On Interreduction of Semi-Complete Term Rewriting Systems

Bernhard Gramlich

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Institut für Computersprachen / Abteilung für Anwendungen der Formalen Logik
Technische Universität Wien, Karlsplatz 13 / E1852, A-1040 Wien, Austria

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Bernhard Gramlich*†
Technische Universität Wien
Institut für Computersprachen
Karlsplatz 13, A-1040 Wien, Austria

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Abstract

For a complete, i.e., confluent and terminating term rewriting system (TRS) it is well-known that simplification (also called interreduction) into an equivalent canonical, i.e., complete and interreduced TRS is easily possible. This can be achieved by first normalizing all right-hand sides of the TRS and then deleting all rules with a reducible left-hand side. Here we investigate the logical and operational preservation properties of the same simplification operations for semi-complete, i.e., confluent and weakly terminating TRSs. Surprisingly, it turns out that for semi-complete TRSs these simplifications are neither operationally harmless nor logically correct: Semi-completeness may get lost and the induced equational theory need not be preserved. We also provide sufficient criteria for the preservation of semi-completeness and of the induced equational theory. In particular, we show that orthogonal TRSs enjoy all these preservation properties. In the general case, for a given semi-complete TRS an equivalent semi-canonical, i.e., semi-complete and interreduced system need not even exist.

Keywords: Theory of computation, term rewriting, confluence, termination, semi-completeness, right-normalization, interreduction.

1 Introduction and Overview

It is well-known that any complete term rewriting system (TRS) $\mathcal{R}$ can be simplified (or interreduced) into a canonical TRS which induces the same equational theory ([Mét83]). This is quite useful in practice because computation

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†E-mail: gramlich@logic.at www: http://www.logic.at/staff/~gramlich

of normal forms in the interreduced TRS may be much more efficient than in the original one. Interreduction of a complete TRS can be performed by first normalizing all right-hand sides, and then omitting all rules with a left-hand side that is reducible by the remaining rules (cf. [Mét83], [Klo92]):

(a) $\mathcal{R}{\downarrow} := \{l \to r{\downarrow_{\mathcal{R}}} \mid l \to r \in \mathcal{R}\}$ (**right-normalization**)

(b) $\mathcal{R}_{del} := \{l \to r \in \mathcal{R} \mid l \text{ is } (\mathcal{R} \setminus \{l \to r\})\text{-irreducible}\}$[1] (**deletion of left-reducible rules**)

If $\mathcal{R}$ is complete then $\mathcal{R}{\downarrow}$ and $\mathcal{R}{\downarrow_{del}}$ are also complete, and their induced equational theories coincide: $\leftrightarrow^*_{\mathcal{R}} = \leftrightarrow^*_{\mathcal{R}{\downarrow}} = \leftrightarrow^*_{\mathcal{R}{\downarrow_{del}}}$. These properties are usually exploited in typical Knuth-Bendix like completion procedures for equational specifications ([KB70]). In actual implementations of completion procedures these simplifications are normally performed during the completion process in an intertwined manner, cf. e.g. [Hue81], [Bac89]. Performing deletion of left-reducible rules before right-normalization may be logically incorrect, i.e., the equational theory may be destroyed. A very simple example is the TRS $\mathcal{R} = \{a \to b, a \to c, b \to c\}$ for which we get $\mathcal{R}{\downarrow} = \{a \to c, b \to c\} = \mathcal{R}{\downarrow_{del}}$ and $\mathcal{R}_{del} = \{b \to c\} = \mathcal{R}_{del}{\downarrow}$.

Of course, completeness of TRSs is very nice and useful, both in theory and practice, but sometimes impossible to achieve. From a logical and computational point of view, the more general class of semi-complete TRSs also enjoys nice properties, in particular the existence and uniqueness of computation results. For efficient computations in such systems preprocessing operations like the above simplifications may be quite useful provided that they do not modify the essential properties of the original system.

Clearly, deletion of left-reducible rules is not only well-defined for complete, but for arbitrary TRSs. Right-normalization is still well-defined for semi-complete TRSs, because for such systems every term has a unique normal form. Hence, it is quite natural to ask the following basic questions due to Kahrs ([Kah95]):

(1) Does right-normalization preserve semi-completeness?

(2) Does right-normalization of semi-complete TRSs preserve the equational theory?

As mentioned in [Kah95] it is easy to see that the preservation of weak termination would be sufficient to settle both questions in the positive (see Lemma 4.1 later on). Moreover, if the answer to question (2) is negative, then one could enforce preservation of the equational theory by adding the potentially missing part, i.e., by defining

$$\mathcal{R}{\downarrow_{ext}} := \mathcal{R}{\downarrow} \cup \{r \to r{\downarrow_{\mathcal{R}}} \mid l \to r \in \mathcal{R}, r \neq r{\downarrow_{\mathcal{R}}}\},$$

but the question

[1] Note that as usual equality of rewrite rules is interpreted as equality modulo variable renamings!

(1a) Does semi-completeness of $\mathcal{R}$ imply semi-completeness of $\mathcal{R}\downarrow_{ext}$?

remains non-trivial ([Kah95]). In view of the above mentioned well-known facts about simplification of complete TRSs and to complement the analogy we add the following questions:

(3) Does deletion of left-reducible rules preserve semi-completeness of right-normalized semi-complete TRSs?

(4) Does deletion of left-reducible rules preserve the equational theory of right-normalized semi-complete TRSs?

Note that, by definition of $\mathcal{R}\downarrow$ and $\mathcal{R}_{del}$, the inclusions $\leftrightarrow^*_{\mathcal{R}\downarrow} \subseteq \leftrightarrow^*_{\mathcal{R}}$ and $\leftrightarrow^*_{\mathcal{R}_{del}} \subseteq \leftrightarrow^*_{\mathcal{R}}$ trivially hold. Hence, preservation of the equational theory means $\leftrightarrow^*_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{R}\downarrow}$ and $\leftrightarrow^*_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{R}_{del}}$, respectively.

Concerning logical correctness, questions (2) and (4) are of course fundamental. From a computational point of view question (1) is certainly the most interesting one. It is closely related to the general question which reduction strategies are normalizing for a given TRS. In many examples right-normalization of a semi-complete, but non-terminating TRS yields a system which is not only semi-complete, but even complete. Somehow surprisingly, we will show that in general both weak termination and (even local) confluence may get lost under right-normalization of semi-complete systems. However, the corresponding counterexample still preserves the equational theory. Yet, we will give another counterexample where even the equational theory is destroyed. Applying well-known modularity results we finally combine these counterexamples into a single one that does not enjoy any of these preservation properties. The remaining questions will also be settled in the negative. Even more generally, we will show that transformation of a semi-complete TRS into an equivalent interreduced semi-complete TRS may be impossible.

