot{t}$, for arbitrary white (black) witnesses $\dot{s}$ and $\dot{t}$ of s and t, respectively.

(6) The union $\mathcal{R}^{\mathcal{F}}$ is confluent.

■

The existence of preserved reducts is obtained for free if destructive steps at level $m \geq 1$ are impossible. This is obviously the case if the composable TRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ are layer preserving (cf. Definition 2.4.17), i.e., if all rules in the union $\mathcal{R}_1 \cup \mathcal{R}_2$ with a top transparent right hand side (i.e., its root is a shared (function) symbol or a variable) also have top transparent left hand sides (i.e., left hand sides with a shared function symbol at the root). Hence, one obtains the following direct consequence of Theorem 4.3.4.

Theorem 4.3.5 (modularity of confluence under layer preservation, [Ohl94a][9])
Confluence is modular for unions of composable, layer preserving TRSs.

Further modularity results for confluence which additionally rely on termination properties will be discussed later on in Chapter 5.

Let us now turn to the other confluence properties mentioned above, namely WCR, NF, UN, UN$^{\to}$, CON, CON$^{\to}$. Obviously, local confluence (WCR) is still modular for constructor sharing as well as composable TRSs (since we still have $\mathrm{CP}(\mathcal{R}_1 \cup \mathcal{R}_2) = \mathrm{CP}(\mathcal{R}_1) \cup \mathrm{CP}(\mathcal{R}_2)$). However, all other positive results from above do not extend, at least not directly. For instance, Example 4.3.2 shows this for NF and UN$^{\to}$. As observed in [Ohl94a], left-linearity is no longer sufficient for the modularity of NF and UN$^{\to}$. A simple counterexample to the modularity of CON and CON$^{\to}$ for constructor sharing (and composable) TRSs is the following.

Example 4.3.6 (CON and CON$^{\to}$ are not modular for (left-linear) constructor sharing TRSs)
The TRSs

$$\mathcal{R}_1 = \left\{ \begin{array}{l} f(a,x,y) \to x \\ f(b,x,y) \to y \end{array} \right.$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{l} c \to a \\ c \to b \end{array} \right\} .$$

are constructor sharing, left-linear and satisfy CON, hence also CON$^{\to}$. However, their union is not even CON$^{\to}$ as witnessed by

$$x \leftarrow f(a,x,y) \leftarrow f(c,x,y) \to f(b,x,y) \to y \ .$$

Indeed, we expect that any serious attempt to extend positive modularity results for these confluence and normal form properties from the disjoint union case to constructor

[9] In [Ohl94a] an extra proof is necessary here for showing weak termination of $\to_{\mathrm{c}}$.

sharing or composable systems must carefully take into account the additional disturbing effect caused by rules which are shared symbol lifting. In particular, this means that besides collapsing rules (i.e., rules with a variable as right hand side) also shared function symbol lifting rules have to receive special attention.

4.4 Conditional Rewrite Systems

First we summarize the known results on modularity of properties of CTRSs which are related to confluence and normal forms for the disjoint union case.

The most important positive result to be mentioned here certainly is Middeldorp's extension of Toyama's Theorem 4.1.2 to CTRSs ([Mid90]).

Theorem 4.4.1 (confluence is modular for disjoint unions of (join / semi-equational) CTRSs, [Mid90])
Confluence is modular for disjoint unions of join CTRSs and of semi-equational CTRSs.

Proof (idea): We only sketch the basic idea and structure of the proof (of the difficult direction $\mathrm{CR}(\mathcal{R}_1^{\mathcal{F}_1}) \wedge \mathrm{CR}(\mathcal{R}_2^{\mathcal{F}_2}) \Longrightarrow \mathrm{CR}(\mathcal{R}^{\mathcal{F}})$, for $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2)$ in [Mid90] for join CTRSs. Essentially, the whole proof is a rather non-trivial reduction to the unconditional case, i.e., to Toyama's Theorem 4.1.2.

First, over $\mathcal{T}(\mathcal{F}, \mathcal{V})$ the relations $\rightarrow_i$ $(i = 1, 2)$ induced by $\mathcal{R}_i$ are defined (according to Definition 2.4.16). Then the following assertions are proved (with the notational convention $\rightarrow = \rightarrow_{\mathcal{R}^{\mathcal{F}}}$).

(1) $\rightarrow_{1,2} \subseteq \rightarrow$.

(2) $\rightarrow_{1,2}$ is confluent.

(3) $\rightarrow \subseteq \downarrow_{1,2}$.

(4) $\leftrightarrow^* \subseteq \downarrow_{1,2}$.

From (1) and (4) one then obtains confluence of $\rightarrow$ as desired.

Now, (1) is obvious by definition of $\rightarrow_{1,2}$.

For proving (2) one defines the disjoint unconditional TRSs $\mathcal{S}_1$, $\mathcal{S}_2$ by

$$\mathcal{S}_i = \{s \rightarrow t \mid s, t \in \mathcal{T}(\mathcal{F}_i, \mathcal{V}), s \rightarrow_i t\}$$

for $i = 1, 2$. Then one shows that the restrictions of $\rightarrow_{\mathcal{S}_i}$, $\rightarrow_i$ and $\rightarrow_{\mathcal{R}_i}$ coincide on $\mathcal{T}(\mathcal{F}_i, \mathcal{V}) \times \mathcal{T}(\mathcal{F}_i, \mathcal{V})$, i.e.:

$$\rightarrow_{\mathcal{S}_i} |_{\mathcal{T}(\mathcal{F}_i,\mathcal{V})\times\mathcal{T}(\mathcal{F}_i,\mathcal{V})} = \rightarrow_i |_{\mathcal{T}(\mathcal{F}_i,\mathcal{V})\times\mathcal{T}(\mathcal{F}_i,\mathcal{V})} = \rightarrow_{\mathcal{R}_i} |_{\mathcal{T}(\mathcal{F}_i,\mathcal{V})\times\mathcal{T}(\mathcal{F}_i,\mathcal{V})} ,$$

and that $\rightarrow_{\mathcal{S}_i}$ and $\rightarrow_i$ also coincide on $\mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$, i.e.:

$$\rightarrow_{\mathcal{S}_i} = \rightarrow_i \ .$$

This yields

$$\rightarrow_{\mathcal{S}_1 \oplus \mathcal{S}_2} = \rightarrow_{\mathcal{S}_1} \cup \rightarrow_{\mathcal{S}_2} = \rightarrow_1 \cup \rightarrow_2 = \rightarrow_{1,2}$$

(on $\mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$), from which, by Theorem 4.1.2, confluence of $\rightarrow_{1,2}$ follows.

The proof of (3) $\rightarrow \subseteq \downarrow_{1,2}$ is also non-trivial and technically involved. The crucial point here is — again in analogy to the unconditional case — that every term has a preserved reduct (w.r.t. $\rightarrow_{1,2}$, however).

Finally, (4) $\leftrightarrow^* \subseteq \downarrow_{1,2}$ is an easy consequence of (2) and (3). ∎

Concerning confluence properties related to normal forms, not very much is known for CTRSs. As to uniqueness of normal forms (UN), Middeldorp succeeded in extending his positive modularity result for disjoint unions of TRSs to the case of semi-equational CTRSs.

Theorem 4.4.2 (UN is modular for disjoint unions of semi-equational CTRSs, [Mid90])
Uniqueness of normal forms (UN) is modular for disjoint unions of semi-equational CTRSs.

Whether this holds also for join CTRSs is still an open problem.

Finally let us consider here the properties local confluence (WCR) and joinability of (all) critical pairs (JCP). Simple examples of [Mid90] show that both properties — which are in general not equivalent for join CTRSs as we recall — are not modular for CTRSs, both for join system and semi-equational ones. Actually, we shall show now that for join CTRSs both WCR and JCP are not even preserved under signature extensions (and even for the case of no extra variables), which at first glance may seem to be very surprising.

Example 4.4.3 (WCR and JCP are not preserved under signature extensions)
Consider the join CTRS $\mathcal{R}^{\mathcal{F}}$ with

$$\mathcal{R} = \left\{ \begin{array}{rcll} f(x,y,z) & \rightarrow & g(x) & \Longleftarrow \quad x \downarrow y, y \downarrow z \\ f(x,y,z) & \rightarrow & g(z) & \Longleftarrow \quad x \downarrow y, y \downarrow z \\ b & \rightarrow & a & \\ b & \rightarrow & c & \\ c & \rightarrow & b & \\ c & \rightarrow & d & \\ g(a) & \rightarrow & g(d) & \\ f(a,x,y) & \rightarrow & f(d,x,y) & \\ f(x,a,y) & \rightarrow & f(x,d,y) & \\ f(x,y,a) & \rightarrow & f(x,y,d) & \end{array} \right.$$

and $\mathcal{F} = \{a, b, c, d, f, g\}$. It is not difficult to show that $\mathcal{R}^{\mathcal{F}}$ has joinable critical pairs and is even locally confluent (but not confluent). More generally, the following properties hold (w.r.t. the signature $\mathcal{F}$):

(1) Terms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ which are equivalent to each other (w.r.t. $\leftrightarrow^*_{\mathcal{R}^{\mathcal{F}}}$) have the same depth (by induction on $depth(s)$ and case analysis according to the applied rule it is straightforward to verify that $s \rightarrow t$ implies $depth(s) = depth(t)$).

(2) If $depth(s) = 1$ and $s \leftrightarrow^* t$ then $g(s) \downarrow g(t)$ and, moreover, $s = t \in \mathcal{V}$ or $s, t \in \{a, b, c, d\}$ (proof by an easy case analysis).

(3) If $depth(s) > 1$ and $s \leftrightarrow^* t$ then $s \downarrow t$ (proof by induction on $depth(s)$ using (1), (2) and a straightforward case analysis concerning the possible shapes of s and t).

This means that $\mathcal{R}^{\mathcal{F}}$ is not only locally confluent but even confluent on all equivalence classes of terms from $\mathcal{T}(\mathcal{F})$ (w.r.t. $\leftrightarrow^*$) except for the equivalence class $[a, b, c, d]$.

Now we add a fresh unary function symbol G, i.e., we consider $\mathcal{R}^{\mathcal{F}'}$ with $\mathcal{F}' = \mathcal{F} \uplus \{G\}$. Then joinability of critical pairs and hence local confluence is lost. To wit, consider for instance the term $f(G(a), G(b), G(d))$ which reduces to two distinct normal forms by one $\mathcal{R}^{\mathcal{F}'}$-step, respectively :

$$g(G(a)) \leftarrow f(G(a), G(b), G(d)) \rightarrow g(G(d))$$

Clearly both $g(G(a))$ and $g(G(d))$ are irreducible. This divergence corresponds to an instance of the critical pair between the first two rules, namely

$$\langle g(x) = g(z) \rangle \impliedby x \downarrow y, y \downarrow z\,.$$

Over the old signature $\mathcal{F}$ every substitution σ which is feasible for this critical pair satisfies $\sigma(g(x)) \downarrow \sigma(g(z))$ whereas this is not the case for the mixed substitution $\tau = \{x \mapsto G(a), y \mapsto G(b), z \mapsto G(d)\}$. Hence the critical pair above is not joinable any more over the extended signature $\mathcal{F}'$.

Note that in the above example $\mathcal{R}^{\mathcal{F}}$ is obviously non-terminating. This is not essential in the following sense. We may replace the 'non-terminating part' of $\mathcal{R}^{\mathcal{F}}$

$$\left\{ \begin{array}{l} b \rightarrow a \\ b \rightarrow c \\ c \rightarrow b \\ c \rightarrow d \end{array} \right.$$

which has joinable critical pairs, hence is locally confluent (it is an unconditional TRS!), by a terminating CTRS with joinable critical pairs which is not confluent, hence necessarily not locally confluent. To this end, we can take for instance the system

$$\left\{ \begin{array}{rcl} h(x) \rightarrow k(b) & \impliedby & k(x) \downarrow h(b) \\ k(a) \rightarrow h(a) & & \\ a \rightarrow b & & \end{array} \right.$$

which has the desired properties (in particular, it is not locally confluent: $h(b) \leftarrow h(a) \rightarrow k(b)$ but $h(b)$ and $k(b)$ are irreducible). Then the remaining construction of $\mathcal{R}^{\mathcal{F}}$ is adapted accordingly.

Example 4.4.4 (JCP is not even preserved for terminating join CTRSs under signature extensions)
Consider the CTRS $\mathcal{R}^{\mathcal{F}}$ with

$$\mathcal{R} = \begin{cases} f(x,y,z) \to g(x) & \Longleftarrow \quad x \downarrow y, y \downarrow z \\ f(x,y,z) \to g(z) & \Longleftarrow \quad x \downarrow y, y \downarrow z \\ h(x) \to k(b) & \Longleftarrow \quad k(x) \downarrow h(b) \\ k(a) \to h(a) & \\ a \to b & \\ g(h(b)) \to g(k(b)) & \\ h(h(b)) \to h(k(b)) & \\ k(h(b)) \to k(k(b)) & \\ f(h(b),x,y) \to f(k(b),x,y) & \\ f(x,h(b),y) \to f(x,k(b),y) & \\ f(x,y,h(b)) \to f(x,y,k(b)) & \end{cases}$$

and $\mathcal{F} = \{a, b, f, g, h, k\}$. It is easy to verify that $\mathcal{R}^{\mathcal{F}}$ is terminating. With some effort one can show (by similar arguments as above) that $\mathcal{R}^{\mathcal{F}}$ has joinable critical pairs. More precisely, the following properties hold:

(1) $\forall s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V}).\, s \leftrightarrow^* t \Longrightarrow depth(s) = depth(t)$ (by induction over the structure of terms and case analysis[10]).

(2) $\forall s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V}), depth(s) = 1.\, s \leftrightarrow^* t \Longrightarrow s \downarrow t$ (by case analysis using (1)).

(3) $\forall s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V}), depth(s) = 2.\, s \leftrightarrow^* t \Longrightarrow g(s) \downarrow g(t)$ (by case analysis using (1),(2)).

(4) $\forall s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V}), depth(s) > 2.\, s \leftrightarrow^* t \Longrightarrow s \downarrow t$ (by structural induction and case analysis using (1)-(3)).

In particular, we note that due to (4) above $\mathcal{R}^{\mathcal{F}}$ is even confluent on all equivalence classes of terms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ (w.r.t. $\leftrightarrow^*$) which have *depth* > 2. But $\mathcal{R}^{\mathcal{F}}$ is not (locally) confluent since we have $h(b) \leftarrow h(a) \to k(b)$ with both $h(b)$ and $k(b)$ irreducible.

Now we add a fresh unary function symbol G, i.e., we consider $\mathcal{R}^{\mathcal{F}'}$ with $\mathcal{F}' = \mathcal{F} \uplus \{G\}$. the new system $\mathcal{R}^{\mathcal{F}'}$ is still terminating (cf. Theorem 5.5.9). But joinability of critical pairs is lost. Consider the critical pair between the first two rules of $\mathcal{R}^{\mathcal{F}'}$,

$$\langle g(x) = g(z) \rangle \ \Longleftarrow \ x \downarrow y, y \downarrow z \,,$$

and the $\mathcal{F}'$-substitution $\tau = \{x \mapsto G(h(b)), y \mapsto G(h(a)), z \mapsto G(k(b))\}$. The corresponding instance of the critical peak is

$$g(G(h(b))) \leftarrow f(G(h(b)), G(h(a)), G(k(b))) \to g(G(k(b)))\,,$$

due to $G(h(b)) \downarrow G(h(a)) \downarrow G(k(b))$, but $g(G(h(b)))$ and $g(G(k(b)))$ are not joinable since they are both irreducible (in $\mathcal{R}^{\mathcal{F}'}$).

For the non-disjoint case, i.e., for unions of constructor sharing or composable CTRSs, no positive result corresponding to the ones mentioned above for confluence and normal form properties (without weak or strong termination assumptions) is currently known.

[10]In particular, one can verify $\forall s \in \mathcal{T}(\mathcal{F}, \mathcal{V}).\, h(s) \to k(b) \iff s = a$.

Further results which require weak or strong termination, in particular concerning semi-completeness ($\mathrm{WN} \wedge \mathrm{CR}$), will be discussed in the next chapter.

Chapter 5

Modularity of Termination Properties

This chapter constitutes the essence of the second major part of the thesis. Modular aspects of termination properties are comprehensively treated here. An overview is provided in Section 5.1. First we give a brief historic account of the crucial papers, ideas, approaches and results that have been obtained up to date. Furthermore basic counterexamples to the modularity of termination in the disjoint union case are collected in a systematic manner. We point out their characteristic features and develop a rough classification of corresponding successful approaches for obtaining positive modularity results for termination. Then, in Section 5.2, the known modularity results for weak termination, weak and strong innermost termination are recapitulated as well as their consequences, for instance concerning semi-completeness. Section 5.3 comprehensively deals with the modularity of various versions of general termination. First we show how, via an abstract structure theorem characterizing minimal counterexamples, many previous results can be generalized and presented in a unifying framework. This powerful abstract structure theorem entails a lot of derived results and criteria for modularity of termination. Then we show how, via a *modular* approach exploiting the modularity of innermost termination and the main results of Chapter 3, further interesting criteria for the preservation of termination and completeness can be obtained relatively easily. And finally, a third basic approach for ensuring modularity of termination is reviewed. In essence, it relies on commutation properties guaranteed by left-linearity and certain uniqueness properties of collapsing reduction. For all three approaches both symmetric and asymmetric criteria for the preservation of termination under disjoint unions are presented. In Section 5.4 it is shown how to extend many previously presented results to combinations of constructor sharing or even of composable systems. Special emphasis is put on the crucial differences of the latter more general combination mechanisms as compared to disjoint unions. And in fact, in most cases the basic ideas and proof techniques for the disjoint union case are also applicable in this more general setting, taking adequately into account the additional sources of complications. Section 5.5 summarizes what corresponding results are known for the conditional rewrite systems. Here we demonstrate in particular that some intuitively appealing assertions are fallacious. Namely, we give counterexamples showing that weak termination as well as weak and strong innermost termination are not even preserved under signature extensions.

5.1 History and Overview

5.1.1 Some History

We begin with some historical remarks.[1] A more detailed discussion of the basic ideas, results and milestones as well as pointers to the relevant literature will follow in the presentation later on.

In his pioneering paper [Toy87b], Toyama did not only establish the modularity of confluence (for disjoint unions), but also demonstrated the non-modularity of termination (see Example 5.1.1 below). This was the starting point of much intensive research devoted to the study of understanding the crucial phenomena responsible for this non-modular behaviour of the termination property as well as of finding interesting sufficient criteria ensuring modularity of termination. In [Toy87a] Toyama collected a couple of further counterexamples and conjectures which gave some orientation of what seemed to be possible and what definitely not. For instance, contrary to a conjecture in [Toy87b], even for confluent TRSs termination (and hence also completeness) turned out not to be modular as shown by a counterexample of Klop & Barendregt in [Toy87a]. Independently, Drosten ([Dro89]) gave a simpler counterexample to the non-modularity of completeness (Example 5.1.2 below is a slightly modified version of the latter). Even for irreducible TRSs (note that the first system in Example 5.1.2 below is not irreducible), completeness is not modular as shown in [Toy87a] by a variation of Klop & Barendregt's counterexample. The same technique can also be applied to Example 5.1.2 below as demonstrated by Middeldorp ([Mid90]).

The first positive results (for weak termination, termination and completeness in the disjoint union case) were obtained by Rusinowitch ([Rus87]), Ganzinger & Giegerich ([GG87]), Middeldorp ([Mid89], [Mid90]), Bergstra, Klop & Middeldorp ([BKM89]), Drosten ([Dro89]), Kurihara & Kaji ([KK90]), and Toyama, Klop & Barendregt ([TKB89]). Modularity of simple termination (for finite systems) was established by Kurihara & Ohuchi ([KO90a]). Results of [Rus87], [Mid89] and [KO90a] were generalized in a unifying manner by the present author in [Gra92a; Gra94a] for the case of finitely branching systems. The latter restriction could be lifted by Ohlebusch ([Ohl94c]). The modularity of completeness for constructor systems was obtained in [MT91; MT93]. This was generalized to overlay systems in [Gra92b; Gra95a] via modularity of innermost termination and sufficient conditions for the equivalence of innermost and general termination. The deep result of [TKB89; TKB95] stating modularity of completeness for left-linear TRSs was further extended and its proof considerably simplified, independently by Marchiori ([Mar95]) and Schmidt-Schauss & Panitz ([SSP94], [SMP95]).

[1] Actually, we don't claim here to give a complete historical survey. This has become non-trivial in the meanwhile since many interesting new papers have been published within the last years.

Non-disjoint unions of (terminating) constructor sharing and, more generally, composable systems were first investigated in [KO92], [MT91; MT93], [Gra92b; Gra92a], [Ohl94a; Ohl94c], and recently also in [KO95a; KO95b], [Ohl95a], [MZ95].

Modularity of termination properties of CTRSs has been studied in [Mid90; Mid93b; Mid93a; Mid94], [Gra93b; Gra95d; Gra94b; Gra96b], [Ohl93; Ohl94a; Ohl95b; Ohl95a].

The preservation behaviour of termination and completeness for hierarchical combinations of (mainly unconditional) TRSs is dealt with in [Kri93; Kri95b], [Der92; Der95], [Gra93a], [Kri94b], [FJ95].

Other preservation results for termination of combined systems can be found e.g. in [BD86], [BL90], [Ges90], [Zan94], [Der95], [FJ95].

5.1.2 Basic Counterexamples

The following counterexamples to modularity of termination that we are going to present now are characteristic in a sense which will be made precise later on.

Example 5.1.1 (termination (SN) is not modular, [Toy87b])
The disjoint TRSs

$$\mathcal{R}_1 = \left\{\ f(a,b,x) \to f(x,x,x) \right.$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{l} G(x,y) \to x \\ G(x,y) \to y \end{array} \right.$$

are terminating,[2] but $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is not, due to the cyclic (hence in particular infinite) derivation

$$\begin{array}{lll} f(a,b,G(a,b)) & \to_{\mathcal{R}_1} & f(G(a,b),G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} & f(a,G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} & f(a,b,G(a,b)) \\ & \to_{\mathcal{R}_1} & \ldots \end{array}$$

Example 5.1.2 (completeness (SN $\wedge$ CR) is not modular, [Dro89])
The disjoint TRSs

$$\mathcal{R}_1 = \left\{ \begin{array}{r} f(a,b,x) \to f(x,x,x) \\ a \to c \\ b \to c \\ f(x,y,z) \to c \end{array} \right.$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{l} K(x,y,y) \to x \\ K(y,y,x) \to x \end{array} \right.$$

[2]For $\mathcal{R}_1$ termination is intuitively obvious, however, a formal proof is not completely trivial (cf. e.g. [Zan94]).

are terminating and confluent hence complete, but again their disjoint union $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ allows cycle:

$$\begin{array}{lll} f(a,b,K(a,c,b)) & \to_{\mathcal{R}_1} & f(K(a,c,b),K(a,c,b),K(a,c,b)) \\ & \to^+_{\mathcal{R}_1} & f(K(a,c,c),K(c,c,b),K(a,c,b)) \\ & \to^+_{\mathcal{R}_2} & f(a,b,K(a,c,b)) \\ & \to_{\mathcal{R}_1} & \cdots \end{array}$$

Example 5.1.3 (SN $\wedge$ CON$^{\to}$ is not modular, [Gra94a])
The disjoint TRSs

$$\mathcal{R}_1 = \left\{ \; f(x,g(x),y) \to f(y,y,y) \right.$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{l} G(x) \to x \\ G(x) \to A \end{array} \right.$$

are terminating. However, in $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ we have the cycle

$$\begin{array}{lll} f(G(g(A)),G(g(A)),G(g(A))) & \to_{\mathcal{R}_2} & f(A,G(g(A)),G(g(A))) \\ & \to_{\mathcal{R}_2} & f(A,g(A),G(g(A))) \\ & \to_{\mathcal{R}_1} & f(G(g(A)),G(g(A)),G(g(A))) \\ & \to_{\mathcal{R}_2} & \cdots \end{array}$$

We observe that these counterexamples to modularity of termination satisfy the following (easily verifiable) properties:

- In all examples one of the systems is duplicating (DUP), i.e., contains a duplicating rule, and not simply terminating, and the other system is collapsing (COL), i.e., contains a collapsing rule, and not non-erasing (NE).
- In Example 5.1.2 both systems are complete, i.e., terminating and confluent (SN $\wedge$ CR), but the first one is not a constructor system and also not an overlay system (OS), and the second one not left-linear (LL).
- In Example 5.1.1 both systems are left-linear (LL), but the second one is not confluent (CR) and also neither consistent (CON) nor consistent with respect to reduction (CON$^{\to}$).
- In Example 5.1.3 both systems are consistent with respect to reduction (CON$^{\to}$) — the first system is even confluent (CR) — however, the first system is not left-linear (LL), and the second one not consistent (CON).

The careful reader may have observed that the counterexample to modularity of completeness, Example 5.1.2 above, involves one system which is not irreducible (namely, $\mathcal{R}_1$). However, this is not essential as shown by Middeldorp [Mid90] (using the same technique as in [Toy87a] for the more complicated counterexample of Klop & Barendregt).

Example 5.1.4 (completeness is not modular for irreducible TRSs, [Toy87a], [Mid90])
The resulting disjoint, irreducible and complete systems are

$$\mathcal{R}_1 = \begin{cases} f(g(x), h(x), y, x) \to f(y, y, y, x) \\ \qquad\qquad\qquad g(a) \to c \\ \qquad\qquad\qquad h(a) \to c \\ \qquad\quad f(x, y, z, a) \to c \end{cases}$$

and

$$\mathcal{R}_2 = \begin{cases} K(x, y, y) \to x \\ K(y, y, x) \to x \end{cases}$$

where, for $t = K(g(a), h(a), h(a))$, the union has a cyclic derivation of the form $f(t, t, t, a) \to^+ f(t, t, t, a)$.

Almost all counterexamples to modularity of termination in the literature (including Examples 5.1.1, 5.1.2, 5.1.4 above) have the property that the minimal rank of infinite derivations in the disjoint union is 3.[3] However, we observe that in Example 5.1.3 the given counterexample has rank 4. In fact, by analyzing there for which mixed terms s, t it is possible that $s \to_{\mathcal{R}} t$ and $s \to_{\mathcal{R}} g(t)$ one can show that the minimal rank of a non-terminating $\mathcal{R}$-derivation is exactly 4. Moreover, Example 5.1.3 can be easily generalized in order to show that the minimal rank of counterexamples may be arbitrarily high.

Example 5.1.5 (minimal counterexamples may have arbitrarily high rank)
Consider the disjoint TRSs

$$\mathcal{R}_1 = \left\{ f(x, g(x), \ldots, g^n(x), y) \to f(y, \ldots, y) \right.$$

for some $n \geq 1$ (f has arity $n + 2$ and $g^n(x)$ stands for the n-fold application of g to x), and

$$\mathcal{R}_2 = \begin{cases} G(x) \to x \\ G(x) \to A \end{cases}$$

Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are clearly terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. For instance, we have the following infinite $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation[4]

$$\begin{array}{rll} f((Gg)^n A, (Gg)^n A, \ldots (Gg)^n A) & \to_{\mathcal{R}_2} & f(A, (Gg)^n A, \ldots, (Gg)^n A) \\ & \to_{\mathcal{R}_2} & f(A, g(Gg)^{n-1} A, \ldots, (Gg)^n A) \\ & \to_{\mathcal{R}_2} & f(A, gA, \ldots, (Gg)^n A) \\ & \vdots & \\ & \to_{\mathcal{R}_2} & f(A, gA, g^2 A, \ldots, g^n A, (Gg)^n A) \\ & \to_{\mathcal{R}_1} & f((Gg)^n A, (Gg)^n A, \ldots, (Gg)^n A) \\ & \to_{\mathcal{R}_2} & \cdots \end{array}$$

[3]Counterexamples of rank 1 or 2 are impossible as will be seen later on.

[4]The notation used here should be self-explanatory. For example, $(Gg)^2(A)$ stands for $G(g(G(g(A))))$.

of rank $2n+2$. Again a careful analysis of possible reductions shows that for this example $2n+2$ is the minimal rank of any conceivable non-terminating $(\mathcal{R}_1 \oplus \mathcal{R}_2)$-derivation. Moreover, it is straightforward to modify the examples in such a way that only finite signatures with function symbols of (uniformly) bounded arities are involved. For instance, one may use a binary f' and the encoding $f'(x_1, f'(x_2, \ldots, f'(x_{n-1}, x_n) \ldots))$ for $f(x_1, \ldots, x_n)$.

This example shows that, when analyzing the reasons for non-modularity of termination in general, there is no hope of being able to restrict attention to terms of rank 3 or any fixed $n \in \mathbb{N}$.

5.1.3 Classification of Approaches

Before going into details now let us give a rough idea of the approaches that have been developed for analyzing the non-modularity of termination and for proving corresponding positive results. Actually, all analyses implicitly or explicitly rely on characteristic properties of minimal counterexamples. Such properties essentially are of the following form:[5]

> If the union of two disjoint terminating systems $\mathcal{R}_1$ and $\mathcal{R}_2$ (having some properties) is non-terminating, then a minimal counterexample in the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ must enjoy certain properties, and consequently $\mathcal{R}_1$ and $\mathcal{R}_2$ must satisfy certain corresponding properties.

Usually, from such characteristic properties one may directly infer corresponding *symmetric* and *asymmetric* positive criteria, i.e., statements of the form

> If $\mathcal{R}_1$ and $\mathcal{R}_2$ are both terminating and both satisfy certain properties, then their disjoint union is terminating, too (and satisfies certain properties)

and

> If $\mathcal{R}_1$ and $\mathcal{R}_2$ are both terminating, one of the systems satisfies some properties and the other one satisfies some other properties, then their disjoint union is terminating, too (and satisfies certain properties)

respectively. As to the current state of the art in the field, we think there are basically three different approaches in which the types of results have respective counterparts concerning the essential proof structures and ideas:

- the **general approach via an abstract structure theorem** where the basic idea is to reduce non-termination in the union to non-termination of a slightly modified generic version of one of the systems,

[5]The formulation here accounts only for the disjoint union case, however, in most cases generalizations to at least constructor sharing and composable systems are possible and natural.

- the **modular approach via modularity of innermost termination** where sufficient criteria for the equivalence of innermost termination (SIN) and general termination (SN) are combined with the (not so difficult to establish) modularity of SIN, and

- the **syntactic approach via left-linearity** which in essence is based on commutation and uniqueness properties for rewriting in left-linear systems.

5.2 Restricted Termination Properties

5.2.1 Weak Termination and Weak Innermost Termination

A first easy result is the following preservation property for normal forms which is a direct consequence of the equality $\rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} = \rightarrow_{\mathcal{R}_1} \cup \rightarrow_{\mathcal{R}_2}$.

Lemma 5.2.1 (preservation of normal forms, [Mid90])
If $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ are disjoint TRSs (with $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$) then $\mathrm{NF}(\mathcal{R}^{\mathcal{F}}) = \mathrm{NF}({\mathcal{R}_1}^{\mathcal{F}}) \cap \mathrm{NF}({\mathcal{R}_2}^{\mathcal{F}})$.

Theorem 5.2.2 (weak termination (WN) is modular, [BKM89], [Dro89], [KK90])
Weak termination is modular for disjoint unions of TRSs.

Proof: The implication $\mathrm{WN}(\mathcal{R}^{\mathcal{F}}) \implies \mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1}) \wedge \mathrm{WN}(\mathcal{R}_2^{\mathcal{F}_2})$ is straightforward. Conversely, following the presentation in [Mid90] we show by induction on $rank(t)$ that every term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ has a normal form w.r.t. $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$.[6] If $rank(t) = 1$ then $\mathrm{WN}(t, \rightarrow)$ follows from the assumptions $\mathrm{WN}(\mathcal{R}_1)$, $\mathrm{WN}(\mathcal{R}_2)$ (and the fact that $\mathcal{R}_1$-rules do not introduce $\mathcal{F}_2$-symbols and vice versa).