The rest of the paper which is a substantially extended version of [Gra97] is structured as follows. First, in Section 2 we provide some basic terminology and no(ta)tions needed. In Section 3 we present all mentioned counterexamples and prove their relevant properties. Finally, in Section 4 we investigate sufficient conditions for the preservation properties corresponding to questions (1)-(4). In particular, we show there that the important class of orthogonal TRSs enjoys all these preservation properties and, hence, allows for safe interreduction.

2 Preliminaries

We assume familiarity with the basic no(ta)tions, terminology and theory of term rewriting (cf. e.g. [DJ90], [Klo92]) but recall some no(ta)tions for the sake of readability. The set of terms over some given signature $\mathcal{F}$ and some (disjoint) countably infinite set $\mathcal{V}$ of variables is denoted by $\mathcal{T}(\mathcal{F},\mathcal{V})$. We write $\mathcal{T}(\mathcal{F})$ for the set of all ground (i.e., variable-free) terms over $\mathcal{F}$. A term rewriting system (TRS) is a pair $(\mathcal{F},\mathcal{R})$ — also denoted by $\mathcal{R}^{\mathcal{F}}$ — consisting of a signature $\mathcal{F}$ and a set $\mathcal{R}$ of rewrite rules over $\mathcal{F}$, i.e., pairs (l,r) — also denoted by $l \to r$ — with

$l, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Here we require that l is not a variable, and that all variables of r occur in l. Instead of $(\mathcal{F}, \mathcal{R})$ we also write $\mathcal{R}$ if $\mathcal{F}$ is clear from the context or irrelevant. The rewrite relation $\rightarrow$ induced by $\mathcal{R}$ (and $\mathcal{R}$ itself) is said to be *confluent* or *Church-Rosser* (CR)[2] if ${}^*\!\leftarrow \circ \rightarrow^* \subseteq \rightarrow^* \circ {}^*\!\leftarrow$, *locally confluent* or weakly Church-Rosser (WCR) if $\leftarrow \circ \rightarrow \subseteq \rightarrow^* \circ {}^*\!\leftarrow$, *strongly confluent* if $\leftarrow \circ \rightarrow \subseteq \rightarrow^= \circ {}^*\!\leftarrow$ ($\rightarrow^=$ denotes the reflexive closure $\rightarrow \cup =$ of $\rightarrow$). $\rightarrow$ (and $\mathcal{R}$) has *unique normal forms* (UN) if $s \leftrightarrow^* t$ for normal forms s, t implies $s = t$. $\rightarrow$ (and $\mathcal{R}$) is *(strongly) terminating* or *strongly normalizing* (SN) if there is no infinite reduction sequence $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$. It is *weakly terminating* or *weakly normalizing* (WN) if every term s has a normal form, i.e., if $s \rightarrow^* s'$ with s' irreducible. A term s is said to be weakly ($\mathcal{R}$-)terminating if it has an ($\mathcal{R}$-) normal form, and ($\mathcal{R}$-)terminating if all reduction sequences issuing from s are finite. A TRS $\mathcal{R}$ is *complete* or *convergent* if it is confluent and terminating, and *semi-complete* if it is confluent and weakly terminating. $\mathcal{R}$ is *interreduced* if $\mathcal{R} = \mathcal{R}{\downarrow} = \mathcal{R}_{del}$. A TRS $\mathcal{R}$ is called *canonical* if it is complete and interreduced, and *semi-canonical* if it is semi-complete and interreduced. If a term s has a unique $\mathcal{R}$-normal form the latter is denoted by $s{\downarrow_{\mathcal{R}}}$. A term is *root-reducible* if it is a *redex*, i.e., an instance $l\sigma$ of the left-hand side of some rule $l \rightarrow r$.[3] In reduction steps $s \rightarrow t$ we sometimes use notations like $s \rightarrow_{l \rightarrow r} t$ to indicate which rule is applied. Furthermore we shall make free use of common context notations.

A critical pair $\langle s, t \rangle$ is *trivial* if $s = t$, and an *overlay* if it is obtained by overlapping two rules at root position. A TRS $\mathcal{R}$ is *non-overlapping* if it has no critical pairs, *almost non-overlapping* if all its critical pairs are trivial overlays, and *weakly non-overlapping* if all its critical pairs are trivial (note that an overlap of a rule with itself at the root position is not considered to be critical). A TRS $\mathcal{R}$ is said to be *strongly non-overlapping* if it remains non-overlapping even after linearizing all left-hand sides.[4] A left-linear TRS $\mathcal{R}$ is called *orthogonal* if it is non-overlapping, *almost orthogonal* if it is almost non-overlapping, and *weakly orthogonal* if it is weakly non-overlapping.

3 Counterexamples

Let us first consider what may happen under right-normalization of semi-complete TRSs, i.e., questions (1) and (2). This is a transformation which is incompatible with the usual (position selection) reduction strategies like (leftmost, parallel) innermost or outermost, namely in the following sense. Applying some rule $l \rightarrow r{\downarrow_{\mathcal{R}}} \in \mathcal{R}{\downarrow}$ (for some $l \rightarrow r \in \mathcal{R}$) to a term $t = C[l\sigma]_p$ means to apply $l \rightarrow r$ to t/p yielding $t' = C[r\sigma]_p$ followed by some derivation below p where the

[2] Actually, confluence (${}^*\!\leftarrow \circ \rightarrow^* \subseteq \rightarrow^* \circ {}^*\!\leftarrow$) and the Church-Rosser property ($\leftrightarrow^* \subseteq \rightarrow^* \circ {}^*\!\leftarrow$) are well-known to be equivalent concepts.

[3] More formally, a redex (in some term s) can be defined by the pair $(p, l \rightarrow r)$ indicating that the subterm $s/p = l\sigma$ (for some σ) in $s = s[l\sigma]_p$ is to be rewritten or *contracted* into $r\sigma$ yielding the result $s[r\sigma]_p$.

[4] More precisely, this means that for every $l \rightarrow r \in \mathcal{R}$ and for every variable x occurring in l all but one occurrences of x in l are replaced by mutually distinct fresh variables.

pattern r of $r\sigma$ is normalized. If $t/p = l\sigma$ is outermost in t, then a redex which is outermost in r (let's say at position q) need not correspond to an outermost redex in $t' = C[r\sigma]_p$ because some new outermost redex in t' (strictly) above pq may have been created, via the surrounding context $C[.]$ or via instantiation with σ. Similarly, if $t/p = l\sigma$ is innermost in t, then a redex which is innermost in r (let's say at position q) need not correspond to an innermost redex in $t' = C[r\sigma]_p$ because some new innermost redex in t' (strictly) below pq may have been created by instantiation. Hence, in general it seems difficult to predict the effect of right-normalization of some semi-complete TRS on a previously existing normalizing reduction strategy (for the original system). But one might be tempted to hope that at least confluence and weak termination are preserved.