If $rank(t) > 1$ then we may assume $t = C[\![t_1, \ldots, t_n]\!]$ such that w.l.o.g. t is top black (i.e., $root(t) \in \mathcal{F}_1$). By induction hypothesis, every t_i $(1 \leq i \leq n)$ has a normal form t_i' (w.r.t. $\mathcal{R}$). Hence we get

$$t = C[\![t_1, \ldots, t_n]\!] \rightarrow^*_{\mathcal{R}} C[t_1', \ldots, t_n'] = C' \{\!\{ s_1, \ldots, s_m \}\!\}$$

for some black context $C' \{\!\{ , \ldots, \}\!\}$ and top white normal forms $s_1, \ldots, s_m$ (w.r.t. $\mathcal{R}$). Choose fresh variables $x_1, \ldots, x_m$ with $\langle s_1, \ldots, s_m \rangle \infty \langle x_1, \ldots, x_m \rangle$. Because $rank(C' \{\!\{ x_1, \ldots, x_m \}\!\}) = 1$, the term $C' \{\!\{ x_1, \ldots, x_m \}\!\}$ has a normal form, say

$$C' \{\!\{ x_1, \ldots, x_m \}\!\} \rightarrow^*_{\mathcal{R}_1} C'' \langle x_{i_1}, \ldots, x_{i_p} \rangle \ .$$

Hence we have the following derivation:

$$t \rightarrow^*_{\mathcal{R}} C' \{\!\{ s_1, \ldots, s_m \}\!\} \xrightarrow{o}{}^*_{\mathcal{R}_1} C'' \langle\!\langle s_{i_1}, \ldots, s_{i_p} \rangle\!\rangle =: t' \ .$$

Clearly, $t' \in \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}})$. By construction we have $t' \in \mathrm{NF}(\xrightarrow{o}_{\mathcal{R}_1})$, and since $s_{i_1}, \ldots, s_{i_p} \in \mathrm{NF}(\mathcal{R}^{\mathcal{F}})$ we also have $t' \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$. The normal form property of Lemma 5.2.1 yields

[6] We remark that a very similar proof by induction over the term structure is possible here, too. Indeed, in some sense structural induction is a refinement of induction over the rank.

$t' \in \mathrm{NF}(\mathcal{R}^{\mathcal{F}})$. We conclude that every term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ has a normal from w.r.t. $\mathcal{R}^{\mathcal{F}}$. ∎

Combining Theorem 5.2.2 with Toyama's Theorem 4.1.2 immediately yields the following.

Theorem 5.2.3 (semi-completeness (CR $\wedge$ WN) is modular)
Semi-completeness is modular for disjoint unions of TRSs.

Theorem 5.2.4 (weak innermost termination (WIN) is modular)
Weak innermost termination is modular for disjoint unions of TRSs.

Proof: The proof is analogous to the one for Theorem 5.2.2. The only additional argument needed (in the induction step of the non-trivial direction) is that[7]

$$C' \{\!\{ x_1, \ldots, x_m \}\!\} \xrightarrow[i]{}^{*}_{\mathcal{R}_1} C'' \langle x_{i_1}, \ldots, x_{i_p} \rangle$$

implies

$$C' \{\!\{ s_1, \ldots, s_m \}\!\} \xrightarrow[i]{}_{\mathcal{R}_1} C'' \langle\!\langle s_{i_1}, \ldots, s_{i_p} \rangle\!\rangle \ ,$$

too (i.e., the instantiated steps are still innermost), for top white normal forms $s_1, \ldots, s_m$ (w.r.t. $\mathcal{R}$). ∎

We remark that modularity of weak termination (Theorem refweak-termination-is-modular) can also be proved via the interesting notion *modular reduction*.

Definition 5.2.5 (modular reduction, [KK90])
Let $\mathcal{R}_1^{\mathcal{F}_1}, \ldots, \mathcal{R}_n^{\mathcal{F}_n}$ be pairwise disjoint TRSs with $\mathcal{R}^{\mathcal{F}}$ denoting their disjoint union (i.e., $\mathcal{R} = \biguplus_{i=1}^{n} \mathcal{R}_i$, $\mathcal{F} = \biguplus_{i=1}^{n} \mathcal{F}_i$). For $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ we define

- $s \rightsquigarrow_{\mathcal{R}_i} t$ if $s \rightarrow^{+}_{\mathcal{R}_i} t$ and t is a normal form w.r.t. $\mathcal{R}_i$.
- $s \rightsquigarrow t$ if $s \rightsquigarrow_{\mathcal{R}_i} t$ for some i, $1 \leq i \leq n$ (the relation $\rightsquigarrow$ is called *modular reduction* in [KK90]).

For any set $X \subseteq \{1, \ldots, n\}$ the relation $\bigcup_{i \in X} \rightsquigarrow_{\mathcal{R}_i}$ is also denoted by $\rightsquigarrow_X$.

Theorem 5.2.6 (modular reduction is terminating, [KK90], [Mid90])
Let $\mathcal{R}_1^{\mathcal{F}_1}, \ldots, \mathcal{R}_n^{\mathcal{F}_n}$ be pairwise disjoint TRSs. The relation $\rightsquigarrow_X$ is terminating for all $X \subseteq \{1, \ldots, n\}$.

Proof: By induction on n, cf. [KK90], [Mid90]. ∎

Note that the modularity of weak termination, Theorem 5.2.2, is an easy consequence of Theorem 5.2.6 above (due to the fact that for pairwise disjoint weakly terminating TRSs $\mathcal{R}_1^{\mathcal{F}_1}, \ldots, \mathcal{R}_n^{\mathcal{F}_n}$, with $\mathcal{R}^{\mathcal{F}} = \bigoplus_{i=1}^{n} \mathcal{R}_i^{\mathcal{F}_i}$, the sets of normal forms of ordinary and of modular reduction coincide on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ ([Mid90]): $\mathrm{NF}(\mathcal{R}) = \mathrm{NF}(\rightsquigarrow)$).

[7]Note that notions for innermost reduction (in the disjoint union) like $\xrightarrow[i]{}_{\mathcal{R}_1}$ are always to be interpreted in the sense that the indicated steps are innermost w.r.t. the union (which in general is different from being innermost w.r.t. one of the component systems!).

5.2.2 Strong Innermost Termination

Interestingly, (strong) innermost termination is modular, too, as shown in [Gra93a; Gra95a].

Theorem 5.2.7 (innermost termination (SIN) is modular)
Innermost termination is modular for disjoint unions of TRSs.

Proof: The implication $\mathrm{SIN}(\mathcal{R}^{\mathcal{F}}) \Longrightarrow \mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1}) \wedge \mathrm{SIN}(\mathcal{R}_2^{\mathcal{F}_2})$ is straightforward. Conversely, assume that there exists a term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ which is not innermost terminating (w.r.t. $\mathcal{R}^{\mathcal{F}}$). Consider a minimal such counterexample, i.e., a term t which is not SIN, but all its proper subterms are SIN. Since t cannot be a variable, it is of the form $t = f(t_1, \ldots, t_n)$, let's say top black (i.e., with $f \in \mathcal{F}_1$), and allows an infinite innermost derivation

$$t = f(t_1, \ldots, t_n) \xrightarrow[i]{}^{*} f(t'_1, \ldots, t'_n) \xrightarrow[i]{}_{\lambda} \cdots ,$$

where $t'_1, \ldots, t'_n$ are all irreducible. Since all top white principal aliens in $t' := f(t'_1, \ldots, t'_n)$ are irreducible, this derivation is an infinite innermost $\mathcal{R}_1$-derivation. Now injective abstraction of all principal top white aliens in t' — i.e., replacing, for $t' = C^b\{\!\{ t_1, \ldots, t_m \}\!\}$, the principal top white aliens t_j by fresh variables x_j with $\langle t_1, \ldots, t_m \rangle \infty \langle x_1, \ldots, x_m \rangle$ — yields an infinite innermost $\mathcal{R}_1$-derivation on pure black terms contradicting innermost termination of $\mathcal{R}_1^{\mathcal{F}_1}$.[8] ∎

5.3 Termination

5.3.1 The General Approach via an Abstract Structure Theorem

A Structural Analysis of Minimal Counterexamples

The crucial point of our approach to be developed is that from a minimal counterexample with all terms let's say top black we can construct an 'almost pure black' counterexample, in a slightly extended version of the black system.

Before formally stating and proving the corresponding abstract structure theorem we

[8] Note that *identifying abstraction*, i.e., taking the *same* fresh variable for all principal aliens, instead of *injective abstraction* as here, does not preserve innermost redexes in general! For instance, take $\mathcal{R}_1 = \{f(x,x) \to a, g(f(x,y)) \to g(f(x,y))\}$ over $\mathcal{F}_1 = \{a, f, g\}$ and $\mathcal{R}_2 = \emptyset$ over $\mathcal{F}_2 = \{A, B\}$. Then the mixed term $g(f(A,B))$ of rank 2 is only reducible at the root (and hence the corresponding step $g(f(A,B)) \to_{\lambda,\mathcal{R}_1} g(f(A,B))$ is innermost). However, identifying abstraction yields (the pure term) $g(f(x,x))$ such that all innermost derivations issuing from it have the form $g(f(x,x)) \xrightarrow[i]{}_{1,\mathcal{R}_1} g(a)$. Hence, $g(f(x,x))$ is innermost terminating despite the fact that $\mathcal{R}_1$ is not (for instance, $g(f(x,y))$ is not even weakly innermost terminating). The general problem is that by identifying abstraction a proper irreducible subterm of the original innermost redex may become reducible thus preventing the same rule application as before from being innermost.

shall now illustrate the essential ideas and construction steps via Example 5.1.2 above. In the union of the disjoint terminating TRSs

$$\mathcal{R}_1 = \begin{cases} f(a,b,x) \to f(x,x,x) \\ a \to c \\ b \to c \\ f(x,y,z) \to c \end{cases}$$

and

$$\mathcal{R}_2 = \begin{cases} K(x,y,y) \to x \\ K(y,y,x) \to x \end{cases}$$

we consider the following infinite derivation (recall that inner reduction steps take place within principal aliens, and outer ones in the top layers):

$$\begin{array}{llll} D: & f(a,b,K(a,b,b)) & \xrightarrow{o}_{\mathcal{R}_1} f(K(a,b,b),K(a,b,b),K(a,b,b)) & (1) \\ & & \xrightarrow{i}_{\mathcal{R}_2} f(a,K(a,b,b),K(a,b,b)) & (2) \\ & & \xrightarrow{i}_{\mathcal{R}_1} f(a,K(c,b,b),K(a,b,b)) & (3) \\ & & \xrightarrow{i}_{\mathcal{R}_1} f(a,K(c,c,b),K(a,b,b)) & (4) \\ & & \xrightarrow{i}_{\mathcal{R}_2} f(a,b,K(a,b,b)) & (5) \\ & & \xrightarrow{o}_{\mathcal{R}_1} \cdots. & \end{array}$$

Obviously, the crucial steps which enable this derivation to be infinite (and even cyclic) are the inner reductions (2)-(5), in particular the steps (2) and (5) which are destructive at level 2. They modify substantially the topmost homogeneous black layer thereby enabling an outer black (i.e., $\mathcal{R}_1$-) reduction step previously not possible. The idea now is to abstract from the concrete form of these inner steps but retain the essential (black) information which permits subsequent outer steps. For that purpose it is sufficient to consider the principal top white, i.e. $\mathcal{F}_2$-rooted, aliens and collect those top black, i.e., $\mathcal{F}_1$-rooted, terms to which the former may reduce. In other words, colour changing derivations issued by principal aliens are essential. The coding of the collected top black successors of some principal top white alien will be achieved by some new function symbol(s) which in a sense serve(s) for abstracting from the concrete form of white layers while keeping only the 'layer separating' information. Since in general also top black aliens hidden in deeper layers may eventually pop up (in possibly modified form), the whole process has to be performed in a recursive fashion in general (which is not necessary in the example). After this abstracting transformation process sequences of inner reduction steps like (2)-(5) above in the original derivation may be simulated by ('deletion' and) 'subterm'[9] steps in the transformed derivation. In order to explain this in more detail let us choose H as a new (varyadic) layer separating function symbol.

[9]or 'projection (embedding)'

Then we get the transformed derivation

$$\begin{array}{llll} D': & f(a,b,H(a,b,c)) & \xrightarrow{o}_{\mathcal{R}_1} f(H(a,b,c),H(a,b,c),H(a,b,c)) & (1') \\ & & \xrightarrow{i}_{\mathcal{R}_2'} f(a,H(a,b,c),H(a,b,c)) & (2') \\ & & \xrightarrow{i}_{\mathcal{R}_2'} f(a,H(b,c),H(a,b,c)) & (3') \\ & & = \quad f(a,H(b,c),H(a,b,c)) & (4') \\ & & \xrightarrow{i}_{\mathcal{R}_2'} f(a,b,H(a,b,c)) & (5') \\ & & \xrightarrow{o}_{\mathcal{R}_1} \ldots & \end{array}$$

where $\mathcal{R}_1$ is as above and $\mathcal{R}_2' = \mathcal{R}_{sub}^H \cup \mathcal{R}_{del}^H$ with[10]

$$\begin{array}{lcl} \mathcal{R}_{sub}^H & = & \{H(x_1,\ldots,x_j,\ldots,x_n) \to x_j | 1 \le j \le n\} \ , \\ \mathcal{R}_{del}^H & = & \{H(x_1,\ldots,x_j,\ldots,x_n) \to H(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n) | 1 \le j \le n\}. \end{array}$$

The top white principal alien $t = K(a,b,b)$ of the top black starting term $s = f(a,b,K(a,b,b))$ of D can be reduced (in finitely many steps) to the top black successors a, b and c. Hence, the abstracting transformation of t yields $H(a,b,c)$ and the whole starting term s is transformed into $f(a,b,H(a,b,c))$. Furthermore, any outer step in D corresponds to an outer step in D' using the same rule. Any inner step in D which is not destructive at level 2, e.g. (3) and (4), corresponds in D' to a (possibly empty) sequence of inner $\mathcal{R}_2'$-steps not destructive at level 2 (here (3') and (4'), respectively). Any inner step in D which is destructive at level 2 (hence collapsing), e.g., (2) and (5), corresponds in D' to an $\mathcal{R}_{sub}^H$-step (here (2') and (5'), respectively).

Observe that the sketched encoding of all top black successors of a top white principal (terminating) alien presupposes that the latter successor set is finite. This is guaranteed if the involved TRSs are finite, or, more generally, finitely branching. In that case one may simply apply Kőnigs Lemma 2.1.16. The following result provides a characterization of the property of TRSs to be finitely branching.

Lemma 5.3.1 (characterization of finitely branching TRSs)
A (possibly infinite) TRS $\mathcal{R}^{\mathcal{F}}$ is finitely branching if and only if for every rule $l \to r \in \mathcal{R}^{\mathcal{F}}$ there are only finitely many different rules in $\mathcal{R}^{\mathcal{F}}$ with the same left hand side l. [11]

Proof: Consider an arbitrary term $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$ and possible $\mathcal{R}^{\mathcal{F}}$-reductions. Clearly, there are only finitely many different left hand sides of rules in $\mathcal{R}^{\mathcal{F}}$ which can match some subterm of s. Hence, the set of one-step-successors of s can be infinite only in the case that there are infinitely many different rules in $\mathcal{R}^{\mathcal{F}}$ with the same left hand side. The only-if-direction of the lemma is trivial. ∎

[10] Note that $\mathcal{R}_{sub}^H = \mathcal{E}mb(\{H\})$ (cf. Definition 2.2.31). For the sake of readability we prefer here the notations $\mathcal{R}_{sub}^H$, $\mathcal{R}_{del}^H$.

[11] Note that rules which can be obtained from one another by renaming variables are considered to be equal!

Corollary 5.3.2 (finitely branching is modular)
The property of being finitely branching is modular for arbitrary finite unions of arbitrary TRSs.[12]

Let us continue now with our example above. In order to stay within the usual setting of fixed-arity function symbols we modify the described transformation by taking a new binary function symbol G and a new constant A instead of the varyadic symbol H. With the correspondence

$$H(t_1, \ldots, t_n) = \begin{cases} A & \text{if} \quad n = 0 \\ G(t_1, G(t_2, \ldots G(t_{n-1}, G(t_n, A)) \ldots)) & \text{if} \quad n > 0 \end{cases}$$

the above construction easily carries over and we obtain the derivation D'':

$$\begin{array}{ll} & f(a, b, G(a, G(b, G(c, A)))) \\ \xrightarrow{o}_{\mathcal{R}_1} & f(G(a, G(b, G(c, A))), G(a, G(b, G(c, A))), G(a, G(b, G(c, A)))) \\ \xrightarrow{i}_{\mathcal{R}_2''} & f(a, G(a, G(b, G(c, A))), G(a, G(b, G(c, A)))) \\ \xrightarrow{i}_{\mathcal{R}_2''} & f(a, G(b, G(c, A)), G(a, G(b, G(c, A)))) \\ = & f(a, G(b, G(c, A)), G(a, G(b, G(c, A)))) \\ \xrightarrow{i}_{\mathcal{R}_2''} & f(a, b, G(a, G(b, G(c, A)))) \\ \xrightarrow{o}_{\mathcal{R}_1} & \ldots . \end{array}$$

Here, $\mathcal{R}_2''$ is to be interpreted as $\mathcal{R}_2'' = \mathcal{R}_{sub}^G$ with

$$\mathcal{R}_{sub}^G \quad = \quad \{G(x, y) \to x,\ G(x, y) \to y\},$$

i.e., deletion rules are not necessary any more. In the following formal presentation we shall use this latter transformation. First we prove some easy structural properties of minimal counterexamples.

Lemma 5.3.3 (structural properties of minimal counterexamples)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be two terminating disjoint TRSs such that

$$D : s_1 \to s_2 \to s_3 \to \ldots$$

is an infinite derivation in $\mathcal{R}_1 \oplus \mathcal{R}_2$ of minimal rank involving only ground[13] terms. Suppose that s_1 is top black, i.e., $\mathcal{F}_1$-rooted. Then all s_i, $1 \leq i$, are top black, and we have:

(a) $rank(D) \geq 3$.

(b) Infinitely many steps in D are outer $\mathcal{R}_1$-steps.

[12]More precisely, this means: If $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \cup \ldots \cup \mathcal{R}_n^{\mathcal{F}_n}$, then $\mathcal{R}^{\mathcal{F}}$ is finitely branching if and only if every $\mathcal{R}_i^{\mathcal{F}_i}$ $(1 \leq i \leq n)$ is finitely branching. Clearly, infinite unions need not preserve the property of being finitely branching.

[13]This may be assumed w.l.o.g. since all variables in a minimal counterexample may be replaced by a constant from $\mathcal{F}_i$ (where i is determined by the 'colour' i of the deepest layer in s_1) yielding a counterexample of the same minimal rank. If $\mathcal{F}_i$ does not contain a constant one may simply add a fresh one. This does not affect the termination behaviour of $\mathcal{R}_i$ and of $\mathcal{R}_1 \oplus \mathcal{R}_2$ as is easily seen.

(c) Infinitely many steps in D are inner $\mathcal{R}_2$-reductions which are destructive at level 2.

Proof: That all s_i are top black is an obvious consequence of the minimality assumption concerning $rank(D)$.

(a) Follows from (c) since whenever $s_i \xrightarrow{i} s_{i+1}$ is destructive at level 2 then $rank(s_i) \geq 3$ (because the s_i's are ground terms).

(b) First we observe that all outer steps in D must be $\xrightarrow{o}_{\mathcal{R}_1}$-steps. Assume for a proof by contradiction that only finitely many steps in D are outer ones. We may further assume w.l.o.g. that no step in D is an outer one. Hence, for $s_1 = C^b[\![t_1, \ldots, t_n]\!]$ all reductions in D are inner ones and take place below one of the positions of the t_i's. Since D is infinite we conclude by the pigeon hole principle that at least one of the t_i's initiates an infinite derivation whose rank is smaller than $rank(D)$. But this is a contradiction to the minimality assumption concerning $rank(D)$.

(c) For a proof by contradiction assume that there are only finitely many steps in D which are destructive at level 2. We may further assume w.l.o.g. that no inner step in D is destructive at level 2. Then, by identifying abstraction, i.e., defining $\widetilde{s_i} = top(s_i)$, any outer step $s_i \xrightarrow{o}_{\mathcal{R}_1} s_{i+1}$ in D yields $\widetilde{s_i} \rightarrow \widetilde{s_{i+1}}$ using the same rule from $\mathcal{R}_1$ and for every inner step $s_i \xrightarrow{i} s_{i+1}$ we have $\widetilde{s_i} = \widetilde{s_{i+1}}$. Since all the s_i's are top black, i.e. $\mathcal{F}_1$-rooted, we can conclude that $\mathcal{R}_1$ is non-terminating which yields a contradiction.

■

Next we formalize the transformation process illustrated above.

Definition 5.3.4 (abstracting transformation)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be two finitely branching terminating disjoint TRSs, $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$ and $n \in \mathbb{N}$ such that for every $s \in \mathcal{T}(\mathcal{F})$ with $rank(s) \leq n$ there is no infinite $\mathcal{R}$-derivation starting with s. Moreover, let $<_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})}$ be some arbitrary, but fixed total ordering on $\mathcal{T}(\mathcal{F}_1 \uplus \{A, G\})$. Then the $\mathcal{F}_2$- (or *white*) *abstraction* is defined to be the mapping

$$\Phi : \mathcal{T}(\mathcal{F})^{\leq n} \uplus \{t \in \mathcal{T}(\mathcal{F})^{n+1} | root(t) \in \mathcal{F}_1\} \quad \longrightarrow \quad \mathcal{T}(\mathcal{F}_1 \uplus \{A, G\})$$

given by

$$\Phi(t) := \begin{cases} t & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_1) \\ A & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_2) \\ C^b[\![\Phi(t_1), \ldots, \Phi(t_m)]\!] & \text{if} \quad t = C^b[\![t_1, \ldots, t_m]\!] \\ CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(t)))) & \text{if} \quad t = C^w[\![t_1, \ldots, t_m]\!] \end{cases}$$

with[14]

$$\begin{array}{rcl}
SUCC^{\mathcal{F}_1}(t) & := & \{t' \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2) | t \to^*_{\mathcal{R}} t' \, , root(t') \in \mathcal{F}_1\} \, , \\
\Phi^*(M) & := & \{\Phi(t) | t \in M\} \quad \text{for} \quad M \subseteq dom(\Phi) \, , \\
CONS(\langle\rangle) & := & A \, , \\
CONS(\langle s_1, \ldots, s_{k+1}\rangle) & := & G(s_1, CONS(\langle s_2, \ldots, s_{k+1}\rangle)) \quad \text{and} \\
SORT(\{s_1, \ldots, s_k\}) & := & \langle s_{\pi(1)}, \ldots, s_{\pi(k)}\rangle \, ,
\end{array}$$

such that $s_{\pi(j)} \leq_{\mathcal{T}(\mathcal{F}_1 \uplus \{A,G\})} s_{\pi(j+1)}$ for $1 \leq j < k$.

Intuitively, for computing $\Phi(t)$ one proceeds top-down in a recursive fashion. Top black layers are left invariant whereas (for the case of top black t) the principal top white subterms are transformed by computing for every such top white subterm the set of possible top black successors, abstracting the resulting terms recursively, sorting the resulting set of abstracted terms and finally constructing again an ordinary term by means of using the new constant symbol A (for empty arguments sets) and the new binary function symbol G (for non-empty argument sets). The sorting process and the total ordering involved here are due to some proof-technical subtleties which will become clear later on. For illustration let us consider again our example from above. Here the white abstraction of the s_i's in the original derivation D yields e.g. (using alphabetical sorting)

$$\begin{array}{rcl}
\Phi(s_1) & = & \Phi(f(a, b, K(a, b, b))) = f(a, b, \Phi(K(a, b, b))) \\
& = & f(a, b, CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(K(a, b, b)))))) \\
& = & f(a, b, CONS(SORT(\Phi^*(\{a, b, c\})))) \\
& = & f(a, b, CONS(SORT(\{a, b, c\}))) \\
& = & f(a, b, CONS(\langle a, b, c\rangle)) = f(a, b, G(a, G(b, G(c, A)))) \, , \\
\Phi(s_3) & = & \Phi(f(a, K(a, b, b), K(a, b, b))) = f(a, \Phi(K(a, b, b), \Phi(K(a, b, b)))) \\
& = & f(a, CONS(SORT(\Phi^*(\{a, b, c\})))^2) \\
& = & f(a, CONS(SORT(\{a, b, c\})), CONS(SORT(\{a, b, c\}))) \\
& = & f(a, CONS(\langle a, b, c\rangle), CONS(\langle a, b, c\rangle)) \\
& = & f(a, G(a, G(b, G(c, A))), G(a, G(b, G(c, A)))) \quad \text{and} \\
\Phi(s_4) & = & \Phi(f(a, K(c, b, b), K(a, b, b))) = f(a, \Phi(K(c, b, b), \Phi(K(a, b, b)))) \\
& = & f(a, CONS(SORT(\Phi^*(\{b, c\}))), CONS(SORT(\Phi^*(\{a, b, c\})))) \\
& = & f(a, CONS(SORT(\{b, c\})), CONS(SORT(\{a, b, c\}))) \\
& = & f(a, CONS(\langle b, c\rangle), CONS(\langle a, b, c\rangle)) \\
& = & f(a, G(b, G(c, A)), G(a, G(b, G(c, A)))).
\end{array}$$

Note that the subterm rewrite step $\Phi(s_3) \to \Phi(s_4)$ reducing $G(a, G(b, G(c, A)))$ to $G(b, G(c, A))$ would not have been possible if we had sorted $\{b, c\}$ as $\langle b, c\rangle$ and $\{a, b, c\}$ as $\langle c, b, a\rangle$.

In the following we shall implicitly use the convention that notions like *rank* or *inner* and *outer* reduction steps have to be interpreted w.r.t. some specific disjoint union which is clear from the context.

[14]Note that the sorting process here is necessary for well-definedness of Φ. It uniquely determines the encoding of sets of terms.

The next lemmas capture the important properties of the above defined abstracting transformation.

Lemma 5.3.5 (Φ is rank decreasing)
Let $\mathcal{R}_1$, $\mathcal{R}_2$ and Φ be given as in Definition 5.3.4. Then, Φ is rank decreasing, i.e. for any $s \in dom(\Phi)$ we have $rank(\Phi(s)) \leq rank(s)$.

Proof: By an easy induction on $rank(s)$ using the definition of Φ. ■

Reduction steps in $(\mathcal{R}_1 \oplus \mathcal{R}_2)$ can be translated in corresponding (sequences of) reduction steps in $\mathcal{R}_1 \oplus \{G(x,y) \to x, G(x,y) \to y\}$ as follows.

Lemma 5.3.6 (properties of the abstracting transformation)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$, $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$, $\mathcal{R}_2' = \mathcal{R}_{sub}^G = \{G(x,y) \to x,\ G(x,y) \to y\}$, n and the (white) $\mathcal{F}_2$-abstraction Φ be given as in Definition 5.3.4. Then, for any $s,t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ with $rank(s) \leq n+1$, $root(s) \in \mathcal{F}_1$ and $s \to_{\mathcal{R}} t$ we have:

(a) If $s \overset{o}{\to}_{\mathcal{R}_1} t$ is not destructive at level 1 then $\Phi(s) \overset{o}{\to}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also not destructive at level 1.

(b) If $s \overset{o}{\to}_{\mathcal{R}_1} t$ is destructive at level 1 then $\Phi(s) \overset{o}{\to}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule, and moreover this step is also destructive at level 1.

(c) If $s \overset{i}{\to}_{\mathcal{R}} t$ is not destructive at level 2 then $\Phi(s) \overset{i}{\to}{}^{*}_{\mathcal{R}_2'} \Phi(t)$ with all steps not destructive at level 2.

(d) If $s \overset{i}{\to}_{\mathcal{R}} t$ is destructive at level 2 then $\Phi(s) \overset{i}{\to}{}^{+}_{\mathcal{R}_2'} \Phi(t)$ such that exactly one of these steps is destructive at level 2.

Proof: Under the assumptions of the lemma assume that $s,t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2)$ are given with $rank(s) \leq n+1$, $root(s) \in \mathcal{F}_1$ and $s \to_{\mathcal{R}} t$.

(a) If $s \overset{o}{\to}_{\mathcal{R}_1} t$ is not destructive at level 1 then we have $s = C\{\{s_1,\ldots,s_m\}\}$, $t = C'\{\{s_{i_1},\ldots,s_{i_k}\}\}$, $1 \leq i_j \leq m$, $1 \leq j \leq k$ for some contexts $C\{\{,\ldots,\}\}$, $C'\{\{,\ldots,\}\}$. By definition of Φ this implies $\Phi(s) = C\{\{\Phi(s_1),\ldots,\Phi(s_m)\}\}$ and $\Phi(t) = C'\{\{\Phi(s_{i_1}),\ldots,\Phi(s_{i_k})\}\}$, hence also $\Phi(s) \overset{o}{\to}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule because of $\langle s_1,\ldots,s_m\rangle \propto \langle \Phi(s_1),\ldots,\Phi(s_m)\rangle$. Clearly, $\Phi(s) \overset{o}{\to}_{\mathcal{R}_1} \Phi(t)$ is not destructive at level 1, too.

(b) If $s \overset{o}{\to}_{\mathcal{R}_1} t$ is destructive at level 1 then we have $s = C[\![s_1,\ldots,s_m]\!]$, $t = s_j$ for some j with $1 \leq j \leq m$ and some context $C[,\ldots,]$. By definition of Φ this implies $\Phi(s) = C[\![\Phi(s_1),\ldots,\Phi(s_m)]\!]$ and $\Phi(t) = \Phi(s_j)$, hence also $\Phi(s) \overset{o}{\to}_{\mathcal{R}_1} \Phi(t)$ using the same $\mathcal{R}_1$-rule because of $\langle s_1,\ldots,s_m\rangle \propto \langle \Phi(s_1),\ldots,\Phi(s_m)\rangle$. Clearly, $\Phi(s) \overset{o}{\to}_{\mathcal{R}_1} \Phi(t)$ is destructive at level 1, too.

(c) If $s \overset{i}{\to}_{\mathcal{R}} t$ is not destructive at level 2 then we have $s = C[\![s_1,\ldots,s_j,\ldots,s_m]\!]$, $t = C[\![s_1,\ldots,s_j',\ldots,s_m]\!]$, $s_j \to_{\mathcal{R}} s_j'$ for some j with $1 \leq j \leq m$ and some context $C[,\ldots,]$. By definition of Φ this implies $\Phi(s) = C[\![\Phi(s_1),\ldots,\Phi(s_j),\ldots,\Phi(s_m)]\!]$

and $\Phi(t) = C[\![\Phi(s_1), \ldots, \Phi(s'_j), \ldots, \Phi(s_m)]\!]$. Since s_j, s'_j are top white, i.e. $\mathcal{F}_2$-rooted, we get $\Phi(s_j) = A = \Phi(s'_j)$ for the case $s_j \in \mathcal{T}(\mathcal{F}_2)$ and $\Phi(s_j) = CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(s_j))))$, $\Phi(s'_j) = CONS(SORT(\Phi^*(SUCC^{\mathcal{F}_1}(s'_j))))$, otherwise. Since $s_j \to_{\mathcal{R}} s'_j$ this implies $SUCC^{\mathcal{F}_1}(s_j) \supseteq SUCC^{\mathcal{F}_1}(s'_j)$ and $SORT(\Phi^*(SUCC^{\mathcal{F}_1}(s_j))) \supseteq SORT(\Phi^*(SUCC^{\mathcal{F}_1}(s'_j)))$, hence $\Phi(s_j) \to^*_{\mathcal{R}'_2} \Phi(s'_j)$ (by Definition of Φ) and also $\Phi(s) \stackrel{i}{\to}{}^*_{\mathcal{R}'_2} \Phi(t)$ with no step destructive at level 2.

(d) If $s \stackrel{i}{\to}_{\mathcal{R}} t$ is destructive at level 2 then we have $s = C[\![s_1, \ldots, s_j, \ldots, s_m]\!]$, $t = C[s_1, \ldots, s'_j, \ldots, s_m]$ with $s_j \to_{\mathcal{R}} s'_j$ colour changing for some j with $1 \leq j \leq m$ and some context $C[, \ldots,]$. By definition of Φ this implies $\Phi(s) = C[\![\Phi(s_1), \ldots, \Phi(s_j), \ldots, \Phi(s_m)]\!]$ and $\Phi(t) = C[\Phi(s_1), \ldots, \Phi(s'_j), \ldots, \Phi(s_m)]$. Moreover, $s'_j \in SUCC^{\mathcal{F}_1}(s_j)$, hence $\Phi(s) \stackrel{i}{\to}{}^+_{\mathcal{R}'_2} \Phi(t)$ (by Definition of Φ). In this derivation there is exactly one (inner) step which is destructive at level 2, namely the last one.

■

Now we are prepared to state and prove the main result of this section. Part (b) of it, i.e., that the finitely branching requirement can be dropped in (a), is due to Ohlebusch ([Ohl94c]).

Definition 5.3.7 (termination preservation under non-deterministic collapses (TPNDC))
A TRS $\mathcal{R}$ is said to be *termination preserving under non-deterministic collapses* (TPNDC for short), if termination of $\mathcal{R}$ implies termination of $\mathcal{R} \oplus \{G(x, y) \to x, G(x, y) \to y\}$.

Theorem 5.3.8 (a general structure theorem for non-modularity of termination)

(a) Let $\mathcal{R}_1, \mathcal{R}_2$ be two disjoint finitely branching TRSs which are both terminating such that their disjoint union $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. Then one of the systems, let's say $\mathcal{R}_1$, is not termination preserving under non-deterministic collapses, i.e., $\mathcal{R}_1 \oplus \{G(x, y) \to x, G(x, y) \to y\}$ is non-terminating, and the other system $\mathcal{R}_2$ is collapsing, or vice versa.

(b) (a) also holds without the requirement that $\mathcal{R}_1$ and $\mathcal{R}_2$ are finitely branching.

Proof: **(a)** We consider a minimal counterexample, i.e., an infinite $\mathcal{R}$-derivation

$$D: \quad s_1 \to s_2 \to s_3 \to \quad \ldots$$

of minimal rank, let's say $n+1$. We may assume w.l.o.g. that all the s_i's are top black ground terms having *rank* $n+1$. Since the preconditions of definition 5.3.4 are satisfied we may apply the white abstraction function Φ to the s_i's. By Lemma 5.3.6 this yields an $\mathcal{R}'$-derivation

$$D': \quad \Phi(s_1) \to^* \Phi(s_2) \to^* \Phi(s_3) \to^* \quad \ldots$$

with $\mathcal{R}' = \mathcal{R}_1 \oplus \mathcal{R}_2{}'$, $\mathcal{R}_2{}' = \mathcal{R}^G_{sub} = \{G(x,y) \to x,\, G(x,y) \to y\}$ (over the disjoint signature $\{A, G\}$), where for any $i \geq 1$ we have

$$\begin{array}{lcl} s_j \xrightarrow{o}_{\mathcal{R}_1} s_{j+1} & \Longrightarrow & \Phi(s_j) \xrightarrow{o}_{\mathcal{R}_1} \Phi(s_{j+1}) \text{ , and} \\ s_j \xrightarrow{i}_{\mathcal{R}} s_{j+1} & \Longrightarrow & \Phi(s_j) \xrightarrow{i}{}^{*}_{\mathcal{R}_2'} \Phi(s_{j+1}) \,. \end{array}$$

Since according to Lemma 5.3.3 (b) infinitely many steps in D are outer ones, the derivation D' is infinite, too. But this means that $\mathcal{R}_1$ is not termination preserving under non-deterministic collapses. Moreover, Lemma 5.3.3 (c) implies that $\mathcal{R}_2$ is collapsing.[15]

(b) For the case that one of the systems is not finitely branching (or both of them) our encoding construction above does not work any more.[16] However, the basic idea of encoding the relevant black information in the top white principal aliens by means of the fresh layer separating binary symbol G (and the additional fresh constant A) such that 'needed' black information can be extracted with the help of the projection rules of $\mathcal{R}^G_{sub}$ is still applicable. This was shown by Ohlebusch ([Ohl94c]). The quite involved, non-trivial construction there proceeds — roughly speaking — lazily (in order to circumvent the mentioned infinity problem) and collects the relevant black information bottom-up layer-to-layer and relative to some fixed initial term. For a detailed presentation the reader is referred to [Ohl94c]. ■

We note that for case (a) above Lemma 5.3.5 shows that the constructed infinite $\mathcal{R}'$-derivation D' has a rank which is less than or equal to the rank of the original minimal counterexample D.

Derived Symmetric and Asymmetric Preservation Criteria for Termination

As immediate consequences of Theorem 5.3.8 we obtain the following symmetric and asymmetric results.

Corollary 5.3.9 (TPNDC suffices for preserving termination)
Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two disjoint terminating TRSs. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are termination preserving under non-deterministic collapses (TPNDC) then their disjoint union is also terminating.

Corollary 5.3.10 (termination is modular for non-collapsing TRSs, [Rus87])
Termination is modular for disjoint unions of non-collapsing TRSs.[17]

[15] This can also be inferred more directly by observing that for non-collapsing $\mathcal{R}_2$ the white abstraction of the (top white) principal subterms of the minimal counterexample always yields the constant A which implies that the transformed infinite derivation is an $\mathcal{R}_1$-derivation contradicting termination of $\mathcal{R}_1$.

[16] An example for such a case is given in [Gra92a; Gra94a].

[17] We observe that this follows already from (the proof) of Theorem 5.3.8(a), i.e., without using (b).

Corollary 5.3.11 (asymmetric version: TPNDC plus NCOL of one system suffices for preserving termination)
The disjoint union of two terminating TRSs is again terminating whenever one of the systems is non-collapsing and termination preserving under non-deterministic collapses.

The next result shows that the class of TRSs which are termination preserving under non-deterministic collapses comprises all non-duplicating TRSs.

Lemma 5.3.12 (NDUP $\Longrightarrow$ TPNDC)
Whenever a TRS is non-duplicating then it is termination preserving under non-deterministic collapses.

Proof: Let $\mathcal{R}_1$ be a non-duplicating and terminating. Consider $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ with $\mathcal{R}_2 = \mathcal{R}^G_{sub} = \{G(x,y) \to x, G(x,y) \to y\}$. We define an ordering $>$ on $\mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ by lexicographically combining $\to^+_{\mathcal{R}_1}$ and the quasi-ordering $\gtrsim_G$, which is given as follows: $s \gtrsim_G t \;:\Longleftrightarrow\; |s|_G \geq_{nat} |t|_G$ (note that the associated strict ordering and equivalence $>_G$ and $\sim_G$, respectively, satisfy: $s >_G t \;:\Longleftrightarrow\; |s|_G >_{nat} |t|_G$ and $s \sim_G t \;:\Longleftrightarrow\; |s|_G = |t|_G$).[18] Now let $>$ be defined by: $s > t$ if either $s >_G t$ or else $s \sim_G t$ and $s \to^+_{\mathcal{R}_1} t$.[19] Since both $\gtrsim_G$ and $\to^+_{\mathcal{R}_1}$ are well-founded,[20] $>$ is well-founded, too. Hence, it suffices to show that $\to_{\mathcal{R}} \subseteq >$. From the form of $\mathcal{R}_2$ we obtain: $s \to_{\mathcal{R}_2} t \Longrightarrow s >_G t$. And the assumption that $\mathcal{R}_1$ is non-duplicating implies: $s \to_{\mathcal{R}_1} t \Longrightarrow s \gtrsim_G t \wedge s \to^+_{\mathcal{R}_1} t$. Since $\to_{\mathcal{R}} = \to_{\mathcal{R}_1} \cup \to_{\mathcal{R}_2}$ this yields $\to_{\mathcal{R}_1 \oplus \mathcal{R}_2} \subseteq >$ as desired. We conclude that $\to_{\mathcal{R}}$ is terminating, and $\mathcal{R}_1$ termination preserving under non-deterministic collapses. ∎

Since the non-duplication property (NDUP) is obviously modular, we immediately obtain the following consequence of Lemma 5.3.12 and Corollary 5.3.11.

Corollary 5.3.13 (SN $\wedge$ NDUP is modular, [Rus87])
Termination is modular for disjoint unions of non-duplicating TRSs.

Combining Lemma 5.3.12 with Corollary 5.3.11 yields

Corollary 5.3.14 (asymmetric version: NDUP plus NCOL of one system suffices for preserving termination, [Mid89])
The disjoint union of two terminating TRSs is again terminating whenever one of the systems is non-collapsing and non-duplicating.

We remark that the powerful Theorem 5.3.8 corresponds nicely to the intuition that the existence of counterexamples to modularity of termination crucially depends on 'non-deterministic collapsing' reduction steps. Hence, Toyama's Example 5.1.1 above is in a sense the simplest conceivable counterexample.

[18] Here, $>_{nat}$ and $\geq_{nat}$ denote the usual orderings on natural numbers.

[19] Observe that $>$ is closed under contexts but not under substitutions (because $>_G$ does not have the latter property).

[20] Note that a quasi-ordering is said to be well-founded if its strict part is well-founded.

Now we shall investigate further sufficient conditions — besides non-duplication — for the property TPNDC.

Given an arbitrary TRS $\mathcal{R}$ it would be desirable to have a method for testing whether $\mathcal{R}$ is TPNDC. But it turns out that this is an undecidable property in general.

Theorem 5.3.15 (TPNDC is undecidable)
The property of TRSs to be termination preserving under non-deterministic collapses is undecidable.

Proof (sketch): This result is an implicit consequence of the proof of the fact that termination is an undecidable property of disjoint unions of terminating TRSs as shown by Middeldorp and Dershowitz (cf. [Mid90]).[21] Roughly speaking the construction proceeds as follows: Given an arbitrary TRS $\mathcal{R}$, another TRS $\mathcal{R}_1$ is constructed by appropriately combining $\mathcal{R}$ with the system $\mathcal{R}_2 := \{f(a,b,x) \to f(x,x,x)\}$ of the basic counterexample 5.1.1 in such a way that $\mathcal{R}_1$ is terminating notwithstanding the fact that $\mathcal{R}$ may be non-terminating. Moreover, choosing $\mathcal{R}_2 := \{G(x,y) \to x,\ G(x,y) \to y\}$, it can be shown that the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is terminating if and only if $\mathcal{R}$ is terminating. Since for arbitrary TRSs termination is undecidable ([HL78]), it follows that the property of TRSs to be termination preserving under non-deterministic collapses is undecidable, too. ■

In order to obtain verifiable sufficient conditions for the property of being termination preserving under non-deterministic collapses we shall now specialize the increasing interpretation (or monotone algebra) method described in Section 2.2.2 and adapt it to the scenario of (disjoint or non-disjoint) unions of TRSs as follows.

For proving termination of $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_2^{\mathcal{F}_2} = (\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$ we apply the increasing interpretation method as follows: Choose D to be $\mathcal{T}(\mathcal{F}_1)$ considered as $\mathcal{F}$-algebra $\mathcal{D}$, where $\mathcal{F}_1$-operations are interpreted as in the term algebra $\mathcal{T}(\mathcal{F}_1)$ and every operation from $\mathcal{F}_2 \backslash \mathcal{F}_1$ is interpreted in some fixed way in terms of $\mathcal{F}_1$-operations, i.e.,

$$f^{\mathcal{D}} \ := \ \lambda x_1, \ldots, x_n \,.f(x_1, \ldots, x_n) \quad \text{for} \quad f \in \mathcal{F}_1$$

and

$$f^{\mathcal{D}} \ := \ \lambda x_1, \ldots, x_n \,.t_f \,, t_f \in \mathcal{T}(\mathcal{F}_1, \{x_1, \ldots, x_n\}) \quad \text{for} \quad f \in \mathcal{F}_2 \backslash \mathcal{F}_1 \,.$$

Hence, the unique homomorphism $\varphi : \mathcal{T}(\mathcal{F}) \to \mathcal{D}$ is given by $\varphi(f) = f^{\mathcal{D}}$. Now define the partial ordering $>_D$ on the domain $D = \mathcal{T}(\mathcal{F}_1)$ of $\mathcal{D}$ by $>_D \ := \ \to_{\mathcal{R}_1}^{+}$. Clearly, $>_D$ is well-founded, hence also $>$ (on $\mathcal{T}(\mathcal{F})$) defined by $s > t$ if $\varphi(s) >_D \varphi(t)$. For showing termination of $\mathcal{R}^{\mathcal{F}}$ via $\to_{\mathcal{R}^{\mathcal{F}}} \subseteq >$ it suffices that $>$ is closed under contexts and that all (ground) instances of rules are orientable with $>$, i.e.:

(a) $\forall s,t \in \mathcal{T}(\mathcal{F}_1)\ \forall f \in \mathcal{F}_2 \backslash \mathcal{F}_1 :\ s \to_{\mathcal{R}_1}^{+} t \implies (\varphi f)(\ldots, s, \ldots) \to_{\mathcal{R}_1}^{+} (\varphi f)(\ldots, t, \ldots)$, and

(b) $\forall l \to r \in \mathcal{R}_2\ \forall \sigma, \sigma\, \mathcal{T}(\mathcal{F}_1)-$ground substitution $:\ \varphi(\sigma l) \to_{\mathcal{R}_1}^{+} \varphi(\sigma r)$, respectively.

[21] Middeldorp states in [Mid90] that this result has been independently obtained by Dershowitz.

Now, it is easily verified that (a) is satisfied whenever φ is a *strict* interpretation for $\mathcal{F}_2$, i.e., for any $f \in \mathcal{F}_2$ we have $V(f(x_1, \ldots, x_n)) \subseteq V(\varphi(f(x_1, \ldots, x_n)))$. Verifying (b) means to show that $\mathcal{R}_2$-rules can be 'strictly simulated' by $\mathcal{R}_1$-rules. Hence we have the following result.

Lemma 5.3.16 (termination by strict simulation)
Let $\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_2^{\mathcal{F}_2}$ be TRSs such that $\mathcal{R}_1$ is terminating. Moreover, let φ be an interpretation of $(\mathcal{F}_1 \cup \mathcal{F}_2)$-operations in terms of $\mathcal{F}_1$-operations which is the identity on $\mathcal{F}_1$ and which is strict on $\mathcal{F}_2$. Then the union $(\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$ is terminating, too, provided that for every rule $l \to r \in \mathcal{R}_2$ we have $\varphi(l) \to_{\mathcal{R}_1}^{+} \varphi(r)$.[22]

An easy consequence of this result is the following 'folklore' fact.

Corollary 5.3.17 (termination is preserved under signature extension)
Whenever a TRS $\mathcal{R}^{\mathcal{F}}$ is terminating then $\mathcal{R}^{\mathcal{F}'}$ is terminating, too, for any enriched signature $\mathcal{F}' \supseteq \mathcal{F}$.

Note that if $\mathcal{F}$ does not contain symbols of arity > 1 and $\mathcal{F}'$ contains a symbol of arity > 1 then this result is not a straightforward consequence of Lemma 5.3.16 (since then the required strict interpretations do not exist) but can be easily proved directly.

Of course, the method for proving termination according to the above lemma is rather restricted, because it requires in a sense that $\mathcal{R}_1 \cup \mathcal{R}_2$ terminates for the same reason as $\mathcal{R}_1$ alone. But in particular for disjoint unions it is well-suited as we shall see now.

Concrete sufficient criteria for modularity of termination are easily obtained by combining the previous considerations with Corollary 5.3.9.

Definition 5.3.18 (non-deterministically collapsing, NDC)
A TRS $\mathcal{R}^{\mathcal{F}}$ is said to be *non-deterministically collapsing* if there exists a term $C[x, y] \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ with $x, y \in \mathcal{V}$ such that $C[x, y] \to^{+} x$ and $C[x, y] \to^{+} y$, i.e., if some term can be reduced to two distinct variables.[23]

Lemma 5.3.19 (NDC $\Longrightarrow$ TPNDC)
If a TRS is non-deterministically collapsing then it is also termination preserving under non-deterministic collapses.

Proof: Let $\mathcal{R}_1^{\mathcal{F}_1}$ be terminating and non-deterministically collapsing. We have to show that the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ with $\mathcal{R}_2 = \{G(x, y) \to x, \, G(x, y) \to y\}$ is terminating. Since $\mathcal{R}_1^{\mathcal{F}_1}$ is non-deterministically collapsing there exists some term $C[x, y] \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ with $x, y \in \mathcal{V}$ such that $C[x, y] \to_{\mathcal{R}_1}^{+} x$ and $C[x, y] \to_{\mathcal{R}_1}^{+} y$. We may further assume w.l.o.g. that x, y are the only variables appearing in $C[x, y]$. Now we interpret the function symbol G strictly by $\varphi f \; = \; \lambda x, y \,.\, C[x, y]$ and simply apply Lemma 5.3.16 the preconditions of which are satisfied. ∎

[22]Here, φ denotes the extension of φ to terms with variables defined in the obvious way.

[23]It is interesting to note that the properties NDC and $\mathrm{CON}^{\to}$ (cf. Definition 4.2.6) are complementary to each other, in the following sense: NDC $\Longleftrightarrow \neg\mathrm{CON}^{\to}$.

According to Corollary 5.3.9 the union of two disjoint terminating TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfying TPNDC is terminating: $\mathrm{SN}(\mathcal{R}_1 \oplus \mathcal{R}_2)$. But then $\mathrm{TPNDC}(\mathcal{R}_2)$ implies $\mathrm{SN}(\mathcal{R}_1 \oplus (\mathcal{R}_2 \oplus \{G(x,y) \to x, G(x,y) \to y\})$ because of Lemma 5.3.19 and Corollary 5.3.9. Hence we can extend Corollary 5.3.9 as follows.

Theorem 5.3.20 (SN $\wedge$ TPNDC is modular)
Termination is modular for disjoint unions of TRSs which are termination preserving under non-deterministic collapses.

As another consequence of combining Corollary 5.3.9 and Lemma 5.3.19 we also get

Corollary 5.3.21 (NDC suffices for preserving termination)
Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two disjoint terminating TRSs. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-deterministically collapsing (NDC) then their disjoint union is terminating, too.

We remark that the property of being non-deterministically collapsing (NDC) is not modular for disjoint unions of TRSs (however, it is of course preserved under the combination of two – even arbitrary – TRSs satisfying NDC). This can be seen from Example 4.2.8.

Further implied asymmetric preservation results (which we refrain to state explicitly) using NDC and NCOL are evident from Theorem 5.3.8 and Lemma 5.3.19.

Next we consider cases where a terminating TRS $\mathcal{R}$ does not necessarily contain collapsing rules but remains terminating when such rules are added.

Definition 5.3.22 (subterm compatible termination refined)
Let $\mathcal{R}^{\mathcal{F}}$ be a TRS and $f \in \mathcal{F}$, $\mathcal{F}' \subseteq \mathcal{F}$. Then, $\mathcal{R}^{\mathcal{F}}$ is said to be *f-subterm compatibly terminating*[24] if $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}^{f}_{sub}$ is terminating.[25] $\mathcal{R}^{\mathcal{F}}$ is *$\mathcal{F}'$-subterm compatibly terminating* if $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}^{\mathcal{F}'}_{sub}$ (with $\mathcal{R}^{\mathcal{F}'}_{sub} = \bigcup_{f \in \mathcal{F}'} \mathcal{R}^{f}_{sub}$) is terminating. $\mathcal{R}^{\mathcal{F}}$ is *subterm compatibly terminating* if $\mathcal{R}^{\mathcal{F}}$ is $\mathcal{F}$-subterm compatibly terminating, i.e., if $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}^{\mathcal{F}}_{sub}$ is terminating.[26]

As a straightforward consequence of Lemma 5.3.19 we get

Corollary 5.3.23 (f-subterm compatible termination — for $arity(f) \geq 2$ — implies SN $\wedge$ TPNDC)
If $\mathcal{R}^{\mathcal{F}}$ is f-subterm compatibly terminating for some $f \in \mathcal{F}$ with $arity(f) \geq 2$ then $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}^{f}_{sub}$ is (terminating and) termination preserving under non-deterministic collapses.

Combining this result with corollary 5.3.9 we obtain the following.

[24] In [Gra94a] this was called *f-simple termination*.

[25] Recall that $\mathcal{R}^{f}_{sub} = \{f(x_1, \ldots, x_j, \ldots, x_n) \to x_j \mid 1 \leq j \leq n\}$.

[26] Note that the latter (alternative) definition of subterm compatible termination (cf. Definition 2.2.34) is justified by Lemma 2.2.44((2) $\iff$ (3)).

Corollary 5.3.24 Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be two (finite) disjoint TRSs with $f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2$ of arity greater than 1 such that $\mathcal{R}_i$ is f_i-subterm compatibly terminating for $i = 1, 2$. Then the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is (f_1- and f_2-subterm compatibly) terminating, too.

If $\mathcal{R}^{\mathcal{F}}$ is a TRS with $arity(f) \leq 1$ for all $f \in \mathcal{F}$ then $\mathcal{R}^{\mathcal{F}}$ is obviously non-duplicating, hence termination preserving under non-deterministic collapses according to Lemma 5.3.12. Thus we obtain the following consequences.

Corollary 5.3.25 (subterm compatible termination implies TPNDC)
If $\mathcal{R}^{\mathcal{F}}$ is subterm compatibly terminating then $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}_{sub}^{f}$ is (terminating and) termination preserving under non-deterministic collapses.

Theorem 5.3.26 (subterm compatible termination is modular)
Subterm compatible termination is modular for disjoint unions of TRSs.

We note that according to Lemma 2.2.44 and Remark 2.2.47 the following properties of a TRS $\mathcal{R}^{\mathcal{F}}$ with $\mathcal{F}$ finite or $\mathcal{R}$ finite[27] are equivalent: subterm compatible termination, termination of $\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}_{sub}^{f}$, subterm compatibility plus termination, subterm compatibility, irreflexivity of $\rightarrow^{+}_{\mathcal{R} \cup \mathcal{R}_{sub}^{f}}$.

Thus, by Theorem 5.3.26 all these properties are modular for disjoint unions of such TRSs, in particular this entails the following (cf. Definitions 2.2.34, 2.2.41):

Theorem 5.3.27 (simple termination is modular for the case of finite signatures)
Simple termination is modular for disjoint unions of TRSs over finite signatures.

This result (together with Lemma 2.2.36) generalizes (for the case of finite signatures) the well-known observation that common classes of precedence based simplification orderings like recursive path orderings or recursive decomposition orderings exhibit a modular behaviour simply by combining the corresponding disjoint precedences.

Another earlier, related result (which also implies Theorem 5.3.27 above) is due to Kurihara & Ohuchi:[28]

Theorem 5.3.28 (subterm compatibility (simplifyingness) is modular, [KO90a])
Subterm compatibility (simplifyingness) or, equivalently, irreflexivity of $\rightarrow^{+}_{\mathcal{R} \cup \mathcal{R}_{sub}^{f}}$, is modular for disjoint unions of (arbitrary) TRSs.

Remark 5.3.29 (Kurihara & Ohuchi's proof technique)
The proof technique used in [KO90a] for Theorem 5.3.28 has some similarity with our abstracting transformation approach for proving Theorem 5.3.8(a). Instead of our white (and black) abstraction function Kurihara & Ohuchi define a mapping called

[27] or such that $\mathcal{R}^{\mathcal{F}}$ introduces only finitely many function symbols

[28] In [KO90a] the notion of *simple termination* means subterm compatibility in our terminology.

'alien-replacement'[29] which is tailored to some specific finite reduction sequence. Moreover their construction is in a sense incremental, but not rank-decreasing in general. To be more precise, consider some finite derivation

$$D: \quad s_0 \to s_1 \to s_2 \to \ldots \to s_m$$

in $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_{sub}^{\mathcal{F}_1} \cup \mathcal{R}_{del}^{\mathcal{F}_1}) \oplus (\mathcal{R}_2^{\mathcal{F}_2} \cup \mathcal{R}_{sub}^{\mathcal{F}_2} \cup \mathcal{R}_{del}^{\mathcal{F}_2})$ with all s_i's top black and such that every $\mathcal{R}$-derivation starting from any (top white) principal alien of s_0 is finite. Then their 'alien replacement' construction for D essentially consists in (recursively) collecting, for any principal alien occurring in D, all direct descendants occurring in D and abstracting them via a new varyadic function symbol. Using this transformation the $\mathcal{R}^{\mathcal{F}}$-derivation D can be translated in a one-to-one manner into a corresponding $(\mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_{sub}^{\mathcal{F}_1} \cup \mathcal{R}_{del}^{\mathcal{F}_1})$-derivation. Now, consider a counterexample (of minimal rank) to irreflexivity of $\to^+_{\mathcal{R} \cup \mathcal{R}_{sub}^f \cup \mathcal{R}_{del}^f}$, i.e., a cyclic $(\mathcal{R} \cup \mathcal{R}_{sub}^f \cup \mathcal{R}_{del}^f)$-derivation

$$s \to \ldots \to s\,,$$

where w.l.o.g. all terms are top black. Then this cyclic derivation can be translated via the mentioned 'alien replacement' ρ into a cyclic $(\mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_{sub}^{\mathcal{F}_1} \cup \mathcal{R}_{del}^{\mathcal{F}_1})$-derivation

$$\rho(s) \to \ldots \to \rho(s)$$

thus contradicting irreflexivity of $\to^+_{\mathcal{R}_1 \cup \mathcal{R}_{sub}^{\mathcal{F}_1} \cup \mathcal{R}_{del}^{\mathcal{F}_1}}$. Note how the finiteness of the considered cyclic derivation avoids any problem due to the possible infinity of the set of successors of some term.

Non-Self-Embedding Systems

In this paragraph we shall consider only finite TRSs $\mathcal{R}^{\mathcal{F}}$ in order to simplify the discussion. We have seen that – as a consequence of Theorem 5.3.8 as well as of Theorem 5.3.28 – simple termination is modular (Theorem 5.3.27). In other words, termination of a disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ can be shown by a simplification ordering if and only if this holds already for $\mathcal{R}_1$ and $\mathcal{R}_2$. According to Kruskal's Tree Theorem 2.2.33 the property of being non-self-embedding implies termination. Furthermore, simple termination is sufficient for being non-self-embedding. Hence, a natural question is to ask whether termination is also modular for non-self-embedding systems, or in slightly modified form: Is the property of being non-self-embedding a modular one? Having again a closer look on Example 5.1.1 it is clear that $\mathcal{R}_1 = \{f(a,b,x) \to f(x,x,x)\}$ is terminating, but cannot be simply terminating, because it is self-embedding as witnessed e.g. by the one-step self-embedding derivation

$$f(a,b,f(a,b,b)) \to_{\mathcal{R}_1} f(f(a,b,b), f(a,b,b), f(a,b,b))\,.$$

Now consider the following modified version of Example5.1.1:

[29] cf. [KO90a], [KO90b] for details; in fact, compared to [KO90a], [KO90b] contains a simplified and clarified version of 'alien replacement'.

Example 5.3.30 (being non-self-embedding is not modular)
Let the disjoint TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ be given by

$$\mathcal{R}_1 = \left\{ \begin{array}{l} f(a,b,x) \to h(x,x,x) \\ h(a,b,x) \to f(x,x,x) \end{array} \right.$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{l} G(x,y) \to x \\ G(x,y) \to y \end{array} \right.$$

Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating and even non-self-embedding as can be easily shown, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ admits for instance the following infinite (and hence self-embedding) derivation:

$$\begin{array}{rll} f(a,b,G(a,b)) & \to_{\mathcal{R}_1} & h(G(a,b),G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} & h(a,G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} & h(a,b,G(a,b)) \\ & \to_{\mathcal{R}_1} & f(G(a,b),G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} & f(a,G(a,b),G(a,b)) \\ & \to_{\mathcal{R}_2} & f(a,b,G(a,b)) \\ & \to_{\mathcal{R}_1} & \ldots \, . \end{array}$$

Moreover, the non-self-embedding system $\mathcal{R}_1$ is not simply terminating. To wit, consider the cyclic $\mathcal{R}_1 \cup \mathcal{R}^f_{sub}$-derivation

$$\begin{array}{rll} f(a,b,f(a,b,b)) & \to & h(f(a,b,b),f(a,b,b),f(a,b,b)) \\ & \to^+ & h(a,b,f(a,b,b)) \\ & \to & f(f(a,b,b),f(a,b,b),f(a,b,b)) \\ & \to^+ & f(a,b,f(a,b,b)) \\ & \to & \ldots \, . \end{array}$$

Thus, we may conclude that termination is not preserved in general under disjoint unions of non-self-embedding TRSs and that the property of being non-self-embedding is not a modular one. Note, that this reveals a gap between simply terminating and non-self-embedding systems. Hence, both implications

$$\mathcal{R} \text{ simply terminating} \quad \Longrightarrow \quad \mathcal{R} \text{ non-self-embedding} \quad \Longrightarrow \quad \mathcal{R} \text{ terminating}$$

cannot be reversed. This is well-known for the latter one (cf. e.g. Dershowitz [Der87]) which seems to be not the case for the first one. Moreover, the gap between non-self-embedding and simply terminating TRSs exists even for TRSs which contain only unary function symbols, hence for string rewriting systems. To this end consider the system

$$\mathcal{R} = \left\{ \begin{array}{l} g(g(x)) \to h(f(h(x))) \\ h(h(x)) \to g(f(g(x))) \end{array} \right.$$

over $\mathcal{F} = \{f,g,h\}$. Here, $\mathcal{R}$ is easily shown to be non-self-embedding but it is not simply terminating because we have for instance the following infinite (cyclic), hence self-embedding derivation in $\mathcal{R} \cup \mathcal{R}^f_{sub}$:

$$g(g(x)) \to h(f(h(x))) \to h(h(x)) \to g(f(g(x))) \to g(g(x)) \to \ \ldots \ .$$

5.3.2 The Modular Approach via Innermost Termination

Symmetric Preservation Criteria for Termination

Combining Theorem 5.2.7 with Theorem 3.3.12 we obtain

Theorem 5.3.31 (SN $\wedge$ WCR $\wedge$ OS is modular)
Termination is modular for disjoint unions of locally confluent overlay TRSs.

Proof: For the non-trivial implication to be proved assume that $\mathcal{R}_1$, $\mathcal{R}_2$ are disjoint, locally confluent and terminating overlay systems. For their disjoint union $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ we obtain (WCR $\wedge$ OS)($\mathcal{R}$) since both local confluence and the property of being an overlay system are modular. Furthermore, since both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating, in particular innermost terminating, $\mathcal{R}$ is innermost terminating, too, by Theorem 5.2.7. Finally, Theorem 3.3.12 yields termination of $\mathcal{R}$ as desired. ■

Corollary 5.3.32 (SN $\wedge$ CR is modular for overlay systems)
Completeness is modular for disjoint unions of overlay TRSs.

Similarly, we obtain the following modularity result for non-overlapping TRSs.

Theorem 5.3.33 (SN $\wedge$ NO is modular)
Termination is modular for disjoint unions of non-overlapping TRSs.

Proof: Analogous to the proof of Theorem 5.3.31, using the facts that NO is modular and that any non-overlapping TRS is in particular a locally confluent overlay system. ■

Corollary 5.3.34 (SN $\wedge$ CR is modular for non-overlapping systems)
Completeness is modular for disjoint unions of non-overlapping TRSs.

Combined with the modularity of innermost termination (Theorem 5.2.7), the refined equivalence results for innermost and general termination obtained in Section 3.4 now easily yield a couple of further modularity results some of which are presented now (more sophisticated versions are possible, too). First we recall some simple modularity facts (which are consequences of the corresponding more general statements of Lemma 4.3.1).

Lemma 5.3.35 (modularity of critical peak properties)
The following critical peak properties are modular for disjoint unions of TRSs:[30]

(1) UIR (uniqueness of innermost reduction, Definition 3.4.1)

(2) AICR (avoidance of innermost-critical steps, Definition 3.4.4)

(3) CPC (critical peak condition, Definition 3.4.13)

[30] As already mentioned, OS and NO are modular, too. Furthermore other critical peak properties like ORTH, WOJCP, WLRJCP etc. are easily seen to be modular, too.

(4) WNO (weakly non-overlapping, Definition 3.4.12)

(5) SLRJCP (strongly left-to-right joinable critical peaks, Definition 3.4.12)

(6) LRJCP (left-to-right joinable critical peaks, Definition 3.4.12)

Now we can generalize Theorem 5.3.33 above as follows.

Theorem 5.3.36 (SN $\wedge$ UIR $\wedge$ AICR is modular)
Termination is modular for disjoint unions of TRSs satisfying UIR and AICR.

Proof: Straightforward by combining Theorems 3.4.11 and 5.2.7 with Lemma 5.3.35. ∎

Theorem 5.3.37 (SN $\wedge$ CPC is modular)
Termination is modular for disjoint unions of TRSs satisfying CPC.