In fact, the following answers question (1) in the negative.

Theorem 3.1 *There exists a semi-complete TRS $\mathcal{R}$ such that $\mathcal{R}\downarrow$ is neither weakly terminating nor locally confluent.*

Proof: Consider the TRS

$$\mathcal{R} = \left\{ \begin{array}{c} a \to b \\ g(b) \to f(a) \\ h(f(x)) \to h(g(x)) \\ h(g(a)) \to c \end{array} \right\} \tag{1}$$

Obviously, right-normalization yields

$$\mathcal{R}\downarrow = \left\{ \begin{array}{c} a \to b \\ g(b) \to f(b) \\ h(f(x)) \to h(g(x)) \\ h(g(a)) \to c \end{array} \right\}$$

We have to show

(i) $\mathcal{R}$ is confluent.

(ii) $\mathcal{R}$ is weakly terminating.

(iii) $\mathcal{R}\downarrow$ is not locally confluent.

(iv) $\mathcal{R}\downarrow$ is not weakly terminating.

The relevant parts of the reduction graphs of $\mathcal{R}$ and $\mathcal{R}\downarrow$ are displayed in Figure 1.

For (i), it suffices to show confluence of $\to_{\widehat{\mathcal{R}}}$ for any $\widehat{\mathcal{R}}$ with $(*)$ $\to_{\mathcal{R}} \subseteq \to_{\widehat{\mathcal{R}}} \subseteq \to^*_{\mathcal{R}}$ because confluence of $\to_{\widehat{\mathcal{R}}}$ satisfying $(*)$ is equivalent to confluence of $\to_{\mathcal{R}}$. Taking for instance $\widehat{\mathcal{R}} := \mathcal{R} \cup \{h(g(b)) \to c, h(f(a)) \to c, h(f(b)) \to c\}$, we have $l \to^+_{\mathcal{R}} r$ for all $l \to r \in \widehat{\mathcal{R}} \setminus \mathcal{R}$ as is easily verified. Hence, $(*)$ holds for $\widehat{\mathcal{R}}$. Moreover, $\widehat{\mathcal{R}}$ is strongly closed ([Hue80]), i.e., for every critical pair $\langle s, t\rangle$ there exist terms u, v such that $s \to^* u \,{}^{=}\!\leftarrow t$ and $s \to^= v \,{}^{*}\!\leftarrow t$. Furthermore, $\widehat{\mathcal{R}}$ is left-linear and right-linear, hence by [Hue80, Lemma 3.2] it is (strongly) confluent. This implies confluence of $\mathcal{R}$ as desired.

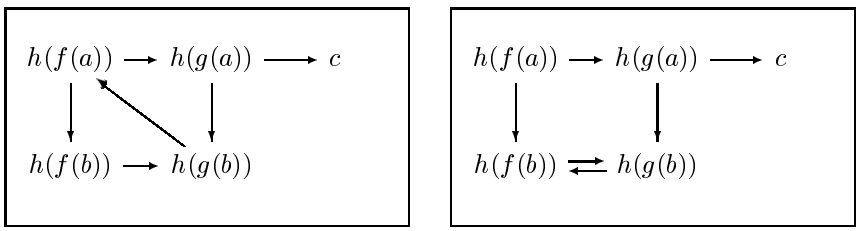

Figure 1: reductions in $\mathcal{R}$ (left) and in $\mathcal{R}{\downarrow}$ (right)

For (ii) we show that all terms t are weakly ($\mathcal{R}$-)terminating, by structural induction and case analysis. In the base case, if t is a variable or a constant (from $\{a, b, c\}$), it obviously has a normal form. In the induction step we have the following cases. If $t = f(t')$ then by induction hypothesis t' has some normal form $\widehat{t'}$. Since there is no rule in $\mathcal{R}$ with f as left-hand side root symbol, $f(\widehat{t'})$ is a normal form of $f(t)$. If $t = g(t')$ then by induction hypothesis t' has some normal form $\widehat{t'}$. If $\widehat{t'} = b$ then $t = g(t')$ reduces to the normal form $f(b)$. Otherwise, i.e., if $\widehat{t'} \neq b$, then $g(\widehat{t'})$ is a normal form of t. Finally, if $t = h(t')$ then by induction hypothesis t' has some normal form $\widehat{t'}$. If $\widehat{t'}$ is of the form $\widehat{t'} = f(t'')$ then, if $t'' = b$ then t reduces to the normal form c otherwise t reduces to the normal form $h(g(t''))$. Otherwise, i.e., if $\widehat{t'} \neq f(t'')$ then t reduces to $h(\widehat{t'})$ which is already in normal form (note that $\widehat{t'} = g(a)$ is impossible since $\widehat{t'}$ is a normal form).
For (iii) we observe that in $\mathcal{R}{\downarrow}$ we have $h(g(b)) \leftarrow h(g(a)) \rightarrow c$. Here, c is irreducible and the only reductions from $h(g(b))$ are of the form $h(g(b)) \rightarrow h(f(b)) \rightarrow h(g(b)) \rightarrow \ldots$. Hence, $\mathcal{R}{\downarrow}$ is not locally confluent.
Moreover, (iv) holds since for instance $h(g(b))$ does not have a normal form in $\mathcal{R}{\downarrow}$. ■

A slight modification of the above counterexample shows that also for the special case of *string rewriting systems* (cf. e.g. [BO93]) semi-completeness is not preserved under right-normalization. To this end one may simply turn the constants a, b, c in $\mathcal{R}$ above into unary function symbols and add some variable x in the corresponding rules. This yields the system $\{a(x) \rightarrow b(x), g(b(x)) \rightarrow f(a(x)), h(f(x)) \rightarrow h(g(x)), h(g(a(x))) \rightarrow c(x)\}$ which corresponds (in the usual way) to the string rewriting system $\{a \rightarrow b, gb \rightarrow fa, hf \rightarrow hg, hga \rightarrow c\}$.

Notice that the above counterexample (1) constitutes a counterexample to question (1a) as well, since

$$\mathcal{R}{\downarrow_{ext}} = \mathcal{R}{\downarrow} \cup \{r \rightarrow r{\downarrow_{\mathcal{R}}} \mid l \rightarrow r \in \mathcal{R}, r \neq r{\downarrow_{\mathcal{R}}}\} = \left\{ \begin{array}{c} a \rightarrow b \\ g(b) \rightarrow f(b) \\ h(f(x)) \rightarrow h(g(x)) \\ h(g(a)) \rightarrow c \\ f(a) \rightarrow f(b) \end{array} \right\}$$

is also neither weakly terminating nor (locally) confluent (for the same reasons as $\mathcal{R}\downarrow$).