Proof: Straightforward by combining Theorems 3.4.11 and 5.2.7 with Lemmas 3.4.14 and 5.3.35. ∎

Theorem 5.3.38 (SN $\wedge$ CR is modular for TRSs satisfying CPC))
Completeness is modular for disjoint unions of TRSs satisfying CPC, in particular for weakly non-overlapping (WNO) TRSs and for TRSs with strongly left-to-right-joinable critical peaks (SLRJCP).

Proof: Analogous to the proof of Theorem 5.3.37, exploiting additionally Theorem 3.4.19 (or, alternatively, Theorem 4.1.2) and the implications WNO $\Longrightarrow$ SLRJCP $\Longrightarrow$ CPC (cf. Theorem 3.4.17). ∎

Let us give an example for illustrating the applicability of Theorem 5.3.38.

Example 5.3.39 (applying Theorem 5.3.38)
Consider the modified version of Example 5.1.2 where

$$\mathcal{R}_1 = \left\{ \begin{array}{c} f(a,b,x) \to f(x,x,x) \\ a \to c \\ f(c,b,x) \to f(x,x,x) \end{array} \right.$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{l} K(x,y,y) \to x \\ K(y,y,x) \to x \end{array} \right.$$

Both systems are terminating and confluent, and moreover have strongly left-to-right joinable critical peaks (SLRJCP) as is easily verified. Hence, Theorem 5.3.38 yields completeness of their disjoint union. We note that none of the previous modularity results is applicable here, since $\mathcal{R}_1$ is neither termination preserving under non-deterministic collapses (TPNDC) nor an overlay system (OS). Furthermore, $\mathcal{R}_2$ is not left-linear, hence also Theorem 5.3.45 (see below) is not applicable.

Asymmetric Preservation Criteria for Termination

A thorough analysis of the essential steps in the proof the modularity of termination for locally confluent overlay systems (Theorem 5.3.31) reveals that if one of the systems is non-collapsing then the overlay requirement for the other system can be dropped. This will be shown next. First we need slightly extended versions of Lemmas 3.3.9 and 3.3.10.

Lemma 5.3.40 (extended version of Lemma 3.3.9)
Suppose $\mathcal{R}$ is a locally confluent TRS. Let $l \to r$ be a rule of $\mathcal{R}$ such that there exists no inside critical peak by overlapping some other (or the same) rule of $\mathcal{R}$ into $l \to r$ properly below the root. Let σ be a substitution such that σl is not complete. Then $\Phi(\sigma l) = (\Phi \circ \sigma)l$.[31] In particular, if additionally σx is complete for all $x \in Var(l)$, then $\Phi(\sigma l) = (\Phi \circ \sigma)l = (\sigma \downarrow)l$ and all proper subterms of $\Phi(\sigma l)$ are irreducible.

Proof: Essentially the same as for Lemma 3.3.9. In fact, the assumption that $l \to r$ allows no inside critical peak by overlapping into it properly below the root captures exactly what is needed for the crucial property, namely that, for any $p \in \mathcal{F}Pos(l)$ with $(\sigma l)/p = \sigma(l/p)$ complete we have $\Phi(\sigma(l/p)) = \sigma(l/p) \downarrow = \sigma'(l/p) \downarrow = \sigma'(l/p)$ where $\sigma' = \{x \mapsto (\sigma \downarrow)x \,|\, x \in Var(l/p)\}$, i.e., $\sigma'(l/p)$ is irreducible. ■

Lemma 5.3.41 (non-empty projection of $\to_{\text{nc}}$-steps)
Suppose $\mathcal{R}$ is a locally confluent TRS. Let $l \to r$ be a rule of $\mathcal{R}$ such that there exists no inside critical peak by overlapping some other (or the same) rule of $\mathcal{R}$ into $l \to r$ properly below the root. Let σ be a substitution such that σl is not complete. If $s = C[\sigma l] \to_{p,\sigma,l \to r} C[\sigma r] = t$ then $\Phi(s) \to^+ \Phi(t)$.

Proof: Analogous to the proof of Lemma 3.3.10, using Lemma 5.3.40 instead of Lemma 3.3.9. ■

Theorem 5.3.42 (asymmetric preservation criterion for termination and completeness based on modularity of innermost termination)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint complete TRSs such that $\mathcal{R}_1$ is additionally a non-collapsing overlay system. Then the disjoint union $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is complete, too.

Proof: Suppose for a proof by contradiction that the disjoint union is $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. Consider an infinite ($\mathcal{R}$-) derivation

$$D : s_1 \to s_2 \to s_3 \to \dots$$

with the additional minimality property that all proper subterms of s_1 are terminating and hence complete. Then D must have the form

$$D : s_1 \to \dots \to s_n \to_\lambda s_{n+1} \to \dots ,$$

[31] Recall that $\Phi \circ \sigma$ denotes the composition of the substitution σ with the parallel normalization mapping for all maximal (parallel) complete subterms Φ from Definition 3.3.4, i.e., the substitution defined by $(\Phi \circ \sigma)x = \Phi(\sigma x)$.

i.e., eventually some step $s_n \to_\lambda s_{n+1}$ is a (first) root reduction step. Clearly, this step is an $\to_{\mathrm{nc}}$-step, and all proper subterms of s_n are complete.

Now consider the case that s_n is top white, i.e., $\mathcal{F}_2$-rooted. All principal top black aliens of s_n are complete and any derivation issuing from them consists of (complete) top black reducts, since $\mathcal{R}_1$ is non-collapsing. Together with the infinity of D this implies that all s_k, $k \geq n$, are top white and that D contains infinitely many outer $\mathcal{R}_2$-steps. But then, by identifying abstraction of all principal top black aliens, we obtain an infinite (pure) $\mathcal{R}_2$-derivation which contradicts termination of $\mathcal{R}_2$.

The other case is that s_n is top black, i.e., $\mathcal{F}_1$-rooted. Since all top white principal aliens of s_n are complete, all $\to_{\mathrm{nc}}$-steps in D after s_n (including $s_n \xrightarrow{o} s_{n+1}$) must be outer $\mathcal{R}_1$-steps, and there must be infinitely many of these. By definition of Φ we know that all top white principal aliens in $\Phi(s_n)$ are irreducible. Hence, applying Φ to D and using Lemma 5.3.41 (which is applicable, because $\mathcal{R}_1$ is a locally confluent overlay system) and Lemma 3.3.8 we obtain the infinite $\mathcal{R}$-derivation

$$\Phi(s_n) \to^* \Phi(s_{n+1}) \to^* \Phi(s_{n+2}) \to^* \dots$$

where all reduction steps are outer $\mathcal{R}_1$-steps. As above, identifying abstraction yields an infinite pure $\mathcal{R}_1$-derivation contradicting termination of $\mathcal{R}_1$.

Thus we may conclude that $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ must be terminating. ∎

Let us give an example for illustrating the applicability of Theorem 5.3.42.

Example 5.3.43 (applying Theorem 5.3.42)
Consider the modified version of Example 5.1.1 where

$$\mathcal{R}_1 = \left\{ \; f(a,b,x) \to f(x,x,x) \right.$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{r} G(x,x) \to x \\ G(A,B) \to A \\ A \to B \end{array} \right.$$

Both systems are terminating and confluent, and moreover $\mathcal{R}_1$ is a non-collapsing overlay system. Hence, Theorem 5.3.42 yields completeness of the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$. We note that none of the previous modularity results is applicable here, since $\mathcal{R}_1$ is (non-collapsing but) not termination preserving under non-deterministic collapses (TPNDC), and $\mathcal{R}_2$ neither is an overlay system (OS) nor does it satisfy the critical pair condition CPC or the non-collapsing property (NCOL). Furthermore, $\mathcal{R}_2$ is not left-linear, hence also Theorem 5.3.45 (see below) is not applicable.

In view of the asymmetric preservation result for termination of Theorem 5.3.42 which relies on the modularity of innermost termination and sufficient conditions for the equivalence of innermost and general termination, one may ask whether analogous asymmetric versions of Theorems 5.3.36, 5.3.37 are possible, too. There, the implication SIN $\Longrightarrow$ SN was guaranteed (cf. Section 3.4) by the properties UIR and AICR. Both properties seem to be crucial for the main constructions and results of Section 3.4). For instance, requiring only UIR does not suffice for preserving termination under disjoint unions as shown by Example 5.1.2 above where it is easy to verify that both systems satisfy UIR. Another, more refined and natural question, analogous to Theorem 5.3.42 above, reads as follows.

> Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint terminating TRSs with unique innermost reduction. Is their disjoint union $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ terminating provided that $\mathcal{R}_1$ is additionally non-collapsing and AICR? (or, more concisely: $(\mathrm{SN} \wedge \mathrm{UIR} \wedge \mathrm{NCOL} \wedge \mathrm{AICR})(\mathcal{R}_1) \wedge (\mathrm{SN} \wedge \mathrm{UIR})(\mathcal{R}_2) \implies \mathrm{SN}(\mathcal{R}_1 \oplus \mathcal{R}_2)$?)

In fact, this does not hold either as demonstrated by the next example.

Example 5.3.44 (AICR seems to be essential, too)
Consider the modified version of Example 5.1.3 where

$$\mathcal{R}_1 = \left\{ \; f(x, g(x), y) \to f(y, y, y) \right.$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{r} G(H(x)) \to x \\ G(H(x)) \to A \\ H(x) \to K(x) \end{array} \right.$$

Both systems are terminating and satisfy UIR as is easily verified. Moreover, $\mathcal{R}_1$ is non-collapsing and satisfies AICR (it is even non-overlapping). However, their disjoint union is non-terminating as witnessed by the cycle

$$f(A, g(A), G(H(g(A)))) \to f(G(H(g(A)))^3) \to^+ f(A, g(A), G(H(g(A)))) \, .$$

According to Theorem 5.3.42 (note that $\mathcal{R}_1$ is a complete non-collapsing overlay system) $\mathcal{R}_2$ cannot be confluent. Indeed, it is neither confluent nor does it satisfy AICR.

It remains open whether other asymmetric preservation results for termination in the spirit of Theorem 5.3.42 but relying on CPC (or, more generally, on UIR, AICR) are possible.

5.3.3 The Syntactic Approach via Left-Linearity

Symmetric Preservation Results for Termination of Left-Linear TRSs

We have already mentioned that completeness is not modular as demonstrated by Example 5.1.2. In this counterexample the disjoint union of the two complete TRSs

$$\mathcal{R}_1 = \left\{ \begin{array}{r} f(a, b, x) \to f(x, x, x) \\ a \to c \\ b \to c \\ f(x, y, z) \to c \end{array} \right.$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{r} K(x, y, y) \to x \\ K(y, y, x) \to x \end{array} \right.$$

is (confluent but) not terminating, since there is the cycle

$$D: \quad f(a,b,K(a,c,b)) \to f(K(a,c,b)^3) \to^+ f(a,b,K(a,c,b))\,.$$

A careful inspection of rewriting in the disjoint union, in particular in the cyclic derivation D above, leads to the conjecture that non-left-linearity of $\mathcal{R}_2$ might be essential for the counterexample.[32] More precisely, the problem in the cyclic derivation above is due to the fact, that colour changing reductions are not uniquely factorizable. We recall that a reduction sequence $s \to^* t$ with s, t ground terms (in the disjoint union) is *colour changing* (cf. Definition 2.4.12) if s is top black and t top white or vice versa. Furthermore note that in a colour changing derivation at least one step must be destructive at level 1 using a collapsing rule). Indeed, in the derivation D the following two colour changing reductions issuing from $K(a,c,b)$ are crucial:

$$K(a,c,b) \to_{\mathcal{R}_1} K(a,c,c) \to_{\mathcal{R}_2} a$$

and

$$K(a,c,b) \to_{\mathcal{R}_1} K(c,c,b) \to_{\mathcal{R}_2} b\,.$$

And a (unique) factorization is not possible, i.e., there is no (top black) term t such that both

$$K(a,c,b) \to^* t \to^* a$$

and

$$K(a,c,b) \to^* t \to^* b$$

hold. This cannot happen in the disjoint union of left-linear TRSs as shown by Toyama, Klop & Barendregt.

Theorem 5.3.45 (completeness is modular for left-linear TRSs, [TKB89; TKB95])
Completeness is modular for disjoint unions of left-linear TRSs.

The proof of this deep result in [TKB89; TKB95] relies on a very intricate analysis of colour changing derivations. The technical main result[33] needed roughly states that if a terminating top black (top white) term s in the disjoint union of two left-linear and complete systems has colour change, i.e., the unique normal form $s\downarrow$ of s is top white (top black), then there exists a unique top white (top black) special subterm of s through which all colour changing reductions issuing from s can be factored (in the terminology of [TKB89; TKB95]: s has exactly one *essential* subterm).

Recently, Marchiori ([Mar95]) and Schmidt-Schauss & Panitz ([SSP94]) independently succeeded in giving a considerably simplified proof of Theorem 5.3.45, using very similar ideas and constructions. In fact, their approach even yields the following extension.

[32] Or, put in a positive manner, that completeness might be modular for (disjoint unions of) left-linear TRSs. This was first conjectured by Klop & Barendregt (1986) as mentioned in a preliminary version of [TKB89].

[33] Yet, the rest of the proof in [TKB89; TKB95] is also non-trivial, since a simple projection technique does not work.

Theorem 5.3.46 (SN $\wedge$ CON$^{\to}$ is modular for left-linear TRSs, [SMP95]) Termination plus consistency with respect to reduction (SN $\wedge$ CON$^{\to}$) is modular for disjoint unions of left-linear TRSs.

Proof (idea): Actually, modularity of CON$^{\to}$ for left-linear systems — which relies on the same proof techniques — has already been mentioned (Theorem 4.2.9). We only sketch the basic ideas and structure of the proof for termination of the disjoint union (cf. [SMP95] for more details). Essentially there are two main properties of left-linear and CON$^{\to}$ TRSs which are extensively exploited.

(1) If $s = C[x, \dots, x]$, where x is the only variable occurring in s and all occurrences of x are displayed, and $s \to^* x$, then the ancestor of the final result x in s is unique.[34] This follows from the fact that, by left-linearity, $s = C[x, \dots, x] \to^* x$ implies $s = C[x_1, \dots, x_n] \to^* x_i$ for (pairwise) distinct variables $x_1, \dots, x_n$ such that, by CON$^{\to}$, the index i is unique.

(2) If $s = \sigma l \to_{\lambda,\sigma,l\to r} \sigma r = t$ and $s \to_{p,\tau,l'\to r'} s'$ for $p \geq q \in \mathcal{V}Pos(l)$ (with $l/q = x$) then $s' = \sigma' l \to_{\lambda,\sigma',l\to r} \sigma' r \;{}_{\tau,l'\to r'}\!\!\leftarrow\!\!\!\!\!\!\Vert\; \sigma r = t$ where $\sigma' y = \sigma y$ for $y \neq x$ and $\sigma' x = (\sigma x)[p\backslash q \leftarrow \tau r']$ (this is possible by left-linearity).[35]

Now the basic idea of the construction in [SMP95] is as follows: Consider in a counterexample of minimal rank (consisting of ground terms, only) in the initial let'say top black term a special subterm $C[\![s_1, \dots, s_m]\!]$ (which is let's say top black) of minimal rank with the property that it allows a colour change but all its principal subterms s_i do not (in other words, any derivation issuing from s_i does not contain a destructive step). Then, according to (1) above, for any colour changing derivation $s \to^* t$, t must have a unique ancestor s_k in s, and moreover $C[x_1, \dots, x_m] \to^* x_k$, hence also $C[\![s_1, \dots, s_m]\!] \to^* s_k \to^* t$. Now the idea is to replace in the initial term the top black special subterm $s = C[\![s_1, \dots, s_m]\!]$ by its (unique) principal top white subterm s_k to which it may collapse and to mimic the original infinite derivation. However, the black information in the top layer of s to be deleted may be essential later on in the original infinite derivation, since it may (by some destructive step(s)) coalesce with other black layers above thereby enabling further steps otherwise not possible. In order to compensate for this potential loss of information the black top layer of s is deleted, i.e., s replaced by s_k as mentioned, but simultaneously the black top layer of s is *piled* above below all top black aliens on the way from the root to s. In fact, this *pile and delete* process as it is called in [Mar95], and the fact that after it the original infinite derivation can still be mimicked is the most difficult part of the proof.[36] In particular, we observe that left-linearity is essential to enable this mimicking (based on property

[34] To make this formally precise one has to introduce notions like *ancestors / descendants* of (sub)terms w.r.t. reduction sequences that are *labelled* by the respective rule applications. We shall not do this here, since we only try to give an intuitive account.

[35] In other words, after rewriting below the variable part of a left-linear rule application, the same rule is still applicable, yielding a reduct that can also be obtained from the original reduct by a corresponding parallel contraction of (possibly duplicated) corresponding redex occurrences within the variable part.

[36] It should be mentioned that a similar, but more complicated *pile and delete* technique is also essential in the second part of the proof of [TKB89; TKB95].

(2) above). Repeating this transformation one eventually obtains an infinite derivation where no destructive step is possible any more. But this means that the black system is non-terminating yielding a contradiction. ∎

Since confluence implies consistency w.r.t. reduction (CR $\Longrightarrow$ CON$^{\rightarrow}$), Theorem 5.3.45 is a corollary of Theorems 5.3.46 and 4.1.2. Moreover, we observe that any non-erasing TRS is necessarily consistent w.r.t. reduction (NE $\Longrightarrow$ CON$^{\rightarrow}$). Hence, since the non-erasing property is obviously modular, the following is also a consequence of Theorem 5.3.46.

Theorem 5.3.47 (termination is modular for left-linear, non-erasing TRSs, [SMP95])
Termination is modular for disjoint unions of left-linear, non-erasing TRSs.

However, without left-linearity, the disjoint union of two terminating, non-erasing TRSs need not be terminating in general, even if the systems are confluent and irreducible. Corresponding (rather involved) counterexamples are given in [Ohl95c].

Finally, we observe that Theorem 5.3.46 combined with Theorem 4.2.7, the implications CR $\Longrightarrow$ CON $\Longrightarrow$ CON$^{\rightarrow}$, and the modularity of left-linearity, also entails the following.

Corollary 5.3.48 (SN $\wedge$ CON is modular for left-linear TRSs)
Termination plus consistency (SN $\wedge$ CON) is modular for disjoint unions of left-linear TRSs.

Asymmetric Preservation Results for Termination

Now let us turn to related asymmetric results. An interesting structural property for disjoint unions of left-linear TRSs is the following.

Theorem 5.3.49 (counterexample properties for left-linear TRSs, [SMP95][37])
Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two disjoint, left-linear and terminating TRSs, and let $\mathcal{R}_1 \oplus \mathcal{R}_2$ be non-terminating. Then one of the systems is CON$^{\rightarrow}$, while the other one is not CON$^{\rightarrow}$.

An asymmetric preservation result for termination corresponding to the symmetric one in Theorem 5.3.45 above is given in [TKB95, Appendix B].

Theorem 5.3.50 (asymmetric version of Theorem 5.3.45, [TKB95])
The disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ of a left-linear, complete TRS $\mathcal{R}_1$ and a non-collapsing, terminating TRS $\mathcal{R}_2$ is terminating.

[37] The proof in [SMP95] is based on analyzing a minimal, let's say top black, counterexample. It is shown that the white system $\mathcal{R}_2$ must be CON$^{\rightarrow}$. Applying Theorem 5.3.46 then yields that $\mathcal{R}_1$ cannot be CON$^{\rightarrow}$.

Recently we have obtained two further related asymmetric preservation results which were presented in [Gra95e]. A detailed account of this work will be the subject of a forthcoming paper.

5.4 Non-Disjoint Unions

In this section we consider modularity and preservation results for termination properties of constructor sharing and of composable TRSs.

5.4.1 Restricted Termination Properties and Semi-Completeness

The modularity of the weak termination properties WN (weak termination), WIN (weak innermost termination) and SIN (innermost termination) from the disjoint union case easily carries over to constructor sharing as well as to composable TRSs.

Theorem 5.4.1 (WN, WIN and SIN are modular for composable TRSs) The properties WN, WIN and SIN are modular for constructor sharing as well as for composable TRSs.

Proof: The proofs are essentially the same as for the disjoint union case. For WN (and implicitly also for WIN) of composable constructor systems this has been shown in [MT91; MT93]. SIN of constructor sharing TRSs is treated in [Gra92b; Gra95a], and the case of composable TRSs (for all three properties) in [Ohl94a; Ohl95a]. ∎

Since weak termination implies the existence of preserved reducts, an immediate consequence of the modularity of weak termination (for composable TRSs) and Theorem 4.3.4 is the modularity of semi-completeness. In [Ohl94a; Ohl94b] this is shown for constructor sharing TRSs,[38] in [Ohl95a] for composable TRSs.[39] For the (more special case) of composable constructor systems the result has been established in [MT91; MT93].

Theorem 5.4.2 (semi-completeness is modular for composable TRSs) Semi-completeness is modular for constructor sharing as well as for composable TRSs.

[38]via weak termination of $\to_c$ which is not really necessary as already mentioned since weak termination of the union already implies the the crucial property that every term has a preserved reduct;

[39]by a direct inductive proof which is also not really necessary by the same argument as before;

5.4.2 Termination of Constructor Sharing / Composable Systems

For constructor sharing / composable TRSs the problematic effect of a layer coalescence in the union is not only caused by (the existence of) collapsing rules, but also by constructor lifting / shared function symbol symbol lifting rules. A simple example demonstrating this is the following.

Example 5.4.3 (shared function symbol lifting rules are problematic, too, [Gra94a])
The TRSs

$$\mathcal{R}_1 = \left\{ \; f(a,b,x) \to f(x,x,x) \right.$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{l} D \to a \\ D \to b \end{array} \right.$$

which share the constructors a and b are both terminating. However, their union is non-terminating as witnessed by the cycle

$$D \,:\, f(a,b,D) \to_{\mathcal{R}_1} f(D,D,D) \to_{\mathcal{R}_2} f(a,D,D)) \to_{\mathcal{R}_2} f(a,b,D)$$

which obviously has rank 2. Note that $\mathcal{R}_2$ is non-collapsing but both its rules are constructor lifting (hence, shared function symbol lifting). This enables the second and third step in D above to be destructive at level 2. Note moreover that whenever a reduction step $s \to t$ is destructive at level 2 then $rank(s) \geq 2$.

Taking into account this additional complication for constructor sharing and for composable TRSs our general structural approach by analyzing minimal counterexamples as presented in Section 5.3.1 easily carries over. In particular, the general structure Theorem 5.3.8(a) can be generalized as follows.

Theorem 5.4.4 (generalized version of Theorem 5.3.8(a), [Gra94a], [KO95a])

Let $\mathcal{R}_1, \mathcal{R}_2$ be two constructor sharing (composable), finitely branching TRSs which are both terminating such that their union $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ is non-terminating. Then one of the systems, let's say $\mathcal{R}_1$, is not termination preserving under non-deterministic collapses, i.e., $\mathcal{R}_1 \oplus \{G(x,y) \to x, G(x,y) \to y\}$ is non-terminating, and the other system $\mathcal{R}_2$ is shared symbol lifting, i.e., not layer preserving, or vice versa.

Proof: For constructor sharing TRSs the proof structure is analogous to the proof of Theorem 5.3.8(a), cf. [Gra94a] for details. The only important difference is that the white (black) abstraction Φ (Definition 5.3.4) has to be adapted, in the sense that for top white (top black) principal aliens not only all top black (top white) but also all top transparent successors have to be collected and recursively abstracted.

For composable TRSs one additionally needs a slightly refined minimality assumption for the considered counterexample as detailed in [KO95a]. ∎

Some straightforward consequences analogous to the disjoint union case are the following (some of which are mentioned in [KO95a]).

Theorem 5.4.5 (derived termination criteria for unions of composable TRSs)

Let $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ be composable, finitely branching TRSs, with $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_2^{\mathcal{F}_2}$. Then the following assertions hold:

(1) If both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating (SN) and termination preserving under non-deterministic collapses (TPNDC),[40] then $\mathcal{R}^{\mathcal{F}}$ is terminating and TPNDC, and vice versa.

(2) If both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating and layer preserving, then $\mathcal{R}^{\mathcal{F}}$ is terminating.[41]

(3) If one of the systems is both TPNDC and layer preserving, then $\mathcal{R}^{\mathcal{F}}$ is terminating.

(4) If both $\mathcal{R}_1$ and $\mathcal{R}_2$ are subterm compatibly terminating, then $\mathcal{R}^{\mathcal{F}}$ is subterm compatibly terminating, and vice versa.[42]

Some further results on preservation of termination for unions of composable TRSs, which involve non-duplication and layer preservation but do not require the systems to be finitely branching, are given in [Ohl94a; Ohl95a].

Interestingly, Theorem 5.3.8(b) (as well as some derived results) cannot be generalized to constructor sharing or composable TRSs,[43] i.e., in Theorem 5.4.4 the finitely branching requirement cannot be dropped.

Example 5.4.6 (finitely branching is essential for constructor sharing / composable TRSs, [Ohl94c])
The constructor sharing TRSs

$$\mathcal{R}_1 = \left\{ \; f_i(c_i, x) \to f_{i+1}(x, x) \,|\, i \geq 1 \; \right\}$$

and

$$\mathcal{R}_2 = \left\{ \; A \to c_i \,|\, i \geq 1 \; \right\}$$

over $\mathcal{F}_1 = \{f_i, c_i \,|\, i \geq 1\}$ and $\mathcal{F}_2 = \{A\} \cup \{c_i \,|\, i \geq 1\}$, respectively, are both terminating (even subterm compatibly terminating because $\mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_{sub}^{\mathcal{F}_1}$ is terminating) and share

[40] which is for instance the case for non-duplicating (NDUP) as well as for non-deterministically collapsing (NDC) systems

[41] This also holds without the assumption that $\mathcal{R}_1$ and $\mathcal{R}_2$ are finitely branching. The reason is — as in the disjoint union case — that a top black (top white) term cannot have top transparent or top white (top black) reducts at all.

[42] By Theorem 2.2.36 this implies in particular: Simple termination is modular for composable TRSs over finite signatures.

[43] The reader familiar with Ohlebusch's proof construction in [Ohl94a] may recognize that it does not extend to the case where there are shared function symbols.

the infinitely many constructors c_i, $i \geq 1$. However, the union is non-terminating (and hence also not subterm compatibly terminating):

$$f_1(c_1, A) \to f_2(A, A) \to f_2(c_2, A) \to f_3(A, A) \to \dots .$$

Note that $\mathcal{R}_2$ is not finitely branching.

Yet, subterm compatibility (simplifyingness) is still modular for constructor sharing and also for composable TRSs as shown by Kurihara & Ohuchi and Ohlebusch, respectively.

Theorem 5.4.7 (extended version of Theorem 5.3.28: subterm compatibility (simplifyingness) is modular for constructor sharing and for composable TRSs, [KO92], [Ohl94a])
Subterm compatibility (or, equivalently, simplifyingness, or irreflexivity of $\to^+_{\mathcal{R}^{\mathcal{F}} \cup \mathcal{R}^{\mathcal{F}}_{sub}}$) is modular for constructor sharing TRSs, and even for composable TRSs.

Proof (idea): The proof of [KO92] for the constructor sharing case is very similar to the disjoint union case of [KO90a], and the proof of [Ohl94a] for composable TRSs works by combining Kurihara & Ohuchi's technique with our abstracting transformation construction for Theorem 5.3.8(a). ■

In contrast to subterm compatible termination the (revised) notion of simple termination (cf. Definition 2.2.41) in the general case (i.e., for possibly infinite signatures) enjoys a better modularity behaviour for composable systems as recently shown by Middeldorp & Zantema.

Theorem 5.4.8 (simple termination is modular for composable TRSs, [MZ95])
Simple termination is modular for composable TRSs.

Proof (idea): Using basic properties of partial well-orderings (PWOs) the difficult direction of this modularity result is proved in [MZ95] by reducing it to the preservation of subterm compatibility (simplifyingness) under the union of composable TRSs (cf. Theorem 5.4.7 above). ■

Next we shall consider modularity and preservation results for termination in the case of constructor sharing and composable TRSs corresponding to Section 5.3.2.

Theorem 5.4.9 (termination and completeness are modular for composable, locally confluent overlay systems, generalized version of Theorem 5.3.31 and Corollary 5.3.32)
Termination and completeness are modular for composable, locally confluent overlay TRSs.

Proof: Straightforward by combining Lemma 4.3.1, Theorem 5.4.1 (modularity of SIN, innermost termination) and Theorem 3.3.12. ■

Similarly we obtain

Theorem 5.4.10 (termination and completeness are modular for composable, non-overlapping TRSs, generalized version of Theorem 5.3.33 and Corollary 5.3.34)
Termination and completeness are modular for composable, non-overlapping TRSs.

Proof: As for Theorem 5.4.9 above. ∎

Theorem 5.4.11 (termination and completeness are modular for composable TRSs satisfying CPC, generalized version of Theorems 5.3.37 and 5.3.38)
Termination and completeness are modular for composable TRSs satisfying CPC, in particular for composable TRSs which are weakly non-overlapping (WNO) or have strongly left-to-right joinable critical peaks (SLRJCP).

Proof: Straightforward by combining Theorems 5.4.1 (modularity of SIN), 3.4.11, 3.4.17 and Lemmas 3.4.14, 4.3.1. ∎

Furthermore we can generalize the asymmetric preservation result of Theorem 5.3.42 for completeness of disjoint unions to the case of composable TRSs.

Theorem 5.4.12 (asymmetric preservation criterion for termination and completeness of composable TRSs, generalized version of Theorem 5.3.42)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be composable, complete TRSs such that $\mathcal{R}_1$ is additionally a layer-preserving overlay system. Then the union $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \cup \mathcal{R}_2^{\mathcal{F}_2}$ is complete, too.

Proof (sketch): The proof structure is the same as for Theorem 5.3.42. However, instead of the non-collapsing property of $\mathcal{R}_1$ its layer preservation is exploited. Moreover, for the case analysis of an assumed minimal counterexample

$$D : s_1 \to \ldots \to s_n \to_\lambda s_{n+1} \to \ldots \ ,$$

where all proper subterms of s_n are complete, the reasoning in the additional case that s_n is top transparent (i.e., then the root symbol of s_n must be a shared function symbol) is as for the case that s_n is top black. ∎

Finally let us briefly discuss the situation with preservation results for termination which were based on uniqueness properties of collapsing reductions and left-linearity (cf. Section 5.3.3).

Theorem 5.3.46, i.e., modularity of SN $\wedge$ CON$^{\to}$ for disjoint unions, does not extend to constructor sharing or composable TRSs. A simple counterexample is given by Example 5.4.3 above. There we had the two constructor sharing TRSs

$$\mathcal{R}_1 = \left\{ \; f(a, b, x) \to f(x, x, x) \right.$$

and

$$\mathcal{R}_2 = \left\{ \begin{array}{l} D \to a \\ D \to b \end{array} \right.$$

which were both terminating, but with their union non-terminating. Indeed, we observe that both systems are also left-linear and consistent with respect to reduction ($CON^{\rightarrow}$). Other counterexamples of [Ohl94a] show that also confluence doesn't help here (note that $\mathcal{R}_2$ is not confluent in the example). Actually, recalling the crucial prerequisites for the sketched proof of Theorem 5.3.46, this doesn't come as a big surprise since collapsing reduction (using rules with a variable as right hand side) is not the only possibility any more of causing a layer coalescence (as it was the case for disjoint unions). Consequently, the consistency property $CON^{\rightarrow}$ (which disallowed reductions to two distinct variables) doesn't account any longer for all problematic cases of 'non-deterministic' layer coalescences.

Yet, bearing this in mind, it seems reasonable to conjecture that the union of constructor sharing (composable) TRSs might be terminating provided that both systems are terminating, left-linear, $CON^{\rightarrow}$ and not shared function symbol lifting (using essentially the same proof as for Theorem 5.3.46. However, this remains to be checked in detail.

Another related but more general idea is to try to adapt the required consistency property appropriately, i.e., by taking into account not only diverging reductions leading to two distinct variables, but also diverging reductions, let's say in $\mathcal{R}_1$, of the form

$$s_1 \;{}^*\!\!\leftarrow s \rightarrow^* s_2 \;,$$

where s is top black and s_1, s_2 are top transparent (which includes both the case of variables as well as of terms with a shared function as root symbol). However, whether this approach for $CON^{\rightarrow}$ via refining it into something like 'consistency with respect to reduction to (distinct) top transparent terms' might be successful for deriving new interesting preservation results for termination, remains open at current.[44]

5.5 Conditional Rewrite Systems

Subsequently we shall study the modularity and preservation behaviour concerning termination properties of combined CTRSs. Again we tacitly assume (if not explicitly stated otherwise) that the CTRSs considered are join systems.

The modularity analysis for CTRSs is much more complicated than for (unconditional) TRSs. For instance, we have already mentioned that — as exhibited in [Mid90] — the fundamental decomposition property of TRSs,

$$s \rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2} t \quad \Longrightarrow \quad s \rightarrow_{\mathcal{R}_1} t \;\vee\; s \rightarrow_{\mathcal{R}_2} t$$

does not hold any more for CTRSs. This is due to the fact that when a rule of one of the systems is applied rules of the other system may be needed in order to satisfy the conditions. Recall that this phenomenon was already a major additional complication in Middeldorp's proof of the modularity of confluence for disjoint CTRSs (cf. Section 4.4).

[44]We had no time yet to investigate this in more detail though it seems promising.

Further problems are caused by extra variables in the conditions which may cause a conflict with proof techniques by induction over the rank, since for these (existentially quantified) extra variables one may substitute arbitrary terms (of arbitrary rank!).

5.5.1 Termination Properties under Signature Extensions

Again the situation is much more complicated for CTRSs than for TRSs since neither weak termination (WN), nor weak innermost termination (WIN) nor strong innermost termination (SIN) are modular in general for CTRSs. For WN and WIN this has been shown by Middeldorp ([Mid90; Mid93b]).

Example 5.5.1 (WN, WIN and SIN are not modular for CTRSs)
Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$

$$\left\{ \; a \to a \quad \Longleftarrow \quad x \downarrow b \wedge x \downarrow c \right.$$

and

$$\left\{ \begin{array}{l} G(x,y) \to x \\ G(x,y) \to y \end{array} \right.$$

over $\mathcal{F}_1 = \{a, b, c\}$ and $\mathcal{F}_2 = \{G, A\}$, respectively. Here, we have $a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a$ by applying the $\mathcal{R}_1$-rule (x is substituted by $G(b, c)$), but neither $a \to_{\mathcal{R}_1} a$ nor $a \to_{\mathcal{R}_2} a$. Hence, a is an $\mathcal{R}_i$-normal form (for $i = 1, 2$) but not a normal form w.r.t. $\mathcal{R}_1 \oplus \mathcal{R}_2$. Obviously, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are strongly (hence also weakly and innermost) terminating but their disjoint union is not. For instance, $a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} \ldots$ is an infinite innermost derivation, and a does not have a normal from w.r.t. $\mathcal{R}_1 \oplus \mathcal{R}_2$.

From the observation in Example 5.5.1 above one might be tempted to conjecture (as it is done in [Mid93b]) that the preservation of normal forms, defined by (with $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$)

$$\mathrm{NFP}(\mathcal{R}_1, \mathcal{R}_2) : \quad \mathrm{NF}(\mathcal{R}^{\mathcal{F}}) \; = \; \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}}) \cap \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}}) \; ,$$

should be a sufficient condition for the modularity of weak termination. But this is also not true in general.[45] The situation is even worse, since – surprisingly – it may happen that a weakly terminating CTRS may become not weakly terminating under the disjoint union with another 'empty' CTRS, i.e., simply by extending the signature without adding new rules.

[45] contradicting Theorem 5.2 in [Mid93b] (cf. also Theorem 4.3.20 in [Mid90]) the proof of which implicitly relies on the incorrect assumption $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1}) \iff \mathrm{WN}({\mathcal{R}_1}^{\mathcal{F}_1 \uplus \mathcal{F}_2})$.

Example 5.5.2 (WN, WIN are not preserved under signature extensions) Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ given by

$$\mathcal{R}_1 = \left\{ \begin{array}{rcl} g(x,y) \to g(x,y) & \Longleftarrow & x \downarrow z \wedge z \downarrow y \\ g(x,x) \to x & & \\ g(x,a) \to c & & \\ g(x,b) \to c & & \\ g(x,g(y,z)) \to c & & \\ g(a,x) \to c & & \\ g(b,x) \to c & & \\ g(g(y,z),x) \to c & & \\ c \to a & & \\ c \to b & & \end{array} \right.$$

over $\mathcal{F}_1 = \{g, a, b, c\}$ and $\mathcal{R}_2 = \emptyset$ over $\mathcal{F}_2 = \{G\}$ (with G unary). It is straightforward to verify that $\mathcal{R}_1^{\mathcal{F}_1}$ is weakly terminating and even weakly innermost terminating. The only potential reason for non-existence of a normal form of some given term from $\mathcal{T}(\mathcal{F}_1, \mathcal{V})$ is the first $\mathcal{R}_1$-rule. But whenever this rule is applicable another rule is applicable, too, which may be preferred (note that without the first rule the system $\mathcal{R}_1$ is even terminating). Now consider the combined system $\mathcal{R}^{\mathcal{F}} = {\mathcal{R}_1}^{\mathcal{F}_1 \uplus \{G\}}$ and the term $g(G(a), G(b))$. In $\mathcal{R}^{\mathcal{F}}$ we get the following cyclic derivation:

$$g(G(a), G(b)) \to_{\mathcal{R}^{\mathcal{F}}} g(G(a), G(b)) \to_{\mathcal{R}^{\mathcal{F}}} g(G(a), G(b)) \to_{\mathcal{R}^{\mathcal{F}}} \dots$$

by applying the first rule (which is indeed applicable since instantiating the extra variable z by $G(c)$ we easily obtain $G(a) \downarrow_{\mathcal{R}^{\mathcal{F}}} G(c)$ and $G(c) \downarrow_{\mathcal{R}^{\mathcal{F}}} G(b)$ as desired). Note moreover that there is no other way of reducing $g(G(a), G(b))$ (all its proper subterms are in normal form w.r.t. $\mathcal{R}^{\mathcal{F}}$, and the second rule is clearly not applicable). Hence, $\mathcal{R}^{\mathcal{F}} = {\mathcal{R}_1}^{\mathcal{F}_1 \uplus \{G\}}$ is neither WIN nor WN although the property $\mathrm{NF}(\mathcal{R}^{\mathcal{F}}) = \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}}) \cap \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}})$ is trivially satisfied.

By slightly modifying Example 5.5.2 (where $\mathcal{R}_1^{\mathcal{F}_1}$ is not SIN !) we can also show that SIN is not preserved under signature extensions in general.

Example 5.5.3 (SIN is not preserved under signature extensions) Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ given by

$$\mathcal{R}_1 = \left\{ \begin{array}{rcl} f(g(x,y)) \to f(g(x,y)) & \Longleftarrow & x \downarrow z \wedge z \downarrow y \\ g(x,x) \to x & & \\ g(x,a) \to c & & \\ g(x,b) \to c & & \\ g(x,f(y)) \to c & & \\ g(x,g(y,z)) \to c & & \\ g(a,x) \to c & & \\ g(b,x) \to c & & \\ g(f(y),x) \to c & & \\ g(g(y,z),x) \to c & & \\ c \to a & & \\ c \to b & & \end{array} \right.$$

over $\mathcal{F}_1 = \{f, g, a, b, c\}$ and $\mathcal{R}_2 = \emptyset$ over $\mathcal{F}_2 = \{G\}$ (with G unary). It is straightforward to verify that $\mathcal{R}_1^{\mathcal{F}_1}$ is SIN (hence also WIN and WN). The crucial point is that an arbitrary infinite $\mathcal{R}_1^{\mathcal{F}_1}$- derivation must contain rewrite steps using the first rule. But whenever this rule is applicable, the contracted redex cannot be innermost, since some proper subterm must then be reducible by the remaining rules which constitute a terminating CTRS. Nevertheless, in the extended system $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1{}^{\mathcal{F}_1 \uplus \{G\}}$ we get the cyclic (hence infinite) innermost $\mathcal{R}^{\mathcal{F}}$-derivation

$$f(g(G(a), G(b))) \underset{i}{\longrightarrow}_{\mathcal{R}^{\mathcal{F}}} f(g(G(a), G(b))) \underset{i}{\longrightarrow}_{\mathcal{R}^{\mathcal{F}}} f(g(G(a), G(b))) \underset{i}{\longrightarrow}_{\mathcal{R}^{\mathcal{F}}} \dots$$

by applying the first rule (instantiating the extra variable z by $G(c)$). Note moreover that there is no other way of reducing $f(g(G(a), G(b)))$ (all its proper subterms are in normal form w.r.t. $\mathcal{R}^{\mathcal{F}}$, and the second rule is clearly not applicable). Hence, $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1{}^{\mathcal{F}_1 \uplus \{G\}}$ is not SIN (and also neither WIN nor WN).[46]

By a thorough analysis of abstraction and innermost reduction properties which allow to project reduction sequences on mixed terms to certain reduction sequences on pure terms we shall show below how the monotonicity of WN, WIN and SIN under signature extensions can be guaranteed. More generally, we also develop sufficient criteria for the modularity of these properties. This analysis heavily relies on some very useful terminology and technical results from [Mid93b].

First we recall the definition of $\to_1$, $\to_2$ and $\to_{1,2}$ (cf. Definition 2.4.16). Intuitively, in a $\to_1$-step $s \to_1 t$ using a (black) rule from $\mathcal{R}_1$ we are only allowed to rewrite in the outermost black layers of the condition terms for verifying the corresponding conditions (using $\mathcal{R}_1$-rules). Due to this restriction, the relation $\to_1$ enjoys a much better behaviour than $\to_{\mathcal{R}_1}$ w.r.t. some desirable technical properties.

Lemma 5.5.4 (injective abstraction is possible for $\xrightarrow{o}_1$-steps, cf. [Mid93b, Proposition 3.5])
Let $\mathcal{R}_1$, $\mathcal{R}_2$ be disjoint CTRSs, s, t be black terms and σ be a top white substitution with $\sigma(s) \xrightarrow{o}_1 \sigma(t)$. Then, for any substitution τ with $\sigma \propto \tau$, we have $\tau(s) \xrightarrow{o}_1 \tau(t)$.

Note that this result implies in particular that a step $\sigma(s) \xrightarrow{o}_1 \sigma(t)$ on mixed terms can be injectively abstracted into a 'pure' step $\tau(s) \to_{\mathcal{R}_1} \tau(t)$ by injectively replacing the maximal top white aliens of $\sigma(s)$ and $\sigma(t)$ by fresh variables. One may wonder whether such an injective abstraction is also possible for an arbitrary outer step $\sigma(s) \xrightarrow{o}_{\mathcal{R}_1} \sigma(t)$, where for verifying the conditions of applied $\mathcal{R}_1$-rules also inner $\mathcal{R}_1$-steps are allowed. This is not the case as shown next.

Example 5.5.5 (injective abstraction is not always possible)
Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ given by

$$\left\{ \begin{array}{rcl} f(x, y) \to x & \Longleftarrow & x \downarrow y \\ a \to b & & \end{array} \right.$$

[46] Note that this example is also a counterexample under semi-equational semantics, i.e., when considering $\mathcal{R}_1$ as a semi-equational CTRS.

over $\mathcal{F}_1 = \{f, a, b\}$ and $\mathcal{R}_2 = \emptyset$ over $\mathcal{F}_2 = \{G\}$ (with G unary). Here we have $f(G(a), G(b)) \xrightarrow{o}_{\mathcal{R}_1} G(a)$ (due to $G(a) \downarrow_{\mathcal{R}_1} G(b)$), but not $f(G(a), G(b)) \xrightarrow{o}_1 G(a)$ since for satisfying $G(a) \downarrow_{\mathcal{R}_1} G(b)$ we need an inner $\mathcal{R}_1$-step. Note that after injective abstraction of $s = f(G(a), G(b))$ and $t = G(a)$ into $f(x, y)$ and x, respectively, the reduction $f(x, y) \rightarrow_{\mathcal{R}_1} x$ is not possible any more.

The above example somehow suggests that by forbidding the possibility of inner reduction steps injective abstraction of reduction steps might still be possible. But even if all maximal top white aliens are irreducible, this is not possible in general.

Example 5.5.6
Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ given by

$$\mathcal{R}_1 = \left\{ \begin{array}{rcl} f(x, y) \rightarrow x & \Longleftarrow & x \downarrow z \wedge z \downarrow y \\ c \rightarrow a & & \\ c \rightarrow b & & \end{array} \right.$$

with $\mathcal{F}_1 = \{f, a, b, c\}$, $\mathcal{R}_2 = \emptyset$, $\mathcal{F}_2 = \{G\}$ (with G unary). Here all proper subterms of $f(G(a), G(b))$ are ($\mathcal{R}^{\mathcal{F}}$-) irreducible and $f(G(a), G(b)) \xrightarrow{o}_{\mathcal{R}_1} G(a)$ but not $f(G(a), G(b)) \xrightarrow{o}_1 G(a)$ since for satisfying $G(a) \downarrow_{\mathcal{R}_1} z \wedge z \downarrow_{\mathcal{R}_1} G(b)$ we have to instantiate the extra variable z in the condition of the first rule by a mixed term of the form $G(u)$, e.g. $G(c)$, and to use inner $\mathcal{R}_1$-steps for establishing $G(a) \downarrow_{\mathcal{R}_1} G(u)$, $G(u) \downarrow_{\mathcal{R}_1} G(b)$. Note again that after injective abstraction of $s = f(G(a), G(b))$ and $t = G(a)$ into $f(x, y)$ and x, respectively, the reduction $f(x, y) \rightarrow_{\mathcal{R}_1} x$ is not possible any more.

Whereas in the above examples an injective abstraction of certain reduction steps on mixed terms to a corresponding reduction step on pure terms is not possible, a non-injective identifying one, which replaces all maximal top white aliens by the same fresh variable, is indeed possible. This is shown next.

Lemma 5.5.7 (identifying abstraction is always possible)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs, s and t be black terms, σ be a top white substitution, and $\widetilde{\sigma}$ be defined by $\widetilde{\sigma}(x) = z$ for all $x \in dom(\sigma)$ where z is a 'fresh' variable, i.e., not occurring in σs. Then we have:

(a) $\sigma(s) \xrightarrow{o}_{\mathcal{R}_1} \sigma(t) \quad \Longrightarrow \quad \widetilde{\sigma}(s) \xrightarrow{o}_{\mathcal{R}_1} \widetilde{\sigma}(t)$, and

(b) $\sigma(s) \xrightarrow{i}_{\mathcal{R}_1} \sigma(t) \quad \Longrightarrow \quad \widetilde{\sigma}(s) = \widetilde{\sigma}(t)$.

Proof: (b) is trivially satisfied by definition of $\xrightarrow{i}$ and of $\widetilde{\sigma}$. We show (a) by induction on n, the depth of rewriting (using the same proof structure as in [Mid93b] for Prop. 3.5, see Lemma 5.5.4 above). In the base case, i.e. for $n \leq 1$, (a) is trivially satisfied. For the induction step, assume $\sigma(s) \xrightarrow{o}_{\mathcal{R}_1} \sigma(t)$ is of depth $n + 1$. Hence we have $\sigma(s) = C[\rho(l)] \xrightarrow{o}_{\mathcal{R}_1} C[\rho(r)]$ for some context $C[]$, some substitution ρ and some rule $l \rightarrow r \Longleftarrow s_1 \downarrow t_1, \ldots, s_m \downarrow t_m$ from $\mathcal{R}_1$ such that $\rho(s_i) \downarrow_{\mathcal{R}_1} \rho(t_i)$ for

$i = 1, \ldots, m$[47] with depth less than or equal to n. Lemma 2.4.15 yields a decomposition $\rho_2 \circ \rho_1$ of ρ such that ρ_1 is black, ρ_2 top white and $\rho_2 \propto \mathrm{id}$. We observe that whenever $\rho_2(u_1) \to_{\mathcal{R}_1} u_2$ for some black term u_1, then there is some black term u_3 with $u_2 = \rho_2(u_3)$. Hence, for any i, $1 \leq i \leq m$, the conversion $\rho(s_i) \downarrow_{\mathcal{R}_1} \rho(t_i)$ has the form $\rho_2(u_{i,1}) \to_{\mathcal{R}_1} \rho_2(u_{i,2}) \to_{\mathcal{R}_1} \ldots \to_{\mathcal{R}_1} \rho_2(u_{i,k_i}) = \rho_2(v_{i,l_i}) \;{}_{\mathcal{R}_1}\!\!\leftarrow \ldots \;{}_{\mathcal{R}_1}\!\!\leftarrow \rho_2(v_{i,2}) \;{}_{\mathcal{R}_1}\!\!\leftarrow \rho_2(v_{i,1})$ (with $u_{i,1} = \rho_1(s_1)$, $v_{i,1} = \rho_1(t_1)$) for black terms $u_{i,1} \ldots u_{i,k_i}$, $v_{i,1}, \ldots, v_{l_i}$. Thus, by using the induction hypothesis and (b) for any single step in $\rho_2(\rho_1(s_i)) \downarrow_{\mathcal{R}_1} \rho_2(\rho_1(t_i))$ it is straightforward to show $\widetilde{\rho_2}(\rho_1(s_i)) \downarrow_{\mathcal{R}_1} \widetilde{\rho_2}(\rho_1(t_i))$ by an additional induction on the length of the conversion $\rho_2(\rho_1(s_i)) \downarrow_{\mathcal{R}_1} \rho_2(\rho_1(t_i))$, for $i = 1, \ldots, m$. ∎

A straightforward consequence of this 'identifying abstraction' lemma is the fact that for rewriting some black term with a black CTRS, considered as CTRS (with the same set of rules) over an extended black-white signature, it is not necessary to instantiate extra variables in the conditions of the (black) rules with non-black terms.

Corollary 5.5.8 ('reduction of pure terms is purely possible')
Let $\mathcal{R}^{\mathcal{F}}$ be a CTRS, $\mathcal{F}'$ be a signature with $\mathcal{F} \subseteq \mathcal{F}'$, and $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Then we have: $s \to_{\mathcal{R}^{\mathcal{F}'}} t \implies s \to_{\mathcal{R}^{\mathcal{F}}} t\,.$

Using Lemma 5.5.7 we are now able to show that at least termination is preserved under signature extensions.

Lemma 5.5.9 (termination is preserved under signature extensions)
Let $\mathcal{R}^{\mathcal{F}}$ be a CTRS and $\mathcal{F}'$ be a signature with $\mathcal{F} \subseteq \mathcal{F}'$. Then the following holds: $\mathrm{SN}(\mathcal{R}^{\mathcal{F}}) \iff \mathrm{SN}(\mathcal{R}^{\mathcal{F}'})$.

Proof: It clearly suffices to show: $\mathrm{SN}(\mathcal{R}^{\mathcal{F}}) \implies \mathrm{SN}(\mathcal{R}^{\mathcal{F}'})$. Hence, assuming $\mathrm{SN}(\mathcal{R}^{\mathcal{F}})$, we prove $\mathrm{SN}(s)$ for all $s \in \mathcal{T}(\mathcal{F}', \mathcal{V})$ by induction on $n = rank(s)$.[48] For $rank(s) = 1$ we get $\mathrm{SN}(s)$ by assumption and Corollary 5.5.8. Let $rank(s) > 1$. If the root symbol of s is a new one, i.e. from $(\mathcal{F}' \setminus \mathcal{F})$, then $\mathrm{SN}(s)$ follows by induction hypothesis, since rewrite steps in the top layer of s are impossible. If $root(s) \in \mathcal{F}$ then – again by induction hypothesis and the fact that $(\mathcal{F}' \setminus \mathcal{F})$-layers cannot collapse – any infinite $\mathcal{R}^{\mathcal{F}'}$-derivation starting with s would have to contain infinitely many outer $\mathcal{R}^{\mathcal{F}'}$-steps. But then identifying abstraction using Lemma 5.5.7 would yield an infinite $\mathcal{R}^{\mathcal{F}}$-derivation contradicting the assumption $\mathrm{SN}(\mathcal{R}^{\mathcal{F}})$. ∎

In order to present sufficient criteria for the preservation of restricted termination properties under signature extensions and – more generally – under disjoint unions, we will introduce now some more notations, in particular for certain innermost reductions steps.

Definition 5.5.10 (more innermost reduction properties)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs and $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2} = (\mathcal{R}_1 \uplus \mathcal{R}_2)^{\mathcal{F}_1 \uplus \mathcal{F}_2}$ be their

[47] Note that ρ may also instantiate extra variables in the conditions of the applied $\mathcal{R}_1$-rule.

[48] Note that we may consider here $\mathcal{R}^{\mathcal{F}'}$ as disjoint union $\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$ with $\mathcal{R}_1^{\mathcal{F}_1} = \mathcal{R}^{\mathcal{F}}$, $\mathcal{R}_2 = \emptyset$ and $\mathcal{F}_2 = \mathcal{F}' \setminus \mathcal{F}$.

disjoint union. If (for $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$) $s \to_{\mathcal{R}^\mathcal{F}} t$ by applying some $\mathcal{R}_1$-rule (where for satisfying the conditions also $\mathcal{R}_2$-rules are allowed) we denote this by $s \to_{\mathcal{R}_1^\mathcal{F}/\mathcal{R}^\mathcal{F}} t$ or simply $s \to_{\mathcal{R}_1/\mathcal{R}} t$. Furthermore, for $j = 1, 2$ we define the innermost reduction properties $\mathrm{IRP}_1(\mathcal{R}_j,\mathcal{R})$, $\mathrm{IRP}_2(\mathcal{R}_j,\mathcal{R})$ and $\mathrm{IRP}_3(\mathcal{R}_j,\mathcal{R})$ by[49]

$$\begin{cases} \mathrm{IRP}_1(\mathcal{R}_j,\mathcal{R}): & \forall s \in \mathcal{T}(\mathcal{F},\mathcal{V}): & s \xrightarrow[i]{}_{\mathcal{R}_j} t & \Longrightarrow & s \xrightarrow[i]{}_j t\,, \\ \mathrm{IRP}_2(\mathcal{R}_j,\mathcal{R}): & \forall s \in \mathcal{T}(\mathcal{F},\mathcal{V}): & s \xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}} t & \Longrightarrow & \exists t': s \xrightarrow[i]{}_{\mathcal{R}_j} t' \\ \mathrm{IRP}_3(\mathcal{R}_j,\mathcal{R}): & \forall s \in \mathcal{T}(\mathcal{F},\mathcal{V}): & s \xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}} t & \Longrightarrow & s \xrightarrow[i]{}_{\mathcal{R}_j} t\,. \end{cases}$$

Note that IRP_1 enables injective abstraction (via Lemma 5.5.4) which will be useful for establishing preservation results under signature extensions. Combined with IRP_2 or the stronger property IRP_3 it will turn out to capture the essence for obtaining modularity results later on.

Using the first innermost reduction property IRP_1 defined above we obtain a sufficient criterion for the preservation of WN, WIN and SIN under signature extensions as follows.

Theorem 5.5.11 (a sufficient condition for the preservation of WN, WIN, SIN under signature extensions)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs with $\mathcal{R}_2 = \emptyset$ (and $\mathcal{R}^\mathcal{F} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2} = \mathcal{R}_1{}^{\mathcal{F}_1 \uplus \mathcal{F}_2}$) such that $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_1^{\mathcal{F}})$ holds. Then we have the following equivalences:

(a) $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1}) \iff \mathrm{WN}(\mathcal{R}_1{}^{\mathcal{F}_1 \cup \mathcal{F}_2})$.

(b) $\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1}) \iff \mathrm{WIN}(\mathcal{R}_1{}^{\mathcal{F}_1 \cup \mathcal{F}_2})$.

(c) $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1}) \iff \mathrm{SIN}(\mathcal{R}_1{}^{\mathcal{F}_1 \cup \mathcal{F}_2})$.

Proof: Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$, $\mathcal{R}^\mathcal{F}$ be given as above satisfying $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_1^{\mathcal{F}})$. The '$\Longleftarrow$'-directions of (a), (b) and (c) are easy by Corollary 5.5.8.

(a) For proving $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1}) \Longrightarrow \mathrm{WN}(\mathcal{R}_1^{\mathcal{F}})$ we proceed by contradiction assuming $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1})$. Let s be counterexample of minimal rank, i.e., $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$ such that $\mathrm{WN}(s, \mathcal{R}_1^{\mathcal{F}})$ does not hold, with $rank(s)$ minimal. The case $rank(s) = 1$ is impossible by the assumption $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1})$, Corollary 5.5.8 and the fact that $\mathcal{R}_2 = \emptyset$. If $rank(s) > 1$ then s has the form $s = C[\![s_1,\ldots,s_m]\!]$. If the top layer $C[,\ldots,]$ is white then by the minimality assumption and $\mathcal{R}_2 = \emptyset$ we get $\mathrm{WN}(s, \mathcal{R}_1^{\mathcal{F}})$, hence a contradiction. Otherwise, in the interesting case where $C[,\ldots,]$ is black, we know by the minimality assumption that every s_i ($1 \le i \le m$) has a normal form w.r.t. $\mathcal{R}_1^{\mathcal{F}}$, let's say t_i. Hence, we get

$$s = C[\![s_1,\ldots,s_m]\!] \to^*_{\mathcal{R}_1^{\mathcal{F}}} C[t_1,\ldots,t_m] = C' \{\!\{ u_1,\ldots,u_n \}\!\}$$

for some black context $C'\{,\ldots,\}$ and top white normal forms u_j w.r.t. $\mathcal{R}_1^{\mathcal{F}}$. Choosing fresh variables $x_1,\ldots,x_n$ injectively, i.e. with $\langle u_1,\ldots,u_n\rangle \infty \langle x_1,\ldots,x_n\rangle$, we have

[49] The notations used here for innermost reduction are slightly ambiguous (for the sake of readability). When writing $s \xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}} t$, $s \xrightarrow[i]{}_{\mathcal{R}_j} t$ or $s \xrightarrow[i]{}_j t$, we always mean that the contracted subterm is an innermost redex of s w.r.t. $\to_\mathcal{R}$ (then it is also an innermost redex of s w.r.t. $\to_{\mathcal{R}_j/\mathcal{R}}$, $\to_{\mathcal{R}_j}$ or $\to_j$, respectively, since $\to_{\mathcal{R}_j/\mathcal{R}}$, $\to_{\mathcal{R}_j}$ and $\to_j$ are subsets of $\to_\mathcal{R}$).

$rank(C'\{x_1,\dots,x_n\}) = 1$, hence again by the minimality assumption $C'\{x_1,\dots,x_n\}$ can be reduced in $\mathcal{R}_1^{\mathcal{F}_1}$ to a normal form $C''\langle x_{i_1},\dots,x_{i_p}\rangle$. Thus we get

$$s \to^*_{\mathcal{R}_1^{\mathcal{F}}} C'\,\{\!\{\, u_1,\dots,u_n \}\!\} \to^*_{\mathcal{R}_1^{\mathcal{F}}} C''\,\langle\!\langle\, u_{i_1},\dots,u_{i_p} \rangle\!\rangle$$

with $u_{i_1},\dots,u_{i_p} \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$. Now, denoting u_{i_j} by v_j, it suffices to show $C''\,\langle\!\langle\, v_1,\dots,v_p \rangle\!\rangle \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$. If this were not the case then there would exist an (outer) innermost $\mathcal{R}_1^{\mathcal{F}}$-reduction step of the form

$$C''\,\langle\!\langle\, v_1,\dots,v_p \rangle\!\rangle \xrightarrow[i]{o}_{\mathcal{R}_1^{\mathcal{F}}} C'''\,\langle\!\langle\, v_{k_1},\dots,v_{k_q} \rangle\!\rangle$$

with $1 \le k_l \le p$, $1 \le l \le q$. But then, due to the assumption $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1},\mathcal{R}_1^{\mathcal{F}})$, we could apply Lemma 5.5.4 which would yield (denoting x_{i_j} by y_j, $1 \le j \le p$)

$$C''\langle y_1,\dots,y_p\rangle \to_{\mathcal{R}_1^{\mathcal{F}_1}} C'''\langle y_{k_1},\dots,y_{k_q}\rangle\ .$$

But this is a contradiction to $C''\langle y_1,\dots,y_p\rangle \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}_1})$. Hence in all cases we have a contradiction.

(b) For proving $\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1}) \implies \mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}})$ we use the same proof structure as for (a), but the argumentation is slightly different. Namely, (using the notations from above) in the interesting case we get

$$s = C[\![s_1,\dots,s_m]\!] \xrightarrow[i]{}^*_{\mathcal{R}_1^{\mathcal{F}}} C[t_1,\dots,t_m] = C'\,\{\!\{\, u_1,\dots,u_n \}\!\}$$

by innermost normalizing the s_i to t_i, $1 \le i \le n$. Furthermore, from

$$C'\{x_1,\dots,x_n\} \xrightarrow[i]{}^*_{\mathcal{R}_1^{\mathcal{F}_1}} C''\langle x_{i_1},\dots,x_{i_p}\rangle$$

(which is possible by the minimality assumption) we obtain, using for any step the same rule at the same position,

$$C'\,\{\!\{\, u_1,\dots,u_n \}\!\} \to^*_{\mathcal{R}_1^{\mathcal{F}}} C''\,\langle\!\langle\, u_{i_1},\dots,u_{i_p} \rangle\!\rangle\ .$$

Note that all $\mathcal{R}_1^{\mathcal{F}}$-reductions here are innermost steps, because otherwise we could conclude, using Lemma 5.5.4 (which is applicable due to the assumption $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1},\mathcal{R}_1^{\mathcal{F}})$), that the corresponding step in

$$C'\{x_1,\dots,x_n\} \xrightarrow[i]{}^*_{\mathcal{R}_1^{\mathcal{F}_1}} C''\langle x_{i_1},\dots,x_{i_p}\rangle$$

is a non-innermost $\mathcal{R}_1^{\mathcal{F}_1}$-step. Hence we have

$$s \xrightarrow[i]{}^*_{\mathcal{R}_1^{\mathcal{F}}} C'\,\{\!\{\, u_1,\dots,u_n \}\!\} \xrightarrow[i]{}^*_{\mathcal{R}_1^{\mathcal{F}}} C''\,\langle\!\langle\, u_{i_1},\dots,u_{i_p} \rangle\!\rangle\ .$$

Now $C''\,\langle\!\langle\, u_{i_1},\dots,u_{i_p} \rangle\!\rangle$ must be $\mathcal{R}_1^{\mathcal{F}}$-irreducible by the same argument as in (a) (exploiting again the assumption $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1},\mathcal{R}_1^{\mathcal{F}})$).

(c) For proving $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1}) \implies \mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}})$ we assume $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1})$ and show by contradiction that for every term $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$ we have $\mathrm{SIN}(s,\mathcal{R}_1^{\mathcal{F}})$. Consider a counterexample which is minimal w.r.t. the subterm relation, i.e., a term s which has an infinite innermost $\mathcal{R}_1^{\mathcal{F}}$-derivation

$$(D)\quad s = s_0 \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}}} s_1 \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}}} s_2 \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}}} \cdots$$

such that all proper subterms of s are strongly innermost terminating (w.r.t. $\mathcal{R}_1^{\mathcal{F}}$). Then, necessarily s is a top black term with $rank(s) > 1$. By the minimality assumption we know that there is some (first) innermost $\mathcal{R}_1^{\mathcal{F}}$-step

$$s_k = C_k[\![t_1, \dots, t_n]\!] \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}}} C' \langle\!\langle t_{i_1}, \dots, t_{i_p} \rangle\!\rangle = s_{k+1}$$

in (D) which is a root reduction step. This implies in particular that all maximal top white aliens of s_k (as well as of all $s_{k'}$, $k' > k$) are $\mathcal{R}_1^{\mathcal{F}}$-irreducible. Hence all steps in

$$s_k \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}}} s_{k+1} \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}}} s_{k+2} \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}}} \cdots$$

are outer steps. Due to the assumption $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_1^{\mathcal{F}}))$ we may apply now Lemma 5.5.4 which yields an infinite innermost $\mathcal{R}_1^{\mathcal{F}_1}$ derivation

$$\widehat{s_k} \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}_1}} \widehat{s_{k+1}} \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}_1}} \widehat{s_{k+2}} \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}_1}} \cdots$$

by injectively abstracting the maximal top white aliens by fresh variables. But this is a contradiction to $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1})$. ■

Using the second innermost reduction property from Definition 5.5.10, we get the following preservation result for weak and weak innermost termination (on the same signature).

Lemma 5.5.12 (a sufficient condition for the preservation of WN and WIN) Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs and $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then we have:

(a) $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R}) \implies [\,\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{WN}(\mathcal{R}_2^{\mathcal{F}}) \iff \mathrm{WN}(\mathcal{R}^{\mathcal{F}})\,]$.