We observe that in counterexample (1) the equational theory is obviously preserved which means that in this case right-normalization is at least logically correct. Next we show that even this logical equivalence need not always be preserved.[5]

Theorem 3.2 *There exists a semi-complete TRS $\mathcal{R}$ such that $\leftrightarrow^*_{\mathcal{R}} \not\subseteq \leftrightarrow^*_{\mathcal{R}\downarrow}$.*
Proof: Consider the TRS

$$\mathcal{R} = \left\{ \begin{array}{c} b \to f(b,b) \\ f(x,x) \to g(b, f(x,b)) \\ g(x,x) \to a \end{array} \right\} \tag{2}$$

Obviously, right-normalization yields

$$\mathcal{R}\downarrow = \left\{ \begin{array}{c} b \to a \\ f(x,x) \to g(a, f(x,a)) \\ g(x,x) \to a \end{array} \right\}$$

We will show the following:

(i) $\mathcal{R}$ is weakly terminating.

(ii) $\mathcal{R}$ is confluent.

(iii) $\leftrightarrow^*_{\mathcal{R}} \not\subseteq \leftrightarrow^*_{\mathcal{R}\downarrow}$.

For (i), we show that all terms t are weakly ($\mathcal{R}$-)terminating, by structural induction and case analysis. In the base case, if t is a variable or a constant (from $\{a, b\}$), it obviously has a normal form. In particular, b reduces to the normal form a. If $t = f(t_1, t_2)$ then by induction hypothesis we get $f(t_1, t_2) \to^*_{\mathcal{R}} f(\widehat{t_1}, \widehat{t_2})$ with $\widehat{t_1}$, $\widehat{t_2}$ being normal forms of t_1 and t_2, respectively. If $\widehat{t_1} \neq \widehat{t_2}$ then $f(\widehat{t_1}, \widehat{t_2})$ is a normal form, too. Otherwise, the latter term can be further reduced to $g(a, f(\widehat{t_1}, a))$. Now, this term is a normal form if $\widehat{t_1} \neq a$. Otherwise, it clearly reduces to the normal form a. If $t = g(t_1, t_2)$ then by induction hypothesis we get $g(t_1, t_2) \to^*_{\mathcal{R}} g(\widehat{t_1}, \widehat{t_2})$ with $\widehat{t_1}$, $\widehat{t_2}$ being normal forms of t_1 and t_2, respectively. In case $\widehat{t_1} \neq \widehat{t_2}$, $g(\widehat{t_1}, \widehat{t_2})$ is already in normal form, otherwise it reduces in one step to the normal form a.
For (ii), we observe that $\mathcal{R}$ is strongly non-overlapping, hence it has unique normal forms (UN) by Chew's Theorem ([Che81], [KV89]). Using (i) and the basic fact UN & WN $\Longrightarrow$ CR, this yields confluence of $\mathcal{R}$.
For proving (iii) it suffices to show that $f(a,a) \leftrightarrow^*_{\mathcal{R}\downarrow} a$ does not hold (because $f(a,a) \leftrightarrow^*_{\mathcal{R}} a$ is obviously true). This is equivalent to show $f(a,a) \not\leftrightarrow^*_{\mathcal{R}'} a$ where $\mathcal{R}' = \{b \to a, g(a, f(x,a)) \to f(x,x), g(x,x) \to a\}$ since $\leftrightarrow^*_{\mathcal{R}\downarrow} = \leftrightarrow^*_{\mathcal{R}'}$. Now, $\mathcal{R}'$

[5] This solves the second main problem in [Kah95], which was left open in [Gra97], also in the negative. However, in contrast to Theorem 3.1 it is open whether such a counterexample also exists for string rewriting systems (somehow, non-left-linearity seems to be an essential feature of counterexample (2)).

is confluent (because it is non-overlapping and terminating, e.g. w.r.t. a recursive path ordering based on $b \succ a$, $g \succ f$, $g \succ a$). But confluence of $\mathcal{R}'$ and the fact that $f(a,a)$ and a are both $\mathcal{R}'$-irreducible imply $f(a,a) \not\leftrightarrow^*_{\mathcal{R}'} a$. Hence we are done. ■

We observe that in the above example the proof of (iii) could also be performed (without using $\mathcal{R}'$) by directly using confluence of $\mathcal{R}\downarrow$.[6] However, proving confluence of $\mathcal{R}\downarrow$ (by structural induction and case analysis) is considerably more complicated than proving confluence of $\mathcal{R}'$ or proving confluence of $\mathcal{R}$ as in (ii). For the latter confluence proof we could exploit Chew's Theorem about unique normal forms of strongly non-overlapping TRSs. This approach is useless for proving confluence of $\mathcal{R}\downarrow$ since this system is not weakly terminating (for instance, $f(a,a)$ has no normal form). In fact, none of the known confluence criteria is applicable here, including the ones of [TO95] and [GOO96] for non-left-linear, non-terminating TRSs. Interestingly, $\mathcal{R}\downarrow$ belongs to a class of TRSs, namely strongly non-overlapping and right-linear systems, for which confluence is a long-standing open problem (cf. Problem 58 in [DJK93], due to Oyamaguchi).

Counterexample (1) above still preserves the equational theory whereas counterexample (2) still preserves confluence of the original semi-complete system. Applying well-known modularity results we will now show that there exist semi-complete TRSs that behave even worse, i.e., that enjoy none of the mentioned preservation properties under right-normalization.

Theorem 3.3 *There exists a semi-complete TRS $\mathcal{R}$ such that $\mathcal{R}\downarrow$ is neither weakly terminating nor locally confluent nor does it satisfy $\leftrightarrow^*_{\mathcal{R}} = \leftrightarrow^*_{\mathcal{R}\downarrow}$.*

Proof: Consider the disjoint union $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$, where $\mathcal{R}_2^{\mathcal{F}_2}$ is counterexample (2) and $\mathcal{R}_1^{\mathcal{F}_1}$ is a renamed version of counterexample (1) such that $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$. By modularity of confluence and of weak termination (cf. [Toy87]) $\mathcal{R}$ is also confluent and weakly terminating, hence semi-complete. Clearly we have $\mathcal{R}\downarrow = \mathcal{R}_1\downarrow \oplus \mathcal{R}_2\downarrow$. Now, $\mathcal{R}\downarrow$ is not weakly terminating and not (locally) confluent because $\mathcal{R}_1\downarrow$ has the respective properties. And $\mathcal{R}\downarrow$ has a strictly smaller equational theory than $\mathcal{R}$, because we have $f(a,a) \leftrightarrow^*_{\mathcal{R}} a$ but $f(a,a) \not\leftrightarrow^*_{\mathcal{R}\downarrow} a$. For the latter we need the fact that disjoint unions yield conservative extensions of the equational theories involved (cf. [Sch89]). Hence: $f(a,a) \leftrightarrow^*_{\mathcal{R}\downarrow} a \iff f(a,a) \leftrightarrow^*_{\mathcal{R}_2} a$. But we have $f(a,a) \not\leftrightarrow^*_{\mathcal{R}_2} a$ as shown in the proof of Theorem 3.2(iii) above. ■

Next we consider problems (3) and (4), i.e., the question whether deletion of left-reducible rules preserves semi-completeness and the equational theory of right-normalized semi-complete TRSs. In fact, this is not the case.

[6] Actually, the simple idea of exploiting confluence of $\mathcal{R}'$ (which is straightforward to establish) for proving (iii) is due to an anonymous referee.

Lemma 3.4 *There exists a semi-complete, right-normalized TRS for which deletion of left-reducible rules neither preserves semi-completeness nor the equational theory.*

Proof: Consider the TRS

$$\mathcal{R} = \left\{ \begin{array}{c} g(a) \to f(a) \\ h(f(x)) \to h(g(x)) \\ h(g(a)) \to b \end{array} \right\} \tag{3}$$

which is easily shown to be semi-complete and which is obviously right-normalized. The only rule with a reducible left-hand side is the last one, hence we obtain

$$\mathcal{R}_{del} = \left\{ \begin{array}{c} g(a) \to f(a) \\ h(f(x)) \to h(g(x)) \end{array} \right\}$$

which is neither weakly terminating (for instance, $h(f(a))$ has no normal form) nor does it satisfy $\leftrightarrow^*_{\mathcal{R}} = \leftrightarrow^*_{\mathcal{R}_{del}}$: For instance, $h(g(a)) \to_{\mathcal{R}} b$ but $h(g(a)) \not\leftrightarrow^*_{\mathcal{R}_{del}} b$. The latter is a consequence of confluence of $\mathcal{R}_{del}$ which in turn follows from orthogonality of $\mathcal{R}_{del}$ ([Ros73], [Hue80]). ■

Note that counterexample (3) above is not an overlay system. Yet, this is not essential. The TRS

$$\mathcal{R} = \left\{ \begin{array}{c} b \to a \\ f(x,x) \to g(a, f(x,a)) \\ g(x,x) \to a \\ f(a,a) \to a \end{array} \right\}$$

obtained by extending the right-normalized version of counterexample (2) by the rule $f(a,a) \to a$ is a right-normalized overlay system. Moreover $\mathcal{R}$ is semi-complete as is not very difficult to verify (both weak termination and confluence can be proved by structural induction and case analysis). Deletion of left-reducible rules yields

$$\mathcal{R}_{del} = \left\{ \begin{array}{c} b \to a \\ f(x,x) \to g(a, f(x,a)) \\ g(x,x) \to a \end{array} \right\}$$

which is not weakly terminating and which has a strictly smaller equational theory than $\mathcal{R}$ (as already shown in the proof of Theorem 3.2 since $\mathcal{R}_{del}$ is identical with the right-normalized version of system (2)).

Actually, the careful reader may have noticed the following property of all counterexamples presented so far: Although interreduction into an equivalent semi-canonical TRS was not possible by right-normalization and deletion of left-reducible rules, there still exists an equivalent semi-canonical (and even canonical) system. For instance,

$$\mathcal{R}_1 = \left\{ \begin{array}{c} a \to b \\ f(b) \to g(b) \\ h(f(x)) \to h(g(x)) \\ h(g(b)) \to c \end{array} \right\}$$

is canonical and equivalent to system (1), and

$$\mathcal{R}_2 = \left\{ \begin{array}{c} f(b,b) \to b \\ g(b, f(x,b)) \to f(x,x) \\ g(x,x)) \to b \\ a \to b \end{array} \right\}$$

is canonical and equivalent to system (2). Apart from the obvious question of how to systematically perform interreduction by other means than right-normalization and (subsequent) deletion of left-reducible rules in such cases, one may ask more generally whether for a semi-complete TRS the existence of an equivalent semi-canonical system is always guaranteed. Unfortunately, even this existential property does not hold in general.

Theorem 3.5 *There is a semi-complete right-normalized TRS such that there exists no equivalent semi-canonical system.*
Proof: Consider the TRS

$$\mathcal{R} = \left\{ \begin{array}{l} f(a,x) \to f(x,a) \\ f(a,a) \to a \end{array} \right\} \tag{4}$$

(over the signature $\mathcal{F} = \{a, f\}$) which is right-normalized and semi-complete, but not interreduced since $\mathcal{R} \neq \mathcal{R}_{del} = \{f(a,x) \to f(x,a)\}$. For a proof by contradiction suppose there exists some semi-canonical TRS $\mathcal{R}'$ such that $\leftrightarrow^*_{\mathcal{R}} = \leftrightarrow^*_{\mathcal{R}'}$. First we will prove the following properties: (1) For every $s \in \mathcal{T}(\mathcal{F})$: $s \to^*_{\mathcal{R}'} a$ and a is $\mathcal{R}'$-irreducible, (2) $\mathcal{R}'$ contains exactly one of the two rules $f(a,x) \to f(x,a)$, $f(x,a) \to f(a,x)$, and (3) $f(a,a) \not\leftrightarrow^*_{\mathcal{R}'} a$. For proving (1), we observe that, by $\leftrightarrow^*_{\mathcal{R}} = \leftrightarrow^*_{\mathcal{R}'}$, the $\leftrightarrow^*_{\mathcal{R}'}$-congruence class of a is $[a]_{\leftrightarrow^*_{\mathcal{R}'}} = \mathcal{T}(\mathcal{F})$. By semi-completeness of $\mathcal{R}'$ all terms in $\mathcal{T}(\mathcal{F})$ must have the same unique $\mathcal{R}'$-normal form. This normal form must be a. Otherwise it would be a term of the form $f(t_1,t_2)$ with $t_1, t_2 \in \mathcal{T}(\mathcal{F})$. But by confluence of $\mathcal{R}'$ this would imply that there exists a rule of the form $a \to u \in \mathcal{R}'$ with $u \in \mathcal{T}(\mathcal{F})$ (such that $u \to^*_{\mathcal{R}'} f(t_1,t_2)$). However, u must contain a as subterm, hence u would be $\mathcal{R}'$-reducible which contradicts the assumption that $\mathcal{R}'$ is interreduced. For proving (2), consider the $\leftrightarrow^*_{\mathcal{R}'}$-congruence classes of $f(a,x)$ and $f(x,a)$. By $\leftrightarrow^*_{\mathcal{R}} = \leftrightarrow^*_{\mathcal{R}'}$ and by the shape of the rules in $\mathcal{R}$ we must have: $[f(a,x)]_{\leftrightarrow^*_{\mathcal{R}'}} = [f(x,a)]_{\leftrightarrow^*_{\mathcal{R}'}} = \{f(s,x) \mid s \in \mathcal{T}(\mathcal{F})\} \cup \{f(x,t) \mid t \in \mathcal{T}(\mathcal{F})\}$. By confluence of $\mathcal{R}'$ at least one of the terms $f(a,x)$, $f(x,a)$ must be $\mathcal{R}'$-reducible, by a rule $l \to r \in \mathcal{R}'$. By (1), l cannot be a, hence it must be of the form $f(a,x)$, $f(x,a)$ or $f(x,y)$. The last case is impossible, since then $[f(x,y)]_{\leftrightarrow^*_{\mathcal{R}'}} = \{f(x,y)\}$ would imply $r = f(x,y) = l$ contradicting the assumption that $\mathcal{R}'$ is interreduced. In the first case, r must be of the form $f(a,x)$ or $f(x,a)$ (because of (1)). By $\mathcal{R}'$-irreducibility of r the only possibility remaining is $r = f(x,a)$. Analogously, in the second case $l = f(x,a)$ we must have $r = f(a,x)$. Now, (3) is a consequence of (1), (2) and the assumption that $\mathcal{R}'$ is semi-canonical: $f(a,a) \leftrightarrow^*_{\mathcal{R}'} a$ is equivalent to $f(a,a) \to^*_{\mathcal{R}'} a$ which is only possible if $f(a,a) \to a \in \mathcal{R}'$. But then $\mathcal{R}'$ is not interreduced since $f(a,a)$ is