(b) $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R}) \implies [\,\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{WIN}(\mathcal{R}_2^{\mathcal{F}}) \iff \mathrm{WIN}(\mathcal{R}^{\mathcal{F}})\,]$.

Proof: Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$, $\mathcal{R}^{\mathcal{F}}$ be given as above satisfying $\mathrm{IRP}_2(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}^{\mathcal{F}})$. The '$\Longleftarrow$'-directions of the equivalences in (a) and (b) are easy by Corollary 5.5.8.

(a) For proving $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{WN}(\mathcal{R}_2^{\mathcal{F}}) \implies \mathrm{WN}(\mathcal{R}^{\mathcal{F}})$ we proceed by contradiction assuming $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}})$, $\mathrm{WN}(\mathcal{R}_2^{\mathcal{F}})$. Let s be counterexample of minimal rank, i.e., $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $\mathrm{WN}(s, \mathcal{R}^{\mathcal{F}})$ does not hold, with $rank(s)$ minimal. The case $rank(s) = 1$ is impossible, due to the assumptions $\mathrm{WN}(\mathcal{R}_j^{\mathcal{F}})$, $\mathrm{IRP}_2(\mathcal{R}_j, \mathcal{R})$ (for $j = 1, 2$). If $rank(s) > 1$ then s has the form $s = C[\![s_1, \dots, s_m]\!]$ with s top black w.l.o.g., and we know by the minimality assumption that every s_i $(1 \le i \le m)$ has a normal form w.r.t. $\mathcal{R}^{\mathcal{F}}$, let's say t_i. Hence, we get

$$s = C[\![s_1, \dots, s_m]\!] \to^*_{\mathcal{R}^{\mathcal{F}}} C[t_1, \dots, t_m] = C' \{\!\{ u_1, \dots, u_n \}\!\}$$

for some black context $C'\{, \dots, \}$ and top white normal forms u_j w.r.t. $\mathcal{R}^{\mathcal{F}}$. By assumption, we can reduce $C' \{\!\{ u_1, \dots, u_n \}\!\}$ to some normal form w.r.t. $\mathcal{R}_1^{\mathcal{F}}$, i.e.

$$s \to^*_{\mathcal{R}^{\mathcal{F}}} C' \{\!\{ u_1, \dots, u_n \}\!\} \to^*_{\mathcal{R}_1^{\mathcal{F}}} C'' \langle\!\langle u_{i_1}, \dots, u_{i_p} \rangle\!\rangle$$

with $C'' \langle\!\langle u_{i_1}, \dots, u_{i_p} \rangle\!\rangle$ irreducible w.r.t. $\to_{\mathcal{R}_2/\mathcal{R}}$. Now, to obtain a contradiction, it suffices to show $C'' \langle\!\langle u_{i_1}, \dots, u_{i_p} \rangle\!\rangle \in \mathrm{NF}(\mathcal{R}^{\mathcal{F}})$. If this were not the case then $C'' \langle\!\langle u_{i_1}, \dots, u_{i_p} \rangle\!\rangle$ would be innermost $\to_{\mathcal{R}_1^{\mathcal{F}}/\mathcal{R}^{\mathcal{F}}}$-reducible, hence by the assumption $\mathrm{IRP}_2(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}^{\mathcal{F}})$ it would also be (innermost) $\to_{\mathcal{R}_1^{\mathcal{F}}}$-reducible, contradicting

$C'' \langle\!\langle u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$. Hence we are done.

(b) For proving $\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1}) \implies \mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}})$ we use the same proof structure as for (a), but the argumentation is slightly different. Namely, (using the notations from above) in the interesting case we get

$$s = C[\![s_1, \ldots, s_m]\!] \xrightarrow[i]{}{}^{*}_{\mathcal{R}_1^{\mathcal{F}}} C[t_1, \ldots, t_m] = C' \{\!\{ u_1, \ldots, u_n \}\!\}$$

by innermost normalizing the s_i to t_i, $1 \leq i \leq n$. Furthermore, from the assumption $\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1})$ we obtain

$$C' \{\!\{ u_1, \ldots, u_n \}\!\} \xrightarrow[i]{}{}^{*}_{\mathcal{R}_1^{\mathcal{F}_1}} C'' \langle\!\langle u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle =: t$$

with t in $\mathcal{R}_1^{\mathcal{F}}$ normal form. Moreover, t is also $\mathcal{R}^{\mathcal{F}}$-irreducible since otherwise, by the assumption $\mathrm{IRP}_2(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}^{\mathcal{F}})$ we would get a contradiction to $t \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$. ■

An equivalent characterization of the precondition in Lemma 5.5.12 is given by the normal form property defined above. More precisely, we obtain the following result.

Lemma 5.5.13 (relating normal form and innermost reduction properties)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs and $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then we have:

$$\mathrm{NF}(\mathcal{R}^{\mathcal{F}}) = \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}}) \cap \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}}) \iff \mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R}) \,.$$

Proof: Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ and $\mathcal{R}^{\mathcal{F}}$ be given as above.
"$\Longrightarrow$": Assuming $\mathrm{NFP}(\mathcal{R}_1, \mathcal{R}_2)$ we have to show $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R})$. Now, w.l.o.g. it suffices to show: $s \xrightarrow[i]{}_{\mathcal{R}_1/\mathcal{R}} t \implies \exists t' : s \xrightarrow[i]{}_{\mathcal{R}_1} t'$. Focusing on the contracted redex in s, let's say at position p, we get $s/p = C \{\!\{ s_1, \ldots, s_n \}\!\} \xrightarrow[i]{}_{\lambda, \mathcal{R}_1/\mathcal{R}} C' \langle\!\langle s_{i_1}, \ldots, s_{i_p} \rangle\!\rangle = t/p$ with s/p irreducible w.r.t. $\mathcal{R}_2^{\mathcal{F}}$. Hence, by $\to_{\mathcal{R}_1/\mathcal{R}}$-reducibility of s and the assumption $\mathrm{NFP}(\mathcal{R}_1, \mathcal{R}_2)$ we know that s/p (and thus also s) must be $\mathcal{R}_1^{\mathcal{F}}$-reducible, too, as desired.
"$\Longleftarrow$": Assuming $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R})$ it suffices to show $\mathrm{NF}(\mathcal{R}_1^{\mathcal{F}}) \cap \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}}) \subseteq \mathrm{NF}(\mathcal{R}^{\mathcal{F}})$. Suppose there exists a term $s \in (\,\mathrm{NF}(\mathcal{R}_1^{\mathcal{F}}) \cap \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}})\,) \setminus \mathrm{NF}(\mathcal{R}^{\mathcal{F}})$. Hence s is $\mathcal{R}^{\mathcal{F}}$-reducible and also $\mathcal{R}^{\mathcal{F}}$-innermost reducible, let's say with an $\mathcal{R}_1$-rule. Thus we have $s \xrightarrow[i]{}_{\mathcal{R}_1/\mathcal{R}} t$ which by $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R})$ implies $s \xrightarrow[i]{}_{\mathcal{R}_1} t'$ for some t'. But this is a contradiction to $s \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$, hence we are done. ■

Now, considering Lemma 5.5.12 above, the property $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R})$ does not yet suffice for the equivalence $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{SIN}(\mathcal{R}_2^{\mathcal{F}}) \iff \mathrm{SIN}(\mathcal{R}^{\mathcal{F}})$. To see this, consider the following slightly modified version of Example 5.5.1.

Example 5.5.14
Consider the disjoint CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ given by

$$\mathcal{R}_1 = \begin{cases} a \to a & \Longleftarrow \quad x \downarrow b \wedge x \downarrow c \\ a \to d & \end{cases}$$

and

$$\mathcal{R}_1 = \begin{cases} G(x, y) \to x \\ G(x, y) \to y \end{cases}$$

over $\mathcal{F}_1 = \{a, b, c, d\}$ and $\mathcal{F}_2 = \{G, A\}$, respectively. Here, both ${\mathcal{R}_1}^{\mathcal{F}_1 \uplus \mathcal{F}_2}$ and ${\mathcal{R}_2}^{\mathcal{F}_1 \uplus \mathcal{F}_2}$ are innermost terminating – and even terminating (it is easily shown that the first $\mathcal{R}_1$-rule is never applicable) – but their disjoint union $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \uplus \mathcal{R}_2)^{\mathcal{F}_1 \uplus \mathcal{F}_2}$ is not innermost terminating due to $a \xrightarrow[i]{}_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \xrightarrow[i]{}_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \xrightarrow[i]{}_{\mathcal{R}_1 \oplus \mathcal{R}_2} \dots$. Nevertheless $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfy the normal form property NFP$(\mathcal{R}_1, \mathcal{R}_2)$: $\mathrm{NF}(\mathcal{R}^{\mathcal{F}}) = \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}}) \cap \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}})$ or equivalently $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R})$ (since a is now both $\mathcal{R}_1^{\mathcal{F}}$- and $\mathcal{R}^{\mathcal{F}}$-reducible due the presence of the rule $a \to d$ in $\mathcal{R}_1$).

Requiring instead of $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R})$ the stronger property $\mathrm{IRP}_3(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_3(\mathcal{R}_2, \mathcal{R})$ accounts for this fact.

Lemma 5.5.15 (a sufficient condition for the preservation of SIN)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs and $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then we have:

$$\mathrm{IRP}_3(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_3(\mathcal{R}_2, \mathcal{R}) \implies [\,\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{SIN}(\mathcal{R}_2^{\mathcal{F}}) \iff \mathrm{SIN}(\mathcal{R}^{\mathcal{F}})\,].$$

Proof: Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ and $\mathcal{R}^{\mathcal{F}}$ be given as above satisfying $\mathrm{IRP}_3(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_3(\mathcal{R}_2, \mathcal{R})$. Now, the "$\Longleftarrow$"-direction is straightforward using Corollary 5.5.8. Vice versa, for proving $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{SIN}(\mathcal{R}_2^{\mathcal{F}}) \implies \mathrm{SIN}(\mathcal{R}^{\mathcal{F}})$ we proceed by contradiction assuming $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}})$ and $\mathrm{SIN}(\mathcal{R}_2^{\mathcal{F}})$. Consider a minimal counterexample, i.e., an infinite innermost $\mathcal{R}^{\mathcal{F}}$-derivation $s_0 \xrightarrow[i]{}_{\mathcal{R}^{\mathcal{F}}} s_1 \xrightarrow[i]{}_{\mathcal{R}^{\mathcal{F}}} s_2 \xrightarrow[i]{}_{\mathcal{R}^{\mathcal{F}}} \dots$ such that no proper subterm of s_0 admits infinite innermost $\mathcal{R}^{\mathcal{F}}$-derivations. By the minimality assumption some step $s_k \xrightarrow[i]{}_{\mathcal{R}^{\mathcal{F}}} s_{k+1}$ in the above derivation must be a root reduction step, let's say using a rule from $\mathcal{R}_1$. But then we know that all subsequent steps must also be $\to_{\mathcal{R}_1/\mathcal{R}}$-steps. Thus, by the assumption $\mathrm{IRP}_3(\mathcal{R}_1, \mathcal{R})$ we can conclude that $s_k \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}}} s_{k+1} \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}}} s_{k+2} \xrightarrow[i]{}_{\mathcal{R}_1^{\mathcal{F}}} \dots$ is an infinite innermost $\mathcal{R}_1^{\mathcal{F}}$-derivation contradicting $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}})$. ■

Next we shall provide sufficient conditions for the innermost reduction properties IRP_1, IRP_2 and IRP_3. Having again a look at Example 5.5.3 we observe that the system $\mathcal{R}_1$ there is non-confluent and has a rule with extra variables which seems to be essential. And indeed, forbidding extra variables or requiring confluence turns out to be crucial as will be shown next.

Lemma 5.5.16 (a first sufficient condition: no extra variables)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs without extra variables, with $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then the innermost reduction properties $\mathrm{IRP}_k(\mathcal{R}_j, \mathcal{R})$ hold for $j = 1, 2$ and $k = 1, 2, 3$.

Proof: It suffices to show the following for $j = 1, 2$:

$$\forall s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) : s \xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}} t \implies s \xrightarrow[i]{}_{\mathcal{R}_j} t \implies s \xrightarrow[i]{}_{j} t$$

This is straightforward by induction on the depth of rewriting, and exploiting the absence of extra variables (since the step $s \xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}} t$ is innermost and due to the absence of extra variables we know that for verifying the conditions of the applied $\mathcal{R}_j$-rule only outer $\to_j$-steps are possible). ■

For the case of confluent CTRSs we need two more technical lemmas from [Mid93b].

Lemma 5.5.17 ([Mid93b, Propositions 3.6, 3.8])
Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two disjoint confluent CTRSs and $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$. Then $\rightarrow_{1,2}$ is confluent and $\downarrow_{1,2}$ coincides with $\leftrightarrow^*_{\mathcal{R}}$.

Lemma 5.5.18 ([Mid93b, Proposition 3.13])
Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two disjoint, confluent CTRSs, $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ and $s_1, \ldots, s_n, t_1, \ldots, t_n$ be black terms. Then, for every substitution σ with $\sigma(s_i) \downarrow_{1,2} \sigma(t_i)$ for $i = 1, \ldots, n$ there exists a substitution τ such that $\sigma \rightarrow^*_{1,2} \tau$ and $\tau(s_i) \downarrow^o_1 \tau(t_i)$ for $i = 1, \ldots, n$.

Lemma 5.5.19 (a second sufficient condition: confluence)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint confluent CTRSs, with $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then the innermost reduction properties $\mathrm{IRP}_k(\mathcal{R}_j, \mathcal{R})$ hold for $j = 1, 2$ and $k = 1, 2, 3$.

Proof: It suffices to show the following for $j = 1, 2$:

$$\forall s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) : s \xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}} t \implies s \xrightarrow[i]{}_{\mathcal{R}_j} t \implies s \xrightarrow[i]{}_j t$$

Now consider a step $s \xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}} t$ using some $\mathcal{R}_j$-rule $l \rightarrow r \Longleftarrow P$ with matching substitution σ (which may also instantiate the extra variables in the conditions P). Hence, we have $\sigma(u) \downarrow_{\mathcal{R}} \sigma(v)$ for all conditions $u \downarrow v$ in P. By Lemma 5.5.17 and Lemma 5.5.18 we obtain the existence of some substitution τ with $\sigma \rightarrow^*_{1,2} \tau$ and $\tau(u) \downarrow^o_j \tau(v)$ for all $u \downarrow v$ in P. Since the step $s \xrightarrow[i]{}_{\mathcal{R}_j/\mathcal{R}} t$ is innermost we know that $\sigma(x)$ is irreducible for all $x \in dom(\sigma) \cap \mathcal{V}(l)$, hence σ and τ coincide on $\mathcal{V}(l)$. Thus we get $\sigma(l) = \tau(l)$, $\sigma(r) = \tau(r)$ which implies $s \xrightarrow[i]{}_j t$ as desired. ∎

Theorem 5.5.20 (sufficient conditions for preservation of WN, WIN, SIN under signature extensions)
The termination properties WN, WIN and SIN are preserved under signature extensions for confluent CTRSs as well as for CTRSs without extra variables.

Proof: Straightforward by combining Theorem 5.5.11 with Lemmas 5.5.12, 5.5.15, 5.5.16 and 5.5.19. ∎

5.5.2 Restricted Termination Properties

By combining Theorem 5.5.11 and the Lemmas 5.5.12, 5.5.15, 5.5.16 and 5.5.19 we now obtain the following modularity results.

Theorem 5.5.21 (sufficient conditions for modularity of WN, WIN, SIN)
Weak termination, weak innermost termination and (strong) innermost termination are modular properties for CTRSs without extra variables (in the conditions) and for confluent CTRSs.

Together with Theorem 4.4.1 this implies in particular the following.

Corollary 5.5.22 (semi-completeness is modular)
Semi-completeness is modular for disjoint unions of CTRSs.

Before turning to other modularity results now let us have a look upon the fact that for disjoint CTRSs $\mathcal{R}_1$, $\mathcal{R}_2$ confluence of $\mathcal{R}_1 \oplus \mathcal{R}_2$ implies confluence of both $\mathcal{R}_1$ and $\mathcal{R}_2$. As already mentioned this is straightforward for systems without extra variables but less obvious if extra variables are allowed. The problem is the following. Let the disjoint union $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ of $\mathcal{R}_1$ and $\mathcal{R}_2$ be confluent. Then we would like to show that for any black term s, whenever we have $s \to^*_{\mathcal{R}_1} t_1$ and $s \to^*_{\mathcal{R}_1} t_2$ (with t_1, t_2 necessarily black, too) then $t_1 \to^*_{\mathcal{R}_1} u$, $t_2 \to^*_{\mathcal{R}_1} u$ for some black term u. By confluence of $\mathcal{R}_1 \oplus \mathcal{R}_2$ we know that there exists (a black term) u with $t_1 \to^*_{\mathcal{R}} u$, $t_2 \to^*_{\mathcal{R}} u$. Now we would like to conclude that all the steps in $t_1 \to^*_{\mathcal{R}} u$, $t_2 \to^*_{\mathcal{R}} u$ are $\mathcal{R}_1$-steps. So, in a sense we would like to be sure that it is not necessary to substitute non-black terms for the extra variables of the black rules (in order to verify their associated, correspondingly instantiated conditions) applied in $t_1 \to^*_{\mathcal{R}} u$, $t_2 \to^*_{\mathcal{R}} u$. This can indeed be proved as will be sketched now. First we need another technical result adapted from [Mid93b].

Lemma 5.5.23
Let $\mathcal{R}_1$, $\mathcal{R}_2$ be disjoint CTRSs such that $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is confluent, and let $s_1, \ldots, s_n$, $t_1, \ldots, t_n$ be black terms. Then, for every substitution σ with $\sigma(s_i) \downarrow_{\mathcal{R}} \sigma(t_i)$ for $i = 1, \ldots, n$ there exists a substitution τ such that $\sigma \to^*_{\mathcal{R}} \tau$ and $\tau(s_i) \downarrow^o_1 \tau(t_i)$ for $i = 1, \ldots, n$.

Proof: We omit a detailed proof here, because it is quite lengthy and completely analogous to the proof of Prop. 3.13[50] in [Mid93b] (cf. Lemma 5.5.18 above). ■

Lemma 5.5.24 (extra variables are harmless for the case of confluence)
Let $\mathcal{R}_1$, $\mathcal{R}_2$ be disjoint CTRSs such that $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is confluent, and let s be a black term with $s \to_{\mathcal{R}_1/\mathcal{R}} t$. Then $s \to_{\mathcal{R}_1} t$ holds, too (by applying the same $\mathcal{R}_1$-rule).

Proof: Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$, s and t as above. Moreover assume that s_1,...,s_n, t_1,...,t_n are the (black) condition terms of the applied $\mathcal{R}_1$-rule $l \to r \Longleftarrow P$ with matching substitution σ and $\sigma(s_i) \downarrow_{\mathcal{R}} \sigma(t_i)$ for $i = 1, ..., n$. Note that – due to the possibility of extra variables in P – we have to take into account that σ may substitute non-black terms for some extra variables. Using Lemma 2.4.15 we can decompose σ into $\sigma_2 \circ \sigma_1$ with σ_1 black, σ_2 top white, $Dom(\sigma_1) \supseteq V(l)$ and such that σ_2 does not affect the σ_1-instances of the left hand side variables, i.e., $\sigma_2(\sigma_1(x)) = \sigma_1(x)$ for all $x \in V(l)$. Then we would like to conclude that there exists a black substitution σ' with $\sigma'(s_i) \downarrow_{\mathcal{R}_1} \sigma'(t_i)$ and $\sigma'(x) = \sigma(x) = \sigma_1(x)$ for all $x \in V(l)$, i.e. the substitutions σ and σ' may differ but only on the extra variables. Now, from $\sigma_2(\sigma_1(s_i)) \downarrow_{\mathcal{R}} \sigma_2(\sigma_1(t_i))$ we know that $\sigma_1(s_i)$, $\sigma_1(t_i)$ are black, hence Lemma 5.5.23 yields a substitution τ with $\sigma_2 \to^*_{\mathcal{R}} \tau$ and $\tau(\sigma_1(s_i)) \downarrow^o_1 \tau(\sigma_1(t_i))$ for $i = 1, \ldots, n$ such that by applying again Lemma 2.4.15 we obtain a decomposition of $\tau \circ \sigma_1$ into $\tau' \circ \sigma'_1$ with σ'_1 black and τ' top white and moreover $\sigma(x) = \sigma_1(x) = \sigma'_1(x) = \tau'(\sigma'_1(x))$ for all $x \in V(l)$. Applying Lemma 5.5.4 now yields $\tau''(\sigma'_1(s_i)) \downarrow^o_1 \tau''(\sigma'_1(t_i))$ for every τ'' with $\tau' \propto \tau''$. We choose τ'' to be defined by $\tau''(x) = z$ for all $x \in Dom(\tau')$ with z some new variable.

[50]Lemma 5.5.23 is obtained from [Mid93b, Proposition 3.13] by replacing $\to_{1,2}$ by $\to_{\mathcal{R}}$ (and accordingly, $\downarrow_{1,2}$ by $\downarrow_{\mathcal{R}}$) and assuming confluence of $\mathcal{R}$ instead of confluence of $\to_{1,2}$.

Hence we get $\tau''(\sigma_1'(s_i)) \downarrow_1^o \tau''(\sigma_1'(t_i))$ with $(\tau'' \circ \sigma_1')$ a black substitution satisfying $\sigma(x) = \sigma_1(x) = \sigma_1'(x) = \tau''(\sigma_1'(x))$ for all $x \in V(l)$. Since $\xrightarrow{o}_1$ is a subset of $\to_{\mathcal{R}_1}$ we are done. This means that there exists a black substitution satisfying the conditions w.r.t. $\mathcal{R}_1$ which coincides with the original (possibly mixed) substitution σ on $V(l)$. ■

As straightforward consequence of Lemma 5.5.24 we obtain the following.[51]

Corollary 5.5.25 (confluence of the disjoint union implies confluence of the component CTRSs)
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs (with extra variables in the conditions allowed) such that $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$ is confluent. Then both $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ are confluent, too.

5.5.3 Termination and Completeness

Let us now come to the question under which conditions termination and completeness are modular for disjoint CTRSs. Combining different results enables us to prove the following.

Theorem 5.5.26 (termination and completeness are modular for non-overlapping CTRSs)
Termination and completeness are modular for disjoint unions of non-overlapping CTRSs.

Proof: Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two disjoint non-overlapping and terminating CTRSs. Applying Theorem 2.3.21 yields confluence of $\mathcal{R}_i$ for $i = 1, 2$. By assumption we know in particular that both systems are innermost terminating. Hence, by Theorem 5.5.21 we get that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is innermost terminating, too. The property of being non-overlapping is obviously modular for CTRSs. Hence $\mathcal{R}_1 \oplus \mathcal{R}_2$ is innermost terminating and non-overlapping. Finally, applying Theorem 3.6.19 yields termination of $\mathcal{R}_1 \oplus \mathcal{R}_2$ which – again by Theorem 2.3.21 – implies confluence of $\mathcal{R}_1 \oplus \mathcal{R}_2$. Hence, $\mathcal{R}_1 \oplus \mathcal{R}_2$ is a confluent, terminating and non-overlapping CTRS. Vice versa, assume that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-overlapping and terminating, hence complete. Then we know that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-overlapping and terminating, hence complete (by Theorem 2.3.21). ■

Note that in the above proof we did not explicitly make use of the modularity of confluence (Theorem 4.4.1). However, this is necessary for the case of conditional overlay systems.

Theorem 5.5.27 (termination and completeness are modular for overlay CTRSs with joinable critical pairs)
Termination and completeness are modular for disjoint unions of conditional overlay systems with joinable critical pairs.

[51] Actually, this easier (and less interesting), yet not completely trivial direction of the modularity of confluence for disjoint CTRSs (cf. Middeldorp's Theorem 4.4.1) is neglected in [Mid90; Mid93b].

Proof: Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two disjoint terminating, conditional overlay systems with joinable critical pairs. Applying Theorem 2.3.21 yields confluence of $\mathcal{R}_i$ for $i = 1, 2$. By assumption we know in particular that both systems are innermost terminating. Hence, by Theorem 5.5.21 we get that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is innermost terminating, too. The property of being a conditional overlay system is obviously modular for CTRSs. Hence $\mathcal{R}_1 \oplus \mathcal{R}_2$ is an innermost terminating, conditional overlay system. Now, in order to be able to apply Theorem 3.6.19 for inferring termination of $\mathcal{R}_1 \oplus \mathcal{R}_2$ we need to establish joinability of all (conditional) critical pairs of $\mathcal{R}_1 \oplus \mathcal{R}_2$. Since both $\mathcal{R}_1$ and $\mathcal{R}_2$ are confluent we know by Theorem 4.4.1 that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is confluent, too. Hence, in particular, all critical pairs of $\mathcal{R}_1 \oplus \mathcal{R}_2$ must be joinable. Applying Theorem 3.6.19 now yields that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is a terminating and confluent conditional overlay system (with joinable critical pairs, of course). Vice versa, assume that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is a conditional overlay system with joinable critical pairs which is terminating, hence confluent and complete. Then we know that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating conditional overlay systems. By Corollary 5.5.25 confluence of $\mathcal{R}_1 \oplus \mathcal{R}_2$ implies confluence of both $\mathcal{R}_1$ and $\mathcal{R}_2$ (hence in particular also joinability of critical pairs). ■

Note that – compared to the unconditional case – the proofs of Theorem 5.5.26 and Theorem 5.5.27 are more complicated. This is due to the fact that both local confluence and joinability of all critical pairs are not modular for CTRSs in general (in fact, not even preserved under signature extensions (cf. Example 4.4.3) as well as innermost termination (cf. Example 5.5.3 above).

Remark 5.5.28 (extensions to semantic versions of being non-overlapping / overlaying are possible, too)
Let us mention that many (but not all) results for non-overlapping as well as for overlaying CTRSs with joinable critical pairs, in particular the latter two modularity results, can be slightly generalized by considering *semantic* versions of the properties NO and OS. We can define that a CTRS is *semantically non-overlapping* (*semantically overlaying*) if it has no feasible critical peak (if all its feasible critical peaks are overlays). For more details concerning such refined versions of NO and OS we refer to [Gra96b].

Next we shall sketch how our general abstraction approach for analyzing non-modularity of termination in the unconditional case (cf. Section 5.3.1) can be extended to the conditional case.

First we recall Example 5.5.14. There, the union of the disjoint terminating CTRSs

$$\mathcal{R}_1 = \left\{ \begin{array}{lcl} a \to a & \Longleftarrow & x \downarrow b \wedge x \downarrow c \\ a \to d & & \end{array} \right.$$

and

$$\mathcal{R}_1 = \left\{ \begin{array}{l} G(x,y) \to x \\ G(x,y) \to y \end{array} \right.$$

was non-terminating with the minimal counterexample

$$a \underset{i}{\longrightarrow}_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \underset{i}{\longrightarrow}_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \underset{i}{\longrightarrow}_{\mathcal{R}_1 \oplus \mathcal{R}_2} \cdots$$

even having rank 1. This cannot happen if we forbid extra variables (in the conditions of the rules).

Lemma 5.5.29 (Lemma 5.3.3 extended, [Gra93b])
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be two terminating disjoint CTRSs without extra variables such that

$$D: \; s_1 \to s_2 \to s_3 \to \dots$$

is an infinite derivation in $\mathcal{R}_1 \oplus \mathcal{R}_2$ of minimal rank (involving only ground terms). Suppose that s_1 is top black, i.e., $\mathcal{F}_1$-rooted. Then all s_i, $1 \leq i$, are top black, and we have:

(a) $rank(D) \geq 3$.

(b) Infinitely many steps in D are outer $\mathcal{R}_1/\mathcal{R}$-steps.[52]

Proof (idea): The proof that all s_i are top black and of (b) is as in the unconditional case. For the proof of (b) one shows that $rank(D) = 1$ and $rank(D) = 2$ are impossible, too (by induction on the depth of rewriting; here the assumption that extra variables are forbidden is crucial). ■

Note, that for unconditional TRSs in any minimal counterexample there are infinitely many inner reduction steps which are destructive at level 2. This property does not hold for CTRSs in general (even without variables).

Example 5.5.30 (minimal counterexamples need not contain destructive steps)
Consider the disjoint CTRSs

$$\mathcal{R}_1 = \Big\{ \; f(x) \to f(x) \quad \Longleftarrow \quad x \downarrow a \wedge x \downarrow b$$

and

$$\mathcal{R}_1 = \begin{cases} G(x,y) \to x \\ G(x,y) \to y \end{cases}$$

Both systems are terminating (even decreasing), but $\mathcal{R}_1 \oplus \mathcal{R}_2$ is not. In fact, the infinite $\mathcal{R}_1 \oplus \mathcal{R}_2$-derivation

$$f(G(a,b)) \to f(G(a,b)) \to f(G(a,b)) \to \dots$$

contains only outer $(\mathcal{R}_1/\mathcal{R})$-steps and hence no steps destructive at level 2.[53]

The extended version of our general structure Theorem 5.3.8(a) for CTRSs reads as follows.[54]

[52]This means that the outer steps are by applying $\mathcal{R}_1$-rules where, however, for verifying the conditions of the applied rules also $\mathcal{R}_2$-rules may be used.
[53]However, we observe that for verifying the instantiated conditions here, namely $G(a,b) \downarrow a$ and $G(a,b) \downarrow b$, destructive steps (using collapsing rules) are needed.
[54]It remains to be investigated whether Ohlebusch's construction for Theorem 5.3.8(b) can also be extended to disjoint unions of CTRSs.

Theorem 5.5.31 (a general structure theorem for non-modularity of termination of CTRSs)
Let $\mathcal{R}_1, \mathcal{R}_2$ be two disjoint finitely branching CTRSs without extra variables which are both terminating such that their disjoint union $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is non-terminating. Then one of the systems, let's say $\mathcal{R}_1$, is not termination preserving under non-deterministic collapses, i.e., $\mathcal{R}_1 \oplus \{G(x,y) \to x, G(x,y) \to y\}$ is non-terminating, and the other system $\mathcal{R}_2$ is collapsing, or vice versa.

Proof: The proof by minimal counterexample is analogous to the one of Theorem 5.3.8(a). In particular, the employed abstraction construction is the same. For the crucial properties of the abstracting transformation one needs an extended version of Lemma 5.3.6 which requires an additional global induction on the depth of rewriting.[55] For details we refer to [Gra93b]. ■

As in the unconditional case this result implies various sufficient criteria for modularity of termination of CTRSs. However, due to lack of space we will not discuss them here but simply refer to [Gra93b] for more details.

Instead let us mention some further known results due to Middeldorp which extend the corresponding results for the unconditional case.

Theorem 5.5.32 (preservation results for termination of CTRSs involving NCOL and NDUP, [Mid90; Mid93b])
Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two disjoint terminating CTRSs. Their union $\mathcal{R}_1 \oplus \mathcal{R}_2$ is terminating if one of the following conditions holds:

(1) Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-collapsing (NCOL).

(2) Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are confluent and non-duplicating (NDUP).

(3) Both systems are confluent and one of them is non-collapsing and non-duplicating.

Preservation results for termination of disjoint (or non-disjoint) unions of CTRSs corresponding to the ones in Section 5.3.3 for TRSs (based on left-linearity and uniqueness properties of collapsing reduction) are not known. Actually, we think that (non-)left-linearity in CTRSs is generally not yet well-understood.

5.5.4 Non-Disjoint Unions

Here we shall only collect the most relevant known results concerning the preservation of termination properties under non-disjoint unions of constructor sharing / composable CTRSs. In fact, there are not very many results up to date. One reason for that certainly is that most proofs are technically fairly involved.

Middeldorp ([Mid93a; Mid94]) has extended the corresponding results of [MT91; MT93] from the unconditional to the conditional case.

[55] For the latter inductive proof the assumption that extra variables are forbidden seems again to be essential.

Theorem 5.5.33 (modularity results for composable conditional constructor systems, [Mid93a; Mid94])
The following properties are modular for composable conditional constructor systems without extra variables (in the conditions of the rules):

(1) Weak termination (WN).

(2) Semi-Completeness (WN $\wedge$ CR).

(3) Completeness (SN $\wedge$ CR).

Ohlebusch ([Ohl94a; Ohl95b]) extended some results on semi-completeness and termination from the disjoint union case to constructor sharing / composable CTRSs.

Theorem 5.5.34 ([Ohl94a; Ohl95b])
Semi-completeness is modular for constructor sharing CTRSs.