$(\mathcal{R}' \setminus \{f(a,a) \to a\})$-reducible. Hence, a semi-canonical TRS $\mathcal{R}'$ equivalent to $\mathcal{R}$ cannot exist. ■

Two aspects of counterexample (4) above are somehow interesting. First of all, we observe that the reduction $f(a,a) \to_{f(a,x)\to f(x,a)} f(a,a)$ which is responsible for the deletion of the rule $f(a,a) \to a$ is not a "proper" one. Hence, one might be tempted to modify the definition of being interreduced by taking $\{l \to r \mid l \to_{\mathcal{R}\setminus\{l\to r\}} l' \Longrightarrow l = l'\}$ instead of $\{l \to r \mid l \text{ is } (\mathcal{R} \setminus \{l \to r\})\text{-irreducible}\}$ for $\mathcal{R}_{del}$ (note that for a complete system $\mathcal{R}$ the two versions of $\mathcal{R}_{del}$ obviously coincide). But even under the resulting more liberal notion of being interreduced Theorem 3.5 would still hold. A simple counterexample is the semi-complete TRS $\{f(a,x,y,z) \to f(x,a,z,y), f(a,a,x,y) \to a\}$ (the proof of the counterexample property for this system is similar to the one for system (4) above).

Secondly, one might also weaken the required equivalence property $\leftrightarrow^*_{\mathcal{R}'} = \leftrightarrow^*_{\mathcal{R}}$ of a given semi-complete TRS $\mathcal{R}$ with a semi-canonical TRS $\mathcal{R}'$ into a conservative extensibility property. More precisely, this would amount to find, for semi-complete $\mathcal{R}^{\mathcal{F}}$, a semi-canonical $\mathcal{R}'^{\mathcal{F}'}$ (over an extended signature $\mathcal{F}' \supseteq \mathcal{F}$) such that $\leftrightarrow^*_{\mathcal{R}'^{\mathcal{F}'}}|_{\mathcal{T}(\mathcal{F},\mathcal{V})\times\mathcal{T}(\mathcal{F},\mathcal{V})} = \leftrightarrow^*_{\mathcal{R}^{\mathcal{F}}}$. In example (4) above, such an $\mathcal{R}'^{\mathcal{F}'}$ indeed exists, namely $\mathcal{R}' = \{f(a,x) \to g(x), f(x,a) \to g(x), g(a) \to a\}$ (over the extended signature $\mathcal{F}' = \mathcal{F} \cup \{g\}$) which is even canonical. However, it is neither clear whether such a system $\mathcal{R}'^{\mathcal{F}'}$ always exists nor how to search for it systematically.

4 Sufficient Conditions

Let us now consider under which conditions the questions (1)-(4) of the introduction can be answered in the affirmative. We start with some simple facts about right-normalization.

Lemma 4.1 *Let $\mathcal{R}$ be semi-complete. Then the following properties hold:*

(i) A term is $\mathcal{R}$-reducible iff it is $\mathcal{R}\downarrow$-reducible.

(ii) $\to_{\mathcal{R}\downarrow} \subseteq \to^*_{\mathcal{R}}$.

(iii) If $\mathcal{R}\downarrow$ is weakly terminating, then $\mathcal{R}\downarrow$ is confluent (hence semi-complete).

(iv) If $\mathcal{R}\downarrow$ is weakly terminating, then $\leftrightarrow^*_{\mathcal{R}} = \leftrightarrow^*_{\mathcal{R}\downarrow}$.

Proof: (i) and (ii) are trivial by definition of $\mathcal{R}\downarrow$. (iii) and (iv) immediately follow by (i) and (ii) from the following property: If $\mathcal{R}\downarrow$ is weakly terminating, then $\leftrightarrow^*_{\mathcal{R}} \subseteq \to^*_{\mathcal{R}\downarrow} \circ {}^*_{\mathcal{R}\downarrow}\!\leftarrow$. To prove this, suppose $s \leftrightarrow^*_{\mathcal{R}} s'$. By $\mathcal{R}\downarrow$ weakly terminating one gets $\mathcal{R}\downarrow$-normal forms t, t' such that $s \to^*_{\mathcal{R}\downarrow} t$ and $s' \to^*_{\mathcal{R}\downarrow} t'$. By (ii) then $t \leftrightarrow^*_{\mathcal{R}} t'$. By (i), t and t' are $\mathcal{R}$-normal forms, which implies $t = t'$ by confluence of $\mathcal{R}$. ■

It is obvious that right-normalization disables certain reduction steps which are possible in the original system. One case where this is harmless is the following.