Theorem 5.5.35 (generalized version of Theorem 5.5.32, [Ohl94a; Ohl95b])
Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two constructor sharing terminating CTRSs. Their union is terminating if one of the following conditions holds:

(1) Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are layer preserving.

(2) Both $\mathcal{R}_1$ and $\mathcal{R}_2$ are confluent and non-duplicating.

(3) Both systems are confluent and one of them is layer preserving and non-duplicating.

Our results on weak termination, weak and strong innermost termination, termination and completeness of disjoint unions of CTRSs presented above extend to the constructor sharing case as follows.

Theorem 5.5.36 (generalized version of Theorem 5.5.21, [Gra96b])
The following properties are modular for constructor sharing CTRSs without extra variables (in the conditions of the rules), as well as for confluent constructor sharing CTRSs:[56]

(1) Weak termination (WN).

(2) Weak innermost termination (WIN).

(3) Strong innermost termination (SIN).

Similarly, the above results on termination and completeness of disjoint unions of non-overlapping and overlay CTRSs extend to the constructor sharing case as well.

Theorem 5.5.37 (generalized version of Theorems 5.5.26, 5.5.27, [Gra96b])
Termination and completeness are modular for constructor sharing non-overlapping CTRSs as well as for constructor sharing overlay CTRSs with joinable critical pairs.

[56] Note that, for confluent constructor-sharing CTRSs, part (1) is equivalent to Theorem 5.5.34 above.

It seems plausible to conjecture that these results as well as (most, if not all) other ones should also hold for composable CTRSs, however, corresponding proof attempts will be — at least technically — challenging.

Chapter 6

Related Topics and Concluding Remarks

Here we give an outline of issues that have not been not explicitly treated or only touched. In Section 6.1 more general hierarchical as well as other types of combining rewrite systems are briefly discussed including the relevant literature. We sketch basic new problems arising for instance when considering hierarchically structured combinations of systems. General aspects of combining abstract reduction systems and known approaches in this field are very briefly dealt with in Section 6.2. Topics and fields which are more or less closely related to the main themes in this thesis but which had to be neglected or omitted in the presentation are finally summarized in Section 6.3.

6.1 Hierarchical and Other Types of Combinations

In the preceding chapters we have mainly dealt with combinations of disjoint, constructor sharing or composable (C)TRSs. Actually, some basic ideas, techniques and results also extend to more general types of combinations, for instance *hierarchical* ones.

Such asymmetric *hierarchical* combinations are obtained by requiring that one system (the *base system*) does not *depend* — in a sense to be made precise — on the other one (the *hierarchical extension*), but possibly vice versa. For instance, constructor sharing combinations may be generalized to hierarchical ones by allowing that the defined function symbols of the base system may occur in (right hand sides, or both right and left hand sides but not as root symbols of the latter, of) rules of the extension but not vice versa. A very simple example of this kind of hierarchical combination is the following.

Example 6.1.1 ([Gra91][1])
Consider the combination of the one-rule base system

$$a \to b$$

[1] In another context, a variant of this example appears also in [Der81].

with the one-rule extension system

$$h(x, x) \to h(a, b)\,.$$

Clearly, both systems are (even simply) terminating (and confluent) however their combination is non-terminating (and even not weakly terminating) due to the cycle

$$h(b, b) \to h(a, b) \to h(b, b)\,.$$

Intuitively it is clear that the problem here comes from the fact that the defined function symbol a of the base system occurs below the defined symbol h of the extension, namely in the right hand side of the second rule. Another problematic effect arising can also be observed in this example: Reduction in the combined system need not be *rank-decreasing* any more. Since, using a natural extension of the definition of *rank* (by considering for instance a to be black, h to be white, and b as well as variables to be transparent), it is obvious that the second rule is strictly rank-increasing. This shows that without further restrictions most results for disjoint, constructor sharing and composable systems are unlikely to extend to arbitrary combinations. In particular, the very useful possibility of performing proofs by induction on the rank of terms is in general not possible for hierarchical combinations.

Somewhat better behaved is another basic example from algebraic specification by (equations and) rewrite rules. Namely, take the usual hierarchical rewrite specification of *multiplication* in terms of a base system specifying *addition*, for instance on natural numbers constructed by *zero* and the *successor* function. Here, the base system is given by

$$\begin{aligned} 0 + y &\to y \\ s(x) + y &\to s(x + y) \end{aligned}$$

and the extension consists of the rules

$$\begin{aligned} 0 \times y &\to 0 \\ s(x) \times y &\to y + (x \times y) \end{aligned}$$

Here, the combined specification for addition and multiplication indeed inherits all nice properties like termination and confluence from the component systems as one would probably expect.

One may also define and consider extended versions of hierarchical combinations which additionally allow for a common shared part thus extending the notion of composable systems.

The first results on termination properties of hierarchical combinations of TRSs were reported independently by Krishna Rao ([Kri92]) and Dershowitz ([Der92]). More thorough investigations by the same authors as well as by others — concerning (innermost, general, weak and simple) termination, completeness and semi-completeness followed subsequently: [Kri93; Kri95b], [Der95], [Gra93a], [Kri94b], [Kri95c]. In particular, in [Kri95b], [Kri95c] an interesting refined notion of hierarchical combination is introduced[2] which generalizes not only combinations of constructor sharing, but also

[2] which is termed *super-hierarchical* there

of composable TRSs.

Another recent interesting paper is by Fernández & Jouannaud ([FJ95]) who besides hierarchical combinations also consider *cap-decreasing* and *alien-decreasing* unions. Cap-decreasing and alien-decreasing unions naturally generalize the cases of disjoint unions of non-collapsing and of non-duplicating systems, respectively.

6.2 Combining Abstract Reduction Systems

Many results on properties of combined TRSs actually rely on corresponding results for abstract reduction systems (ARSs), in particular on various *commutation* properties. Commutation properties are in a sense *rearrangement* properties for reduction sequences. In the early literature (cf. e.g. [Hin64], [Ros73], [Sta75]) commutation (more precisely, its symmetric version, sometimes also called *commutation with*) is mainly used for giving confluence criteria for combined ARSs in terms of commutation conditions. Later on, asymmetric versions of commutation, in particular *commutation over* and *quasi-commutation over*, have been defined and used to derive abstract criteria for termination as well as for other related properties involving both termination and confluence (cf. e.g. [BD86]). Further interesting papers dealing with related proof techniques for properties of ARSs include among others [Klo80] (ARSs, *combinatory reduction systems*), [Hue80] (an early survey of the state of the art for ARSs and TRSs), [Der83] (construction mechanisms for well-founded orderings), [Ges90] (*relative termination*), [Toy92] (*balanced weak Church-Rosser property*), [Oos94a] (confluence by *decreasing diagrams*).

For TRSs, results on termination and confluence properties (of single or combined systems) based on commutation techniques via additional syntactic conditions like left- and right-linearity and critical pair properties appear e.g. in [RV80], [Hue80], [Der81], [BD86], [Toy88], [Ges90], [BL90], [TKB89; TKB95], [Pre94], [Der95], [Oos95].

6.3 Related Fields

Finally, we would like to mention and briefly discuss various interesting topics and fields, which are more or less closely related to the main subjects of this thesis but which have at most been touched. Even this list is not exhaustive, and we concentrate on the most relevant aspects.

- **String Rewriting Systems (SRS):** Various results presented and mentioned in this thesis might have interesting applications and consequences for string rewriting systems, a special case of TRSs. This should be investigated in more detail (cf. e.g. [BO93], [Kur90], [Wra92]).

- **Forward / Overlap Closures:** Some results presented in Chapter 3 are closely related to properties of (forward) closures of TRSs ([Geu89], [DH95]) which can also be used to derive modularity results. Forward / overlap closures have been introduced as an interesting syntactical means for termination proofs (cf. [Der81], [GKM83], [Zha91]).

- **Extending /Combining Orderings:** Positive modularity results for termination of combined systems somehow always have a counterpart in corresponding extensions / combinations of appropriate well-founded orderings on terms. First promising results on such ordering extensions and combinations have been obtained in [Rub95]. More efforts in this direction are likely to produce further interesting insights and results.

- **Combining Equation Systems / Unification Theory:** Combining equational theories and corresponding matching / unification algorithms has been an active and fruitful area of research (cf. e.g. [Sch89], [Nip89], [JK91], [BS92],[BS94]). And it is well-known that for instance collapsing rules are also problematic within an equational setting. Hence, it might be worth investigating whether modularity results in term rewriting (as treated here) and in bi-directional rewriting, i.e., equational term rewriting and unification theory, could have a mutually fertilizing effect.

- **Typed Rewriting:** By imposing a *type* (or *sort)*) discipline certain semantic aspects of computation by rewriting / equational reasoning can be better modeled than by unsorted systems. For instance, *many-sorted* and (various forms of) *order-sorted* versions of equational reasoning are well-established nowadays. Consequently, modularity, preservation and decomposition results for many-and order-sorted versions of rewriting would also be very useful. Moreover, these typing mechanisms can also be applied for the purpose of termination and confluence proofs in the untyped (unsorted) setting. Some works along this line of reasoning are [GG87], [Pol92], [Gna92], [Zan94]. However, many interesting questions in this field are still open.

- **Function Definition Formalisms:** General term rewriting provides an elegant and powerful formalism for specifying recursive functions (algorithms) in a rather abstract way, with a well-defined denotational semantics. Other, more operationally and algorithmically oriented formalisms have been widely used, too ([BM79]). Typically arising problems there involve termination proofs for algorithms and inductive theorem proving tasks ([Wal94a; Wal94b], [Gie95]). In these latter function definition formalisms the employed evaluation mechanism corresponds to innermost term rewriting, and the imposed specification discipline means specification by locally confluent constructor (hence overlaying or even non-overlapping) systems. Thus, our results in Chapter 3 on innermost and general termination close the gap between the latter formalisms and term rewriting. For fully exploiting the potential benefits thereof, however, more detailed investigations seem to be necessary.

- **Rewriting Modulo:** Almost all modularity results presented apply only for ordinary rewriting. Of course, it would be interesting to try to generalize (at least some of) these results to the more general case of rewriting modulo some equational theory. Some progress along this line of reasoning is reported in [FJ95], [Rub95].

- **Term Graph Rewriting:** In implementations of term rewriting often various forms of *graph rewriting* are used. The basic idea here is to represent terms as *directed acyclic graphs (DAGs)* thereby enabling a flexible amount of structure sharing. Hence, shared subterms may be simultaneously rewritten. In general, the corresponding versions of *term graph rewriting systems* have other properties than TRSs, also with respect to their modularity behaviour. For modularity and other results in this setting we refer to [Plu93b; Plu93a; Plu94], [KO95b], [FJ95], [Kri95a]. Graph rewriting is surveyed in [Cou90], term graph rewriting in [SPe93].

- **Combining First- and Higher-Order Rewriting:** The combination of (first-order) TRSs and various versions of λ-calculi and the investigation of confluence and termination properties of the resulting systems is a rapidly expanding, interesting field of research. Many papers in this direction have appeared in the last few years, for instance: [BT88], [BTG89; BTG91; BTG94], [Oka89], [Bar90], [JO91], [BF93b; BF93a], [DK94].

Appendix A

Proofs

Proof: **(of Theorem 3.6.1)**
For the sake of readability, let us recall the statement to be proved (under the assumption that $\mathcal{R}$ is a CTRS with OS($\mathcal{R}$) and JCP($\mathcal{R}$)):

> Let s be a term with SN(s). For all terms t, u, v and sets Π of mutually disjoint positions, such that $s \to^* t$ and $u = C[s]_\Pi \to^* v$, we have $C[t]_\Pi \downarrow v$.

We shall proceed by proving a slightly more general statement, making use of the encoding of conditions by an equality predicate eq. [1] For this purpose we extend the CTRS $\mathcal{R}$ over the given signature $\mathcal{F}$ as follows. The signature $\mathcal{F}$ is extended into $\mathcal{F}' := \mathcal{F} \uplus \{eq, true\}$, with $true$ a new constant of some new sort and eq a new binary function symbol of the same new sort with arguments of the 'old' sort. $\mathcal{R}$ is extended into $\mathcal{R}' := \mathcal{R} \uplus \{eq(x,x) \to true\}$ where x is a variable of the 'old' sort. This extension $\mathcal{R}'$ (over $\mathcal{F}'$) of $\mathcal{R}$ (over $\mathcal{F}$) is conservative in the sense of Remark 2.3.6. Moreover, $\mathcal{R}'$ obviously also satisfies OS($\mathcal{R}'$) and JCP($\mathcal{R}'$).

Now consider the following statement (I):

> Let $\bar{s} \in \mathcal{T}(\mathcal{F},\mathcal{V})$ with SN($\bar{s}$). For all terms $t \in \mathcal{T}(\mathcal{F},\mathcal{V})$, $u, v \in \mathcal{T}(\mathcal{F}',\mathcal{V})$ and all sets Π of mutually disjoint positions, such that $\bar{s} \to^*_{\mathcal{R}} t$ and $u = C[\bar{s}]_\Pi \to^*_{\mathcal{R}'} v$, we have $C[t]_\Pi \downarrow_{\mathcal{R}'} v$.

According to the properties of the extension as described in Remark 2.3.6 (and since $\mathcal{R}'$ also satisfies OS($\mathcal{R}'$) and JCP($\mathcal{R}'$)), this statement implies the above original one to be proved.

Actually, instead of (I) we shall prove the following strengthened version (II):

> Let $\bar{s} \in \mathcal{T}(\mathcal{F},\mathcal{V})$ with SN($\bar{s}$). For all terms $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$ with $\bar{s} >_1 s$, all $t \in \mathcal{T}(\mathcal{F},\mathcal{V})$, all $u, v \in \mathcal{T}(\mathcal{F}',\mathcal{V})$ and all sets Π of mutually disjoint positions, such that $s \to^*_{\mathcal{R}} t$ and $u = C[s]_\Pi \to^*_{\mathcal{R}'} v$, we have $C[t]_\Pi \downarrow_{\mathcal{R}'} v$.

[1] Cf. Remark 2.3.6.

Here the partial ordering $>_1$ (depending on $\bar{s}$) is defined by $>_1 := (\to \cup \rhd)^+{}_{\text{below}\,\bar{s}}$, where — for some binary (ordering) relation R — $R_{\text{below}\,a}$ is defined by $R \cap (\{b | aR^*b\}^2)$. Note that $>_1$ (in (II) above) is well-founded due to SN($\bar{s}$). We shall prove (II) by induction, using the following complexity measure[2] $M(s,t,u,v,\Pi)$ for $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$ with $\bar{s} >_1 s$, $t \in \mathcal{T}(\mathcal{F},\mathcal{V})$, $u, v \in \mathcal{T}(\mathcal{F}',\mathcal{V})$ and Π some set of mutually disjoint positions, such that $u = C[s]_\Pi \to^*_{\mathcal{R}'} v$:

$$M(s,t,u,v,\Pi) = \langle s, n, k \rangle \,,$$

where

$$n = n(u,v) = \min\{m \,|\, u \overset{m}{\longrightarrow}{}^* \; v\}$$

and

$$k = k(u,v,n) = \min\{l \,|\, u \overset{n}{\longrightarrow}{}^l \; v\} \,.$$

These triples $\langle s,n,k \rangle$ are compared using the lexicographic combination $\succ = \mathit{lex}(>_1, >_2, >_3)$ with $>_2 = >_3 = > = >_{\mathbb{N}}$ (and $>_1$ as above). Now, by well-foundedness of $>_1$ (and of $>_{\mathbb{N}}$) $\succ$ is well-founded, too.

For a proof of (II) by contradiction, we may assume that there exists a minimal counterexample w.r.t. $\succ$, i.e., s, t, u, v, Π as above with

$$(a) \quad s \to^*_{\mathcal{R}} t \;\wedge\; u = C[s]_\Pi \to^*_{\mathcal{R}'} v$$

and

$$(b) \quad \neg\,(\, C[t]_\Pi \downarrow_{\mathcal{R}'} v\,)$$

such that the corresponding complexity measure $M(s,t,u,v,\Pi) = \langle s,n,k \rangle$ (with $\langle s,n,k \rangle$ as above) is minimal w.r.t. $\succ$.[3] In order to obtain a contradiction we proceed by case analysis showing that the counterexample above cannot be minimal. Subsequently, for the sake of readability we shall omit the subscripts in notations like $s \to^*_{\mathcal{R}} t$ and $u \to^*_{\mathcal{R}'} v$ (since they are always clear from the context).

If $u = v$ (i.e. $n = k = 0$) or $s = t$ we are done since (b) is violated. Otherwise, let $s \to s' \to^* t$. If we can show that $C[s']_\Pi \downarrow v$ holds then by induction (the first component of the measure decreases) we get $C[t]_\Pi \downarrow v$ because we have $s \to s'$, hence $s >_1 s'$. But this is a contradiction to (b). We shall distinguish the following cases:

(1) Proper subterm case (see Figure 1): If the first step $s \to s'$ reduces a proper subterm of s, i.e. $s \to_p s'$ for some $p > \lambda$, then we have

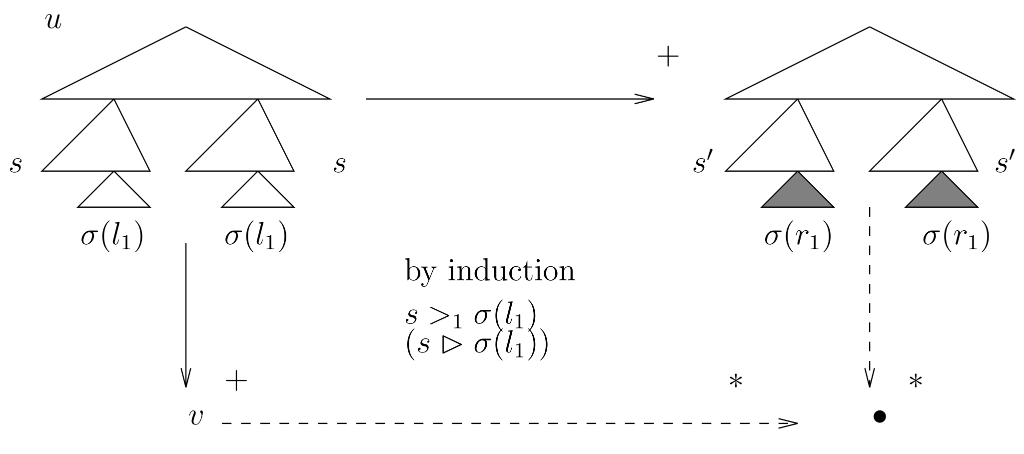
$$\begin{aligned} C[s]_\Pi &= C[C'[s/p]_p]_\Pi = (C[C'[]_p]_\Pi)[s/p]_{\Pi p} \to^+ C[s']_\Pi \\ &= (C[C'[]_p]_\Pi)[s'/p]_{\Pi p} \end{aligned}$$

[2]Actually, this measure does not depend on t.

[3]Note that instead of requiring here minimality of $\langle s,n,k \rangle$ w.r.t. to $\succ$ (the definition of which depends on $\bar{s}$) it would also suffice to require only minimality w.r.t. $\succ_s = \mathit{lex}((\to \cup \rhd)^+{}_{\text{below}\,s}, >_2, >_3)$. For a detailed discussion of this aspect and a theoretical foundation of the underlying parameterized version of the well-founded induction principle the reader is referred to Appendix B.

with $s = C'[s/p]_p$ for some context $C'[]_p$, hence $C[s']_\Pi \downarrow v$ as desired by induction (the first component of the measure decreases because $s \rhd s/p = \sigma(l_1)$ implies $s >_1 s/p$).

Figure 1 : Proper subterm case



(2) Otherwise, we may suppose

$$s \to_{\lambda,\sigma,l_1 \to r_1 \Longleftarrow P_1} s', \text{ i.e., } s = \sigma(l_1)\ ,\ s' = \sigma(r_1) \text{ and } \sigma(P_1) \downarrow$$

for some rule $l_1 \to r_1 \Longleftarrow P_1 \in \mathcal{R}$ and some substitution σ. Moreover assume

$$u = C[s]_\Pi \xrightarrow{n}_{q,\tau,l \to r \Longleftarrow P} u' \xrightarrow{n}{}^{k-1} v\ ,$$

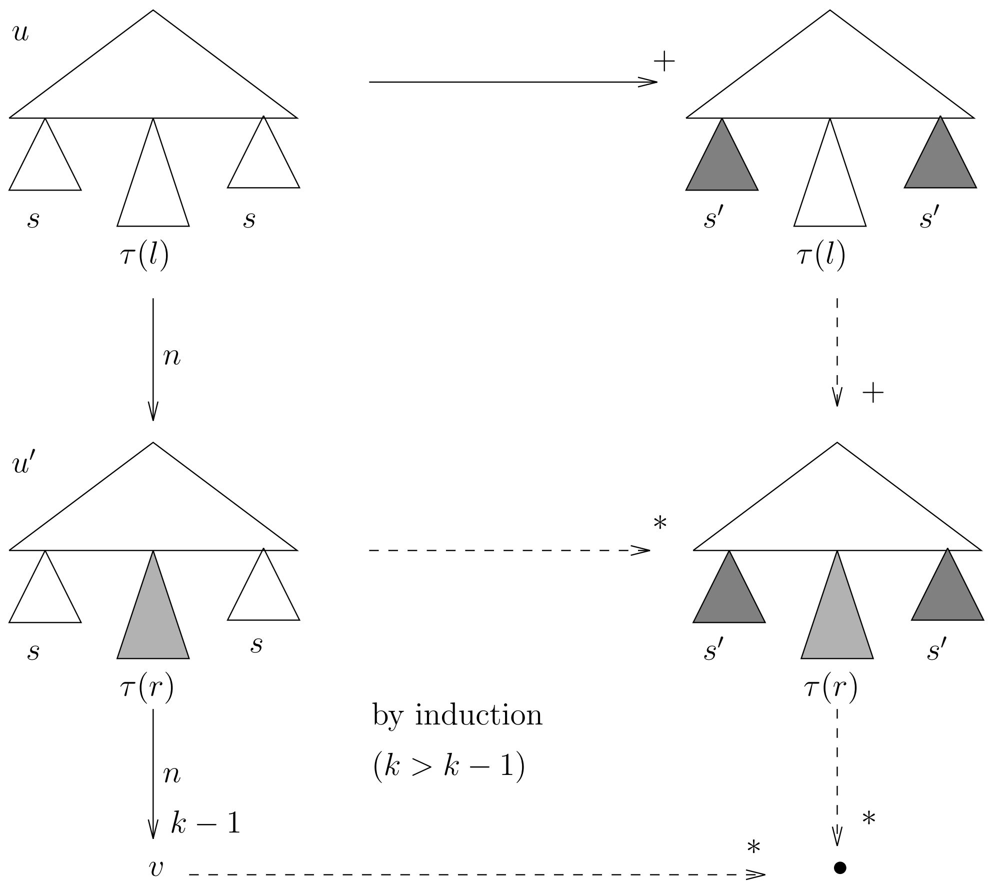
i.e., $C[s]_\Pi/q = \tau(l)$, $u'/q = \tau(r)$ and $\tau(P)\downarrow_{n-1}$, for $n \geq 1$ minimal with $u \xrightarrow{n}{}^{*} v$ and $k \geq 1$ minimal with $u \xrightarrow{n}{}^{k} v$.[4]

Then we have to distinguish the following four subcases according to the relative positions of q and Π:

(2.1) $q \mid \Pi$ (disjoint peak, see Figure 2.1): Then we have $u = C[s]_\Pi \xrightarrow{n}_q C'[s]_\Pi = u' \xrightarrow{n}{}^{k-1} v$, $C'[s]_\Pi \to^* C'[s']_\Pi$ and $C[s]_\Pi \to^* C[s']_\Pi \to_q C'[s']_\Pi$ for some context $C'[,\ldots,]$, hence by induction (the measure decreases in the second or third component) $C'[s']_\Pi \downarrow v$ and thus $C[s']_\Pi \to C'[s']_\Pi \downarrow v$ as desired.

[4]Note that $n' = \min\{m \mid u' \xrightarrow{m}{}^{*} v\}$, $k' = \min\{l \mid u' \xrightarrow{n'}{}^{l} v\}$ and n, k are related as follows: either $n' < n$ or else $n' = n$ and $k' < k$. For that reason, one may apply the induction hypothesis in situations of the form $u' = C'[s]_{\Pi'} \to^* v$, $s \to^* t'$ for inferring $C'[t']_{\Pi'} \downarrow v$, because of a decrease of the measure in the second or third component.

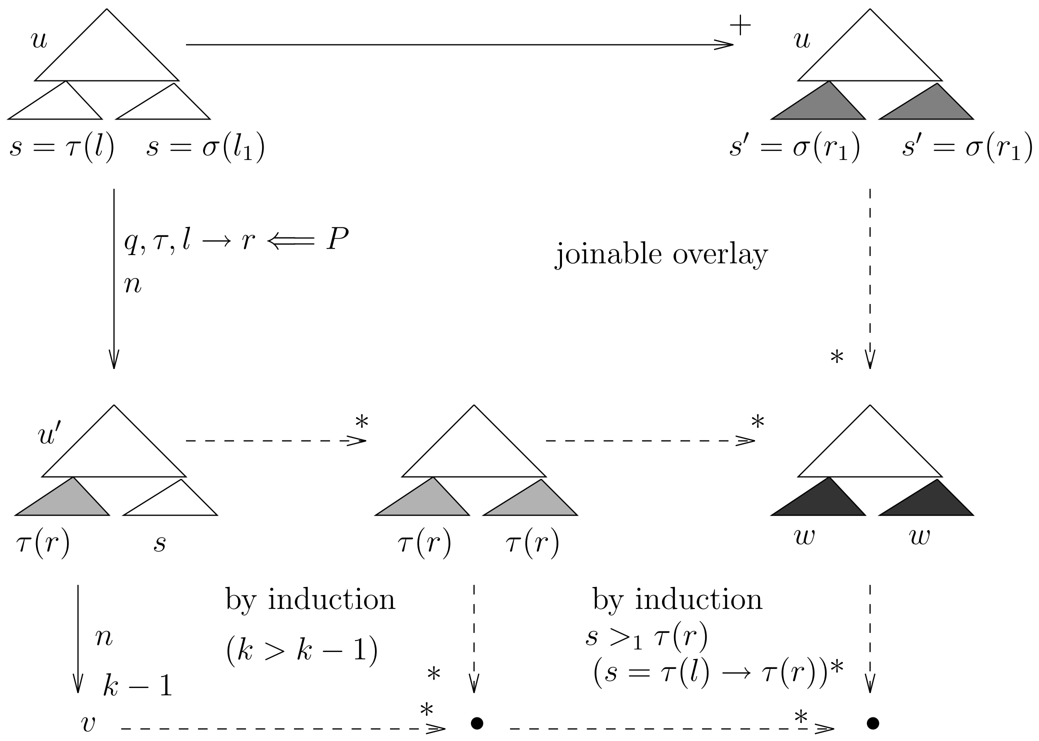
Figure 2.1 : Disjoint peak case



(2.2) $q \in \Pi$ (critical peak, see Figure 2.2): In this case we have a critical peak which is an instance of a critical overlay of $\mathcal{R}$, i.e. $s = \sigma(l_1) = \tau(l)$. Since all conditional critical pairs are joinable (overlays) we know that there exists some term w with $s = \sigma(l_1) \to \sigma(r_1) = s' \to^* w$ and $s = \tau(l) \to \tau(r) \to^* w$. Obviously, we have $u = C[s]_\Pi \to_{l \to r \Leftarrow P} (C[\tau(l)]_\Pi)[q \leftarrow \tau(r)] = u' \to^*_{l \to r \Leftarrow P} C[\tau(r)]_\Pi$. For $|\Pi| = 1$ we obtain $\Pi = \{q\}$ and $(C[\tau(l)]_\Pi)[q \leftarrow \tau(r)] = u' = C[\tau(r)]_\Pi \to^* v$. Otherwise, we have $C[\tau(l)]_\Pi[q \leftarrow \tau(r)] \xrightarrow{n}{}^{k-1} v$. Hence, by induction (the measure decreases in the second or third component) we obtain $C[\tau(r)]_\Pi \downarrow v$ Moreover, $\tau(r) \to^* w$ yields $C[\tau(r)]_\Pi \to^* C[w]_\Pi$ which by induction (the measure decreases in the first component due to $s = \tau(l) \to \tau(r)$, hence $s >_1 \tau(r)$) implies $C[w]_\Pi \downarrow v$. Thus, $C[s']_\Pi = C[\sigma(r_1)]_\Pi \to^* C[w]_\Pi \downarrow v$ because of $\sigma(r_1) \to^* w$. Hence we get $C[s']_\Pi \downarrow v$ as desired.

The remaining case is that of a *variable overlap*, either above or in some subterm $C[s]_\Pi/\pi = s$ ($\pi \in \Pi$) of $C[s]_\Pi$. Note that a critical peak which is not an overlay cannot occur due to $\mathrm{OS}(\mathcal{R})$.

Figure 2.2 : Critical peak (overlay)



(2.3) $q < \pi$ for some $\pi \in \Pi$ (variable overlap above, see Figure 2.3): Let Π' be the set of positions of those subterms $s = \sigma(l_1)$ of $u/q = \tau(l)$ which correspond to some $u/\pi = s$, $\pi \in \Pi$. Formally, $\Pi' := \{\pi' \mid q\pi' \in \Pi\}$. Moreover, for every $x \in dom(\tau)$, let $\Delta(x)$ be the set of positions of those subterms s in $\tau(x)$ which are rewritten into s' in the derivation $u = C[s]_\Pi \to^+ C[s']_\Pi$, i.e. $\Delta(x) := \{\rho' \mid \exists \rho : l/\rho = x \wedge \rho\rho' \in \Pi'\}$. Then τ' is defined by $\tau'(x) := \tau(x)[\rho' \leftarrow s' \mid \rho' \in \Delta(x)]$ for all $x \in dom(\tau)$. Obviously, we have $\tau(x) \to^*_{l_1 \to r_1 \Leftarrow P_1} \tau'(x)$ for all $x \in dom(\tau)$. Thus we get

$$u = C[s]_\Pi = C'[\tau(l)]_q \xrightarrow{n}_{l \to r \Leftarrow P} C'[\tau(r)]_q = u' \xrightarrow{n}{}^{k-1} v$$

for some context $C'[]_q$,

$$u = C[s]_\Pi \to^*_{l_1 \to r_1 \Leftarrow P_1} C[s']_\Pi \to^*_{l_1 \to r_1 \Leftarrow P_1} C''[\tau'(l)]_q$$

for some context $C''[]_q$, and

$$u' = C'[\tau(r)]_q \to^*_{l_1 \to r_1 \Leftarrow P_1} C''[\tau'(r)]_q\,.$$

Moreover we have

$$C'''[\tau'(l)]_q \rightarrow_{q,\tau',l\rightarrow r\Leftarrow P} C'''[\tau'(r)]_q$$

by induction (due to $\tau'(P)\downarrow$ as shown below) and finally

$$C[s']_\Pi \rightarrow^* C'''[\tau'(r)]_q \downarrow v$$

as desired by induction (the measure decreases in the second or third component).

It remains to prove the claim $\tau'(P)\downarrow$. This means that we have to show $\tau'(z_1) \downarrow \tau'(z_2)$ for all $z_1 \downarrow z_2 \in P$. If P is empty or trivially satisfied (i.e., $n \leq 1$) we are done. Otherwise, we know by assumption that $\tau(z_1) \downarrow \tau(z_2)$ for all $z_1 \downarrow z_2 \in P$ in depth at most $n-1$. This means $eq(\tau(z_1), \tau(z_2)) \stackrel{n-1^*}{\longrightarrow} true$. By construction of τ' we know $\tau(z_1) \rightarrow^*_{l_1\rightarrow r_1\Leftarrow P_1} \tau'(z_1)$, $\tau(z_2) \rightarrow^*_{l_1\rightarrow r_1\Leftarrow P_1} \tau'(z_2)$. Moreover, $eq(\tau(z_1), \tau(z_2))$ is of the form $E[s]_Q$, for some context $E[]_Q$, such that $E[s]_Q \rightarrow^* E[s']_Q = eq(\tau'(z_1), \tau'(z_2))$. By induction (the measure decreases in the second component due to $n > n-1$) we obtain $E[s']_Q = eq(\tau'(z_1), \tau'(z_2)) \downarrow true$, hence $eq(\tau'(z_1), \tau'(z_2)) \rightarrow^* true$ (since $true$ is irreducible) which means that $\tau'(z_1)$ and $\tau'(z_2)$ are joinable (without using the rule $eq(x,x) \rightarrow true$). This finishes the proof of the claim $\tau'(P)\downarrow$. Summarizing we have shown $C[s']_\Pi \downarrow v$ as desired.