Theorem 4.2 *Let $\mathcal{R}$ be a semi-complete TRS such that $\mathcal{R}\downarrow$ satisfies: $(*)$ no non-variable subterm of a right-hand side in $\mathcal{R}\downarrow$ is unifiable with a left-hand side in $\mathcal{R}\downarrow$. Then $\mathcal{R}\downarrow$ is semi-complete.*

Proof: Let $\mathcal{R}$ satisfy the above assumptions. By Lemma 4.1 it suffices to show weak termination of $\mathcal{R}\downarrow$. We show by structural induction that every term t has a normal form (in $\mathcal{R}\downarrow$). First we consider the base case: If t is a variable or irreducible constant then we already have a normal form. If t is a reducible constant, then t is a left-hand side of $\mathcal{R}$ and of $\mathcal{R}\downarrow$. Consequently, t reduces in one $\mathcal{R}\downarrow$-step to a normal form. In the induction step t is of the form $t = f(t_1, \ldots, t_n)$. By induction hypothesis, every t_i $(1 \leq i \leq n)$ has some normal form $\widehat{t_i}$ in $\mathcal{R}\downarrow$, hence $t = f(t_1, \ldots, t_n) \to^*_{\mathcal{R}\downarrow} f(\widehat{t_1}, \ldots, \widehat{t_n}) =: \widehat{t}$. If $\widehat{t}$ is irreducible, we are done. Otherwise, we must have $\widehat{t} = l\sigma$ for some rule $l \to r \in \mathcal{R}$ and some irreducible substitution σ, i.e., with $\sigma(x)$ irreducible for every $x \neq \sigma(x)$ (note that σ may be assumed to be irreducible because all proper subterms of $\widehat{t}$ are irreducible). Hence, $\widehat{t} = l\sigma \to_{\mathcal{R}\downarrow} (r\downarrow_{\mathcal{R}})\sigma$. The latter term is irreducible by assumption $(*)$ and irreducibility of σ. Hence, $t = f(t_1, \ldots, t_n)$ reduces in $\mathcal{R}\downarrow$ to the normal form $(r\downarrow_{\mathcal{R}})\sigma$ as desired. Thus we are done. ■

Note that for instance counterexample (1) in the proof of Theorem 3.1 violates the condition $(*)$, since the non-variable subterm $g(x)$ of the right-hand side $h(g(x))$ of the $\mathcal{R}\downarrow$-rule $h(f(x)) \to h(g(x))$ is unifiable with the left-hand side $g(b)$.

We observe that for the preservation of weak termination in Theorem 4.2 confluence of $\mathcal{R}$ is not really necessary. More precisely, if $\mathcal{R}$ is weakly terminating then we may consider any system $\mathcal{R}'$, which is obtained by replacing in $\mathcal{R}$ every right-hand side by one of its $\mathcal{R}$-normal forms. If $\mathcal{R}'$ satisfies $(*)$ above, then it is innermost terminating, hence in particular weakly terminating (this is easily proved by structural induction in a way very similar to the above proof). Moreover, in such a TRS $\mathcal{R}'$ the contractum of an innermost redex in some term is irreducible. This means that the length of innermost derivations in $\mathcal{R}'$ is bounded by the size of the initial term.

For this reason, the class of TRSs to which Theorem 4.2 applies seems to be rather restricted. Moreover, in some sense the condition $(*)$ in Theorem 4.2 is not very satisfactory because it is formulated in terms of the resulting system and not of the original one.

Another much more interesting sufficient condition relies on a deep result of O'Donnell about normalization of reduction strategies. More precisely, we shall exploit the fact that parallel-outermost reduction and — more generally — outermost-fair reduction are normalizing for almost orthogonal TRSs ([O'D77]). The latter result has recently been extended to the higher-order case by van

Raamsdonk ([Raa96]).[7] First let us explain some notions needed here and subsequently. A reduction strategy is said to be *normalizing* if for any term t possessing a normal form reduction (of t) according to the strategy eventually ends in a normal form. A (possibly infinite) reduction sequence (where for every step the redex contracted is specified) is said to be *outermost-fair* if every outermost redex occurrence is eventually eliminated. In almost orthogonal TRSs outermost redex occurrences can only be eliminated and created by contracting redex occurrences which are outermost themselves (this property does not hold for weakly orthogonal systems).

Theorem 4.3 *For almost orthogonal semi-complete TRS, right-normalization preserves semi-completeness as well as the equational theory.*

Proof: Let $\mathcal{R}$ be semi-complete and almost orthogonal. To show semi-completeness of $\mathcal{R}{\downarrow}$ and the preservation of the equational theory it suffices by Lemma 4.1 to prove that $\mathcal{R}{\downarrow}$ is weakly terminating. For an arbitrary term s we construct the $\mathcal{R}{\downarrow}$-reduction (parallel reduction by contraction of outermost redexes is denoted by $\underset{\circ}{\Rightarrow}$)

$$D' : \; s =: s_1 \; \underset{\circ}{\Rightarrow}_{\mathcal{R}\downarrow} s_2 \; \underset{\circ}{\Rightarrow}_{\mathcal{R}\downarrow} s_3 \; \underset{\circ}{\Rightarrow}_{\mathcal{R}\downarrow} \cdots$$

where s_{k+1} is obtained from s_k by the parallel contraction of all (parallel) outermost $\mathcal{R}{\downarrow}$-redexes in s_k. If some outermost root-reducible subterm s_k/p of s_k is root-reducible by several distinct rules then we choose an arbitrary applicable one. In any case, the result of contracting s_k/p is unique, since $\mathcal{R}$ is almost orthogonal. By definition of $\mathcal{R}{\downarrow}$, we have $l \to_{\mathcal{R}} r \to^*_{\mathcal{R}} r'$ for every $l \to r \in \mathcal{R}$, $l \to r' \in \mathcal{R}{\downarrow}$. Thus we may refine D' into an $\mathcal{R}$-derivation

$$D : \; s =: s_1 \; \underset{\circ}{\Rightarrow}_{\mathcal{R}} s'_2 \to^*_{\mathcal{R}} s_2 \; \underset{\circ}{\Rightarrow}_{\mathcal{R}} s'_3 \to^*_{\mathcal{R}} s_3 \; \underset{\circ}{\Rightarrow}_{\mathcal{R}} \cdots$$

where s_{k+1} is obtained from s_k by the parallel contraction of all (parallel) outermost $\mathcal{R}$-redexes in s_k (yielding s'_{k+1}) followed by the $\mathcal{R}$-normalization of the right-hand side patterns of the applied $\mathcal{R}$-rules,[8] i.e., if $s_k = C[l_1\sigma_1, \ldots, l_n\sigma_n]$ (with $l_i \to r_i \in \mathcal{R}$) where all outermost $\mathcal{R}$-redexes are displayed then

$$s_k \; \underset{\circ}{\Rightarrow}_{\mathcal{R}} C[r_1\sigma_1, \ldots, r_n\sigma_n] = s'_{k+1} \to^*_{\mathcal{R}} C[(r_1{\downarrow_{\mathcal{R}}})\sigma_1, \ldots, (r_n{\downarrow_{\mathcal{R}}})\sigma_n] = s_{k+1} \,.$$

Now, for every k every outermost $\mathcal{R}$-redex of s_k is eliminated in the step $s_k \; \underset{\circ}{\Rightarrow}_{\mathcal{R}} s'_{k+1}$.[9] Hence, D is in particular an outermost-fair $\mathcal{R}$-derivation. Since outermost-fair reduction is normalizing for almost orthogonal TRSs and $\mathcal{R}$ is semi-complete and almost orthogonal, D must end with an $\mathcal{R}$-normal form s_n (for some n). By definition of $\mathcal{R}{\downarrow}$ we know that s_n is also a normal form w.r.t. $\mathcal{R}{\downarrow}$. Consequently, D' must also end in the ($\mathcal{R}$- and $\mathcal{R}{\downarrow}$-) normal form s_n. Hence,

$$D' : \; s =: s_1 \; \underset{\circ}{\Rightarrow}_{\mathcal{R}\downarrow} s_2 \; \underset{\circ}{\Rightarrow}_{\mathcal{R}\downarrow} \cdots \; \underset{\circ}{\Rightarrow}_{\mathcal{R}\downarrow} s_n$$

[7] Note that *outermost-fair* reduction ([Raa96]) is called *eventually outermost* in [O'D77].

[8] For uniqueness one may assume for every rule $l \to r \in \mathcal{R}$ one fixed normalizing $\mathcal{R}$-derivation $r \to^*_{\mathcal{R}} r{\downarrow_{\mathcal{R}}}$.

[9] Note that, for $k \geq 2$, all outermost redexes of all intermediate terms occurring in $s'_k \to^*_{\mathcal{R}} s_k$ are eliminated at the latest in the step $s_k \; \underset{\circ}{\Rightarrow}_{\mathcal{R}} s'_{k+1}$.

is a normalizing $\mathcal{R}\downarrow$-derivation for s as desired. ■

We observe that the right-normalized version of a semi-complete orthogonal TRS is not only semi-complete but by construction also orthogonal. This preservation of orthogonality does not extend to almost orthogonality, i.e., the right-normalized version of a semi-complete, almost orthogonal TRS is semi-complete, but need not be almost orthogonal (however, it is still an overlay system). For instance, right-normalization of the complete, almost orthogonal TRS

$$\mathcal{R} = \left\{ \begin{array}{c} f(x, g(a)) \to h(x) \\ f(g(a), g(x)) \to h(g(x)) \\ h(g(x)) \to b \end{array} \right\}$$

yields

$$\mathcal{R}\downarrow = \left\{ \begin{array}{c} f(x, g(a)) \to h(x) \\ f(g(a), g(x)) \to b \\ h(g(x)) \to b \end{array} \right\}$$

which is complete, but obviously not almost orthogonal. This observation may be interesting because almost all known results about reduction strategies for orthogonal TRSs also hold for almost orthogonal ones, but not for more general classes of TRSs.

Actually, considering the proof of Theorem 4.3 it is obvious that instead of normalization of outermost-fair reduction *hyper normalization*[10] of parallel-outermost reduction may also be used as essential property (this is exactly what is needed in the construction). And it seems plausible ([Mid97b]) that hyper normalization of parallel-outermost reduction for weakly orthogonal TRSs can be proved by using ideas of [SR93], [OR94], [Mid97a]. Yet, this remains to be checked in detail. Concerning normalization of outermost-fair reduction we are not aware of any further positive results (besides the one mentioned above). For instance, it seems to be open whether outermost-fair reduction is also normalizing for weakly orthogonal TRSs. Note that weak orthogonality is only slightly more general than almost orthogonality.[11]

Finally, we investigate conditions under which deletion of left-reducible rules is well-behaved, in particular w.r.t. the preservation of the equational theory. In the previous section we have seen that even for overlay systems things may go wrong. One case where this simplification is harmless is the following.

[10] We say that a strategy $\to_s$ (i.e., satisfying $\to_s \subseteq \to^*$) whose normal forms are also $\to$-normal forms is *hyper normalizing* if every term that is weakly terminating under $\to$ terminates under $\to^* \circ \to_s \circ \to^*$.

[11] *Note added in final version*: In fact, very recently van Oostrom has shown in [Oos99] that outermost-fair (reduction) strategies are normalizing for weakly orthogonal TRSs (the result in [Oos99] is even more general in the sense that it holds for the more general class of weakly orthogonal *fully extended higher-order pattern rewrite systems*). Hence, the statement of our Theorem 4.3 also holds for weakly orthogonal semi-complete TRSs.

Lemma 4.4 *For any weakly non-overlapping TRS $\mathcal{R}$, $\mathcal{R}$ and $\mathcal{R}_{del}$ induce the same (one-step) rewrite relation.*
Proof: By definition of $\mathcal{R}_{del}$ it suffices to show $l \to_{\mathcal{R}_{del}} r$ for all $l \to r \in \mathcal{R}$. Let $l \to r \in \mathcal{R}$ and suppose by induction w.r.t. the strict part $\trianglerighteq$ of the encompassment ordering $\trianglerighteq$ (cf. [DJ90]) that $l' \to_{\mathcal{R}_{del}} r'$ holds for all $l' \to r' \in \mathcal{R}$ with $l \trianglerighteq l'$. If $l \to r \in \mathcal{R}_{del}$ then we are obviously done. Otherwise, l must be $(\mathcal{R} \setminus \{l \to r\})$-reducible. Consider a rule $l' \to r' \in \mathcal{R} \setminus \{l \to r\}$ that reduces l such that l' is minimal w.r.t. $\trianglerighteq$ (among all left hand sides of rules in $\mathcal{R} \setminus \{l \to r\}$ that reduce l). Since $\mathcal{R}$ is weakly non-overlapping, we have $l \to_{l' \to r'} r$. If l' is a renaming of l then, again by $\mathcal{R}$ weakly non-overlapping, $l \to r$ and $l' \to r'$ are the same rules, which contradicts $l' \to r' \in \mathcal{R} \setminus \{l \to r\}$. Hence, we must have $l \trianglerighteq l'$. By inductive hypothesis this implies $l' \to_{\mathcal{R}_{del}} r'$, hence also $l \to_{\mathcal{R}_{del}} r$ as was to be shown. ■

By combining Theorem 4.3, the fact that right-normalization of semi-complete orthogonal TRSs trivially preserves orthogonality, and Lemma 4.4, we finally get the following.

Theorem 4.5 *For semi-complete orthogonal TRSs, right-normalization as well as subsequent deletion of left-reducible rules preserve semi-completeness and the equational theory.*

This result may be particularly useful for actual computations, because for semi-complete orthogonal TRSs right-normalization and (subsequent) deletion of left-reducible rules also preserve orthogonality. Consequently, strategies which are known to be normalizing for orthogonal systems like parallel-outermost can safely be used to compute in the interreduced system, too.

5 Conclusion

We have investigated the operational and logical preservation properties of the typical simplifying transformations for complete TRSs, right-normalization and (subsequent) deletion of left-reducible rules, in the context of semi-complete TRSs. Somehow surprisingly, it has turned out here, that in general these simplifications are neither operationally harmless nor logically correct. However, we could prove that at least the important class of orthogonal TRSs enjoys all these preservation properties and, hence, allows for safe interreduction. In the general case, for a given semi-complete TRS an equivalent semi-canonical system need not even exist.

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