(2.4) $\pi < q$ for some $\pi \in \Pi$ (variable overlap below, see Figure 2.4): Remember that we have $u/\pi = \sigma(l_1) = s$ and $u/q = \tau(l)$. Now let q', q'', q''', Π', Π'' and contexts $C'[]_q$, $D[]_{q'''}$, $D'[]_{\Pi'}$, $D''[]_{\Pi''}$ be (uniquely) defined by $u = C'[\tau(l)]_q$, $q = \pi q'$, $q' = q''q'''$, $l_1/q'' = x \in \mathcal{V}$, $\sigma(x) = D[\tau(l)]_{q'''}$, $\Pi' = \{\pi' \mid l_1/\pi' = x\}$, $\Pi'' = \{\pi'' \mid r_1/\pi'' = x\}$, $\sigma(l_1) = D'[D[\tau(l)]_{q'''}]_{\Pi'}$, $\sigma(r_1) = D''[D[\tau(l)]_{q'''}]_{\Pi''}$. Moreover let σ' be the substitution on $V(l_1)$ defined by

$$\sigma'(y) = \begin{cases} \sigma(y)\,,\ y \neq x \\ D[\tau(r)]_{q'''}\,,\ y = x, \sigma(x) = D[\tau(l)]_{q'''}\,. \end{cases}$$

Then we get

$$\begin{aligned} C[s]_\Pi &= C[\sigma(l_1)]_\Pi \\ &= C[D'[D[\tau(l)]_{q'''}]_{\Pi'}]_\Pi \rightarrow^+_{\sigma,l_1\rightarrow r_1\Leftarrow P_1} C[s']_\Pi = C[\sigma(r_1)]_\Pi \\ &= C[D''[D[\tau(l)]_{q'''}]_{\Pi''}]_\Pi \rightarrow^*_{\tau,l\rightarrow r\Leftarrow P} C[D''[D[\tau(r)]_{q'''}]_{\Pi''}]_\Pi \\ &= C[\sigma'(r_1)]_\Pi\,, \end{aligned}$$

and

$$\begin{aligned} C[s]_\Pi &= C[\sigma(l_1)]_\Pi = C[D'[D[\tau(l)]_{q'''}]_{\Pi'}]_\Pi \\ &= C'[\tau(l)]_q \rightarrow_{q,\tau,l\rightarrow r\Leftarrow P} u' \\ &= C'[\tau(r)]_q \rightarrow^*_{\tau,l\rightarrow r\Leftarrow P} C[D'[D[\tau(r)]_{q'''}]_{\Pi'}]_\Pi = C[\sigma'(l_1)]_\Pi\,. \end{aligned}$$

By induction (the first component decreases due to $s \rhd \tau(l)$) we obtain

$$C[s]_\Pi \quad = \quad C[\sigma(l_1)]_\Pi \rightarrow^+ C[\sigma'(l_1)]_\Pi \downarrow v\,,$$

hence $s >_1 \tau(l)$). Moreover, we get

$$C[\sigma'(l_1)]_\Pi \rightarrow^*_{\sigma', l_1 \rightarrow r_1 \Leftarrow P_1} C[\sigma'(r_1)]_\Pi$$

since $\sigma'(P_1) \downarrow$ is satisfied by induction (the first component decreases due to $s \rhd \tau(l)$, hence $s >_1 \tau(l)$).[5] Furthermore we have $C[\sigma'(r_1)]_\Pi \downarrow v$ by induction (again the first component decreases due to $s = \sigma(l) \rightarrow^+ \sigma'(l_1)$, hence $s >_1 \sigma'(l_1)$). Summarizing we have shown

$$C[s']_\Pi \rightarrow^* C[\sigma'(r_1)]_\Pi \downarrow v$$

as desired.

Thus, for all cases we have shown $C[s']_\Pi \downarrow v$ yielding a contradiction to (b), hence we are done. ■

[5] Note that – in contrast to case 2.3 – a 'proper *eq*-reasoning' can be avoided here (by applying the induction hypothesis twice), since the inductive argument needed does not involve the *depth* of rewriting.

Figure 2.3 : Variable overlap above

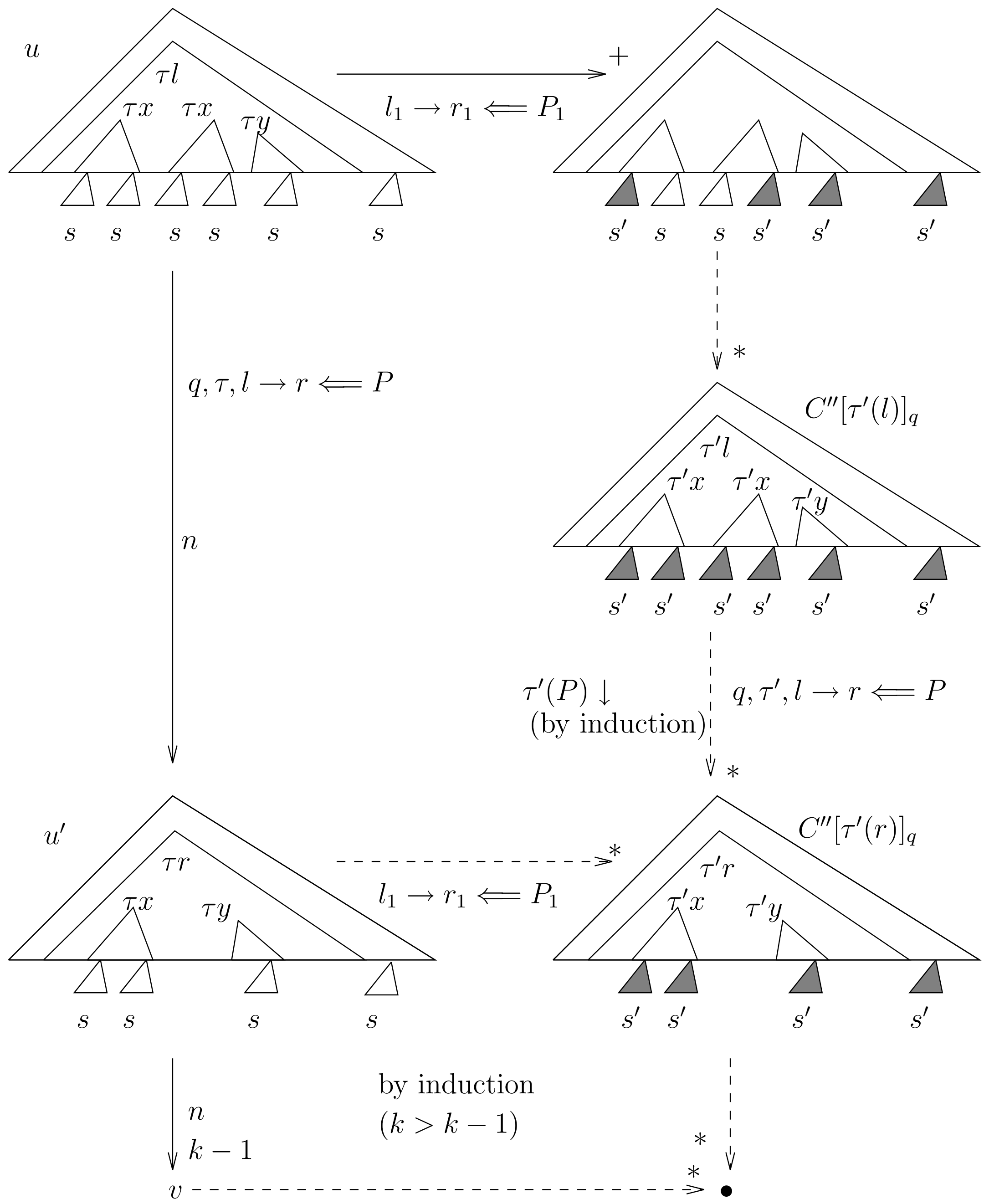
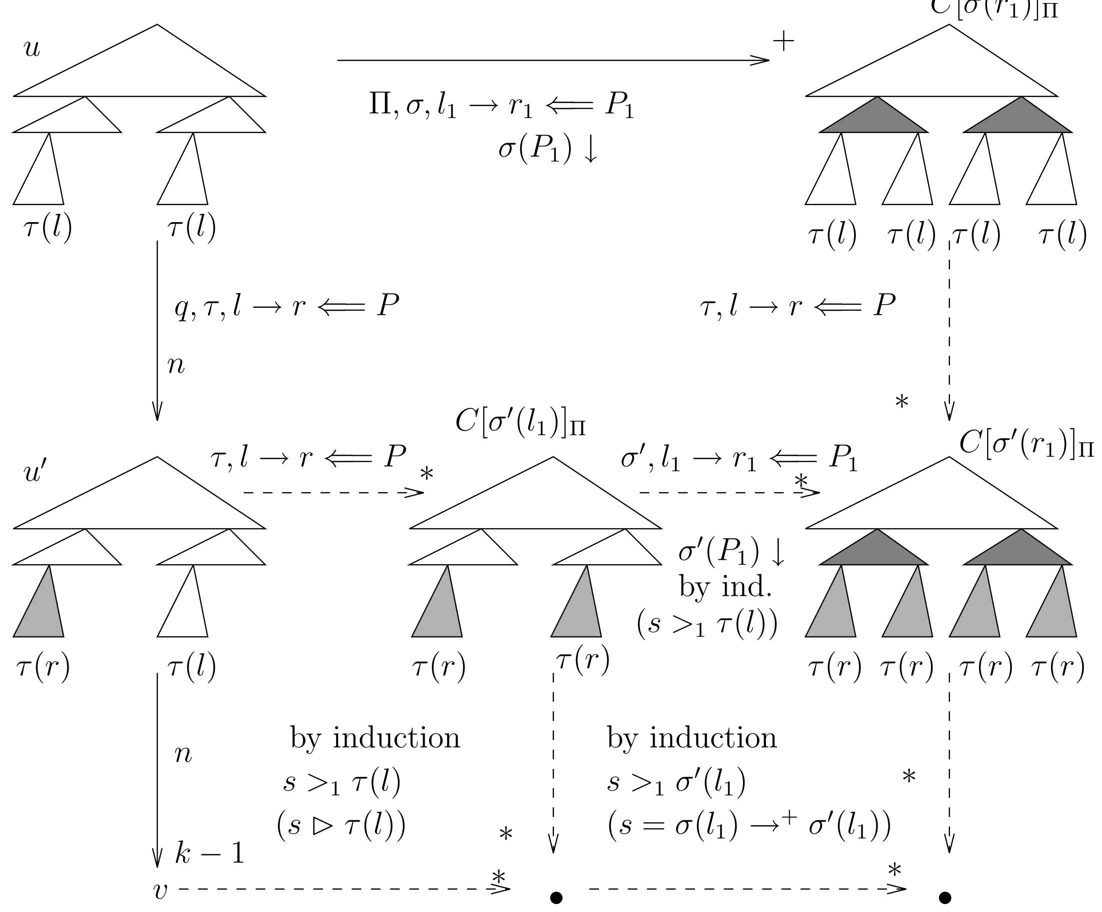


Figure 2.4 : Variable overlap below



Appendix B

A Parameterized Version of the Well-Founded Induction Principle

The well-known and powerful proof principle by well-founded induction says that for verifying $\forall x : P(x)$ for some property P it suffices to show $\forall x : [[\forall y < x : P(y)] \implies P(x)]$, provided $<$ is a well-founded partial ordering on the domain of interest. Here we investigate a more general formulation of this proof principle which allows for a kind of parameterized partial orderings $<_x$ which naturally arises in some cases. More precisely, we develop conditions under which the parameterized proof principle $\forall x : [[\forall y <_x x : P(y)] \implies P(x)]$ is sound in the sense that $\forall x : [[\forall y <_x x : P(y)] \implies P(x)] \implies \forall x : P(x)$ holds, and give counterexamples demonstrating that these conditions are indeed essential.[1]

Usually, in proofs by well-founded induction (cf. e.g. [Coh65], [Fef77], [MW93]) one tries to verify

$$\forall x : P(x) \tag{B.1}$$

by showing

$$\forall x : [\,[\forall y < x : P(y)\,] \implies P(x)\,] \tag{B.2}$$

where $<$ is a fixed well-founded partial ordering on the domain of interest. In fact, $<$ need not be a partial ordering. Any well-founded or terminating relation suffices.

Definition B.0.1 (cf. e.g. [Wec92]) Let R be a (binary) relation on a set A.

- Let B be a non-empty subset of A. An element $b \in B$ is said to be *R-minimal* (or simply *minimal*) if, for all $a \in A$, bRa implies $a \notin B$.
- R is called *well-founded* (or *Noetherian*) if every non-empty subset of A has a minimal element.[2]
- R is called *terminating* if there is no infinite sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n R a_{n+1}$ for all $n \in \mathbb{N}$.

[1] An extended version of the results in this appendix has been published in [Gra95c].

Well-foundedness and termination are equivalent notions. Well-foundedness obviously implies termination, and the reverse direction is also easy but requires the *Axiom of (Dependent) Choice* (cf. e.g. [Jec77]).

Theorem B.0.2 (cf. e.g. [Wec92])
A (binary) relation is well-founded if and only if it is terminating.

In practice, i.e., when trying to apply the general principle of proof by well-founded induction, it often occurs that an appropriate well-founded partial ordering either is not available or unknown, or – if some partial ordering seems to be an obvious candidate – its well-foundedness is not guaranteed or somehow depends on the property to be proved. To illustrate this situation consider the following local version of Newman's Lemma:

> Every terminating element $a \in A$ in a locally confluent ARS $\mathcal{A} = (A, \rightarrow)$ is confluent (or more succinctly: $\forall\, s \in A\, :\, \mathrm{SN}(s) \wedge \mathrm{WCR}(\mathcal{A}) \implies \mathrm{CR}(s)$).[3]

A typical proof of this lemma might looks as follows:

Proof: Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS with $\mathrm{WCR}(\mathcal{A})$, and let $s \in A$ with $\mathrm{SN}(s)$ be given. Let $Q(x)$ (for $x \in A$) be defined by

$$Q(x) := \mathrm{CR}(x)\,.$$

We proceed by induction over x w.r.t. the ordering $> := >_s := \rightarrow^+ |_{G(s)\times G(s)}$, where $G(s) := \{t \in A \mid s \rightarrow^* t\}$, showing

$$\forall\, x \in A, x \leq s\, :\, Q(x)\,.$$[4]

Observe that we have $x > y \iff s \rightarrow^* x \rightarrow^+ y$, and $x \geq y \iff s \rightarrow^* x \rightarrow^* y$ (for the reflexive ordering $\geq$ induced by the strict partial ordering $>$). By the assumption $\mathrm{SN}(s)$ we know that $> = \rightarrow^+ |_{G(s)\times G(s)}$ is well-founded. Now, assuming $s \geq x$ and $y\ {}^*\!\leftarrow x \rightarrow^* z$, we have several cases. If y or z equals x, we are done. This includes the case that x is a minimal element w.r.t. $>$, i.e. irreducible. Otherwise, there exist $y', z' \in A$ with $y\ {}^*\!\leftarrow y' \leftarrow x \rightarrow z' \rightarrow^* z$. By $\mathrm{WCR}(x)$, which follows from $\mathrm{WCR}(\mathcal{A})$,[5] we know that there exists some u with $y' \rightarrow^* u\ {}^*\!\leftarrow z'$. By induction hypothesis for y' ($x > y'$ due to $s \rightarrow^* x \rightarrow y'$) we conclude that there exists some v with $y \rightarrow^* v\ {}^*\!\leftarrow u$ and the induction hypothesis for z' ($x > z'$ due to $s \rightarrow^* x \rightarrow z'$) yields the existence of some w with $y' \rightarrow^* u \rightarrow^* v \rightarrow^* w\ {}^*\!\leftarrow z\ {}^*\!\leftarrow z'$. Summarizing we get $y \rightarrow^* v \rightarrow^* w\ {}^*\!\leftarrow z$

[2] Following common usage, we call a partial ordering relation denoted by $<$ *well-founded* / *terminating* if $> := <^{-1}$ is well-founded / terminating.

[3] Note that this formulation is slightly weaker than Theorem 2.1.26 (yet, the corresponding statements are equivalent).

[4] Note that in general the statement $\forall x \in A, x \leq s\, :\, Q(x)$ is stronger than $Q(s)$ (although here both are equivalent)! In fact, it is well-known that in proofs by induction it is often easier to prove a stronger statement than the original one since this also provides stronger induction hypotheses.

[5] Obviously, the assumption $\mathrm{WCR}(\mathcal{A})$ in this local version of Newman's Lemma can even be weakened to $\mathrm{WCR}(\mathcal{G}(a))$ where $\mathcal{G}(s) = (G(s), \rightarrow |_{G(s)\times G(s)})$ is the *sub-ARS* of $\mathcal{A}$ determined by the element s (cf. Theorem 2.1.26).

and hence $\mathrm{CR}(x)$ as desired. ■

Note that in the above proof the ordering used for showing

$$\forall x \in B : \, Q(x)$$

with $B = \{x \in A \,|\, x \leq_s s\}$ depends on s and its well-foundedness was assumed before. A careful inspection of the proof which was done via the instantiated scheme

$$\forall x \in B : \, [\, \forall y <_s x : \, Q(y)\,] \implies Q(x) \tag{B.3}$$

reveals that in the induction step we could have also used the induction hypotheses

$$\forall y <_x x : \, Q(y)$$

instead of

$$\forall y <_s x : \, Q(y)$$

with $<_x$ defined by

$$u >_x v \iff s \to^* x \to^* u \to^+ v \, , \tag{B.4}$$

i.e., according to the instantiated scheme

$$\forall x \in B : \, [\, \forall y <_x x : \, Q(y)\,] \implies Q(x) \, . \tag{B.5}$$

Similarly, when defining $Q'(x)$ (for $x \in A$) by

$$Q'(x) \; := \; [\, \mathrm{SN}(x) \implies \mathrm{CR}(x)\,]$$

the dependence on well-foundedness of the applied partial ordering is incorporated in $Q'(x)$. Then, proving the local version of Newman's Lemma above amounts to showing

$$\forall x \in A : \, Q'(x)$$

which one might be tempted to accomplish by showing

$$\forall x \in A : \, [\, \forall y \in A, y <_x x : \, Q'(y)\,] \implies Q'(x) \tag{B.6}$$

with $<_x$ defined by $(u, v \in A)$:

$$u >_x v \iff x \to^* u \to^+ v \, . \tag{B.7}$$

Here, the proof of (B.6) is analogous to the proof of (B.5).[6]

Note that proceeding as sketched above presupposes in general correctness of the following induction principle which is parameterized by a family of (strict partial) orderings $<_x$:

$$\forall x : \, [\forall y <_x x : \, P(y)] \implies P(x) \tag{B.8}$$

The correctness of (B.8) is expressed by

$$[\, \forall x : \, [\, [\forall y <_x x : \, P(y)] \implies P(x)\,]\,] \quad \implies \quad [\, \forall x : \, P(x)\,] \tag{B.9}$$

and obviously depends on properties of the involved ordering relations $<_x$. As already mentioned, a careful inspection of the above proof for the local version of Newman's Lemma shows that essentially the same proof can be used for establishing (B.8) with

[6] From an intuitive point of view one would usually prefer to proceed according to (B.3) (or (B.5)) since there the well-foundedness assumption and the statement to be proved are clearly separated, and thus easier to understand.

P instantiated appropriately (by Q) and $<_x$ defined by (B.4). Hence, in this special case the induction scheme (B.8) is correct, i.e., (B.9) holds. So one may ask in general, under what conditions concerning the applied family of orderings $<_x$ and the involved predicate $Q(x)$ is (B.8) a correct induction principle as expressed by (B.9)? That correctness is not assured in general, can be seen from the following counterexamples.

Example B.0.3 Let $G = \{a, b\}$ be a set of two elements and $<_a$, $<_b$ be two partial orderings on G given by $<_a := \{(b, a)\}$, $<_b := \{(a, b)\}$. Moreover let Q be some unary predicate on G such that $\neg Q(a)$ and $\neg Q(b)$ hold, i.e. Q is neither satisfied for a nor for b. Then the induction principle (B.8) with P instantiated by Q becomes

$$\forall x \in G : [\forall y \in G, y <_x x : Q(y)] \implies Q(x) \tag{B.10}$$

which is equivalent to

$$[\,[\forall y <_a a : Q(y)\,] \implies Q(a)\,] \wedge [\,[\forall y <_b b : Q(y)\,] \implies Q(b)\,]$$

which in turn is, by definition of $<_a$, $<_b$, equivalent to

$$[\,Q(b) \implies Q(a)\,] \qquad \wedge \qquad [\,Q(a) \implies Q(b)\,]\,. \tag{B.11}$$

Note that due to the assumptions $\neg Q(a)$, $\neg Q(b)$ we obviously have that (B.11) holds. However,

$$\forall x \in G : Q(x)$$

is false, hence the instantiated version of the parameterized induction scheme (B.8) is incorrect, i.e.,

$$[\forall x \in G : [\,[\forall y \in G, y <_x x : Q(y)] \implies Q(x)\,]\,] \implies [\forall x \in G : Q(x)\,]$$

is false.

Note that in the above example the parameterized ordering relations $<_a$, $<_b$ are clearly well-founded, but the ordering information of $<_a$, $<_b$ is 'contradictory'. The latter is not the case in the following example.

Example B.0.4 Let $G = \{a_0, a_1, a_2, \ldots\}$ be a countably infinite set with ordering relations $<_{a_i}$ (for $i \geq 0$) defined by

$$\begin{array}{ccccccccc}
a_0 & >_{a_0} & a_1 & >_{a_0} & a_2 & >_{a_0} & a_3 & >_{a_0} & a_4 & \ldots \\
 & & a_1 & >_{a_1} & a_2 & >_{a_1} & a_3 & >_{a_1} & a_4 & \ldots \\
 & & & & a_2 & >_{a_2} & a_3 & >_{a_2} & a_4 & \ldots \\
 & & & & & & \ldots & \ldots & \ldots & \ldots\,,
\end{array}$$

i.e. $<_{a_i}$ is defined by $<_{a_i} := \{(a_k, a_j) \mid k > j \geq i\}$. Moreover, for some unary predicate Q on G let $\neg Q(a_i)$ hold for all $i \geq 0$. Then the induction principle (B.8) with P instantiated by Q becomes

$$\forall x \in G : [\forall y \in G, y <_x x : Q(y)] \implies Q(x) \tag{B.12}$$

which holds since the induction hypothesis $[\forall y \in G, y <_x x : Q(y)]$ is never satisfied (note that for any $a_j \in G$ there exists $a_k \in G$ (choosing e.g. $k = j + 1$) with $a_k <_{a_j} a_j$ but not $Q(a_k)$).

Note that in this example the ordering information is somehow consistent, but $<_{a_i}$ is clearly not well-founded.

The previous examples motivate the following abstract correctness conditions for the parameterized induction principle (B.8):

$$\forall x, y : \; [\, y <_x x \implies <_y \subseteq <_x \,] \tag{B.13}$$

and

$$\forall x : \; [\, \neg P(x) \implies <_x \text{ is well-founded. }] \tag{B.14}$$

We shall show now that if both the "compatibility" condition (B.13) and the "well-foundedness" condition (B.14) hold, then the parameterized induction principle (B.8) is correct as expressed by (B.9).

Theorem B.0.5 The principle of parameterized (well-founded) induction (B.8) is correct, i.e. (B.9) holds, provided that the "compatibility" condition (B.13) and the "well-foundedness" condition (B.14) are satisfied.

We shall present two alternative proofs for this result. The first one is a more direct one and works by contradiction, and the second one essentially shows that (B.8) is equivalent to the usual principle of well-founded induction using one fixed uniform well-founded ordering.

Proof: (by contradiction)
Assume that (B.13) and (B.14) are satisfied, and assume that (B.8) holds, but not (B.9). Hence, there exists some x with $\neg P(x)$, let's say x_0. Condition (B.14) implies that $<_{x_0}$ is well-founded. Now (B.8) implies in particular

$$[\, \forall y <_{x_0} x_0 \; : \; P(y) \,] \implies P(x_0)$$

which, due to $\neg P(x_0)$, yields the existence of some $x_1 <_{x_0} x_0$ with $\neg P(x_1)$. Choosing $x = x_1$ in (B.8) and using $\neg P(x_1)$ we know that there is some $x_2 <_{x_1} x_1$ with $\neg P(x_2)$, and so on.[7] Hence, by (ordinary) induction (on the ordering of the natural numbers) we can conclude that (for every $i \geq 0$) there exists some x_i with $\neg P(x_i)$ and

$$x_0 >_{x_0} x_1 >_{x_1} x_2 >_{x_2} x_3 >_{x_3} x_4 \ldots .$$

Applying repeatedly condition (B.13) we get

$$x_0 >_{x_0} x_1 >_{x_0} x_2 >_{x_0} x_3 >_{x_0} x_4 \ldots .$$

But this means that $<_{x_0}$ is not well-founded, contradicting condition (B.14). ■

Proof: (by ordinary well-founded induction)
Assume that (B.13) and (B.14) are satisfied. Then we define a binary relation $<$ as follows:

$$u < v \quad :\Longleftrightarrow \quad \neg P(v) \wedge u <_v v \,. \tag{B.15}$$

Next we show that $<$ is a well-founded partial ordering, i.e., it is irreflexive, transitive, and well-founded. Irreflexivity of $<$ follows from irreflexivity of $<_u$ for all u. For

[7] Note that this actually requires the *Axiom of Choice*.

showing transitivity we have to show that $u < v$ and $v < w$ implies $u < w$. By definition of $<$ the assumption yields $u <_v v <_w w$. Using (B.13) we get $u <_w v <_w w$ which, by transitivity of $<_w$, implies $u <_w w$. From $v < w$ we get $\neg P(w)$, hence together this yields $u < w$. For proving well-foundedness of $<$ (by contradiction) assume that

$$u_0 > u_1 > u_2 > u_3 > \dots$$

is an infinite decreasing $>$-chain. This implies

$$\forall i \geq 0 \, : \, \neg P(u_i)$$

and

$$u_0 >_{u_0} u_1 >_{u_1} u_2 >_{u_2} u_3 >_{u_3} u_4 \dots$$

which, again by the compatibility condition (B.13), yields

$$u_0 >_{u_0} u_1 >_{u_0} u_2 >_{u_0} u_3 >_{u_0} u_4 \dots \, .$$

But this means that $<_{u_0}$ is not well-founded contradicting (B.14). Hence, we conclude that $<$ is indeed a well-founded partial ordering, for which the principle of well-founded induction (B.2) is correct. Thus, substituting the definition of $<$ into (B.2) we obtain

$$\forall \, x : \, [\, [\forall y : \neg P(x) \wedge y <_x x \implies P(y)\,] \implies P(x)\,]$$

which is equivalent to

$$\forall \, x : \, [\, [\forall y : P(x) \vee \neg(y <_x x) \vee P(y)\,] \implies P(x)\,]$$

and to

$$\forall \, x : \, [\, [\, [\forall y : y <_x x \implies P(y)\,] \, \vee \, P(x)\,] \implies P(x)\,]$$

hence yielding

$$\forall \, x : \, [\, [\forall y, y <_x x : \, P(y)\,] \implies P(x)\,] \, .$$

Thus, correctness of the ordinary well-founded induction principle (B.2) implies correctness of the parameterized (well-founded) induction principle (B.8) under the conditions (B.13) and (B.14) as was to be shown. ∎

The counterexamples (B.0.3) and (B.0.4) above demonstrate that the "compatibility" condition (B.13) and the "well-foundedness" condition (B.14) cannot be dropped without loosing correctness of the principle of parameterized (well-founded) induction (B.8) in general. In our introductory Example above we observe that these two conditions are indeed satisfied. In fact, with $>_u$ defined by

$$x >_u y \iff s \to^* u \to^* x \to^+ y \; ,$$

compatibility means

$$x >_x y \implies >_x \supseteq >_y$$

or equivalently

$$x >_x y \implies [\forall u, v : \, u >_y v \implies u >_x v\,]$$

which holds, since $x >_x y$ and $u >_y v$ imply $s \to^* x \to^* x \to^+ y$, $s \to^* y \to^* u \to^+ v$, hence $s \to^* x \to^+ y \to^* u \to^+ v$ and thus $u >_x v$. The well-foundedness condition (B.14) is also satisfied, since every $>_x$ is well-founded by the global assumption SN(s).

Although the second proof of Theorem B.0.5 reveals that (B.8) is not more powerful than ordinary well-founded induction, the parameterized induction principle (B.8) has the advantage that one may directly work with (B.8), i.e. with a family of ordering relations, which may arise quite naturally in certain cases. The only thing to be verified for correctness is to ensure that the abstract properties (B.13) and (B.14) are satisfied. Working directly with with (B.8) may be useful (from a conceptual point of view) for instance in inductive proofs by some counterexample x which is assumed to be minimal w.r.t. some well-founded ordering $>$, in the sense that w.l.o.g. x may be assumed to be minimal w.r.t. some (naturally defined) $>_x$ (instead of minimal w.r.t. $>$). This may be beneficial for the sake of better understanding the essence of the involved inductive reasoning, in particular in cases where the whole inductive proof is very complicated .

Finally let us mention that the two conditions (B.13) and (B.14) are only one possibility for guaranteeing correctness of (B.8). Indeed, let us consider the following modification of Example B.0.4.

Example B.0.6 Let $G = \{a_0, a_1, a_2, \ldots\}$ be a countably infinite set with ordering relations $<_{a_i}$ (for $i \geq 0$) defined by

$$>_{a_i} := \{(a_i, a_{i+1})\} \ .$$

Moreover, for some unary predicate Q on G let $\neg Q(a_i)$ hold for all $i \geq 0$. Then the induction principle (B.8) with P instantiated by Q becomes

$$\forall x \in G : \; [\forall y \in G, y <_x x : \; Q(y)] \implies Q(x) \tag{B.16}$$

which holds since the induction hypothesis $[\forall y \in G, y <_x x \; : \; Q(y)]$ is never satisfied (note that for any $a_j \in G$ we have $a_{j+1} <_{a_j} a_j$ but not $Q(a_{j+1})$). Hence,

$$[\forall x : \; [\, [\forall y <_x x : \; P(y)] \implies P(x)\,]\,] \quad \implies \quad [\forall x : \; P(x)\,]$$

is obviously incorrect in this case.

In this example, the ordering relations $(>_{a_i})_{i \geq 0}$ are all well-founded, and compatible in the sense that combining any two $>_{a_i}$, $>_{a_j}$ of them (or even finitely many $>_{a_k}$) still yields a well-founded relation. However, the problem is, that $\bigcup_{i \geq 0} >_{a_i}$ is not well-founded any more. In fact, the crucial point for correctness of (B.8) is that an infinite sequence of the form

$$x_0 >_{x_0} x_1 >_{x_1} x_2 >_{x_2} x_3 >_{x_3} x_4 \ldots \ .$$

issuing from some counterexample x_0 (i.e. with $\neg P(x_0)$) is impossible (cf. the (first) proof of Theorem B.0.5). To ensure this property, one might require instead of (B.13), (B.14) the following more general condition:

$$\forall x_0 : \; [\neg P(x_0) \implies (\bigcup_x >_x)_{\text{below } x_0} \quad \text{is well-founded}\,] \tag{B.17}$$

where, for some binary relation R, $R_{\mathrm{below}\,y}$ is given by

$$R_{\mathrm{below}\,y} \;=\; R \cap \{(u,v) \mid yR^*uRv\} = R|_{\{z \mid yR^*z\}^2} \;.$$

Then the proof(s) of the modified version of Theorem B.0.5 go through as well,[8] just as before.

In order to ensure correctness of (B.8) as a general scheme – and not only of specific instances of (B.8) as considered above and in particular in Theorem B.0.5 – one simply has to require well-foundedness of

$$\bigcup_x >_x \;.$$

[8] Note that any well-founded (binary) relation can be turned into a well-founded (strict partial) ordering, simply by taking the transitive closure. Thus, for proofs by well-founded induction, it does not really matter whether the underlying well-founded relation is an ordering or not, since transitivity can always be enforced.

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Symbols