A Note on Minimal Counterexamples to Modularity of Termination (Preliminary Version)

Bernhard Gramlich and Klaus Györgyfalvay

August 2012

TECHNISCHE UNIVERSITÄT WIEN

Technical Report E1852-2012-02

Theory and Logic Group, Institute of Computer Languages (E185/2)
TU Wien, Favoritenstraße 9, A-1040 Wien, Austria

A Note on Minimal Counterexamples to Modularity of Termination (Preliminary Version)

Bernhard Gramlich and Klaus Györgyfalvay
Institute of Computer Languages (E185/2), TU Wien, Austria
{gramlich,klausgy}@logic.at

August 2012

Abstract

Termination is well-known to be a non-modular property of term rewriting systems in general. We analyze the complexity of showing non-modularity in terms of the rank of minimal counterexamples (to the modularity of termination). Our main result is that for left-linear terminating systems the rank of minimal counterexamples may be arbitrarily high. We also show the same result for terminating systems which are confluent.

1 Introduction and Basics

Modular aspects in term rewriting have now been studied for about 30 years, with an impressive amount of results, insights and very fruitful developments. In general, arbitrary combinations of term rewritings systems (TRSs) have a very bad modularity behaviour. For termination, combining the terminating one-rule systems consisting of $a \to b$ and $b \to a$, respectively, yields a non-terminating (even cyclic) system. Similarly, the combination of the confluent one-rule systems consisting of $a \to b$ and $a \to c$, respectively, gives a non-confluent system. For this reason, a very special case of combinations of TRSs, and dually of decompositions of TRSs, has been analyzed in depth, namely that of disjoint unions (with disjoint sets of function symbols and hence also disjoint sets of rules). In such combinations of systems there is almost no interaction between the two systems, except via the shared variables. Still, the analysis of this special type of combinations has turned out to be very much non-trivial, but fruitful, deep and fundamental for any less restrictive type of combinations. Informally, a property $\mathcal{P}$ of TRSs is *modular* (w.r.t. disjoint unions) if for any disjoint TRSs we have that both of them enjoy $\mathcal{P}$ iff their disjoint union enjoys $\mathcal{P}$.

In the sequel we generally assume familiarity with term rewriting, cf. e.g. [1, 2], but for the sake of readability will introduce some basics. Then, in Section 2 we discuss non-modularity results, counterexamples and sufficient criteria for modularity of especially the termination property. Finally, in the main Section 3 we will present new (families of) counterexamples of arbitrarily high complexity which sheds some new light on the complexity of proving both positive and negative modularity results.

1.1 Basic Notions and Notations in Term Rewriting and Modularity

1.1.1 Abstract Rewriting

An *abstract reduction system (ARS)* is a pair $\mathcal{A} = \langle A, \rightarrow \rangle$ consisting of a set A and a reduction (or rewrite) relation, i.e., a binary relation $\rightarrow \subseteq AxA$ for which we use infix notation.A *reduction sequence* or *derivation* (in $\mathcal{A}$) is a (finite or infinite) sequence $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \ldots$. For $b \rightarrow a$ we also write $a \leftarrow b$. The symmetric, transitive, transitive-reflexive and symmetric-transitive-reflexive closures of $\rightarrow$ are denoted by $\leftrightarrow$, $\rightarrow^+$, $\rightarrow^*$ and $\leftrightarrow^*$, respectively. If $a \rightarrow^* b$ we say that a *reduces* or *rewrites* to b and we call b a *reduct* of a. By $a \rightarrow^m b$ we mean that a reduces to b in m steps. Accordingly $a \rightarrow^{\leq n} b$ means $a \rightarrow^m b$ for some $m \leq n$. Two elements $a, b \in A$ are *joinable*, denoted by $a \downarrow b$, if there exists an element $c \in A$ with $a \rightarrow^* c \leftarrow^* b$. An element $a \in A$ is *irreducible* or *a normal form* if there is no $b \in A$ with $a \rightarrow b$. An element $a \in A$ *has a normal form* if there exists a normal form $b \in A$ with $a \rightarrow^* b$. In that case b is called a *normal form of* a. The set of all normal forms of $\mathcal{A}$ is denoted by NF($\mathcal{A}$) or simply NF($\rightarrow$).

$\mathcal{A} = \langle A, \rightarrow \rangle$ is said to be *weakly normalizing* (or *weakly terminating*) (WN) if every element of A has a normal form. $\mathcal{A}$ is *strongly normalizing* or *terminating* (SN) if there is no infinite reduction sequence $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \ldots$, i.e., of every reduction sequence eventually ends in some normal form. $\mathcal{A}$ is *confluent* or *Church-Rosser* (CR)[1] if for all $a, b, c \in A$ with $b \leftarrow^* a \rightarrow^* c$ we have $b \downarrow c$. $\mathcal{A}$ is *locally confluent* or *weakly Church-Rosser (WCR)* if for all $a, b, c \in A$ with $b \leftarrow a \rightarrow c$ we have $b \downarrow c$.

$\mathcal{A}$ *has the normal form property* (NF) if for all $a, b \in A$ with $a \leftrightarrow^* b$ and $b \in$ NF($\mathcal{A}$) we have $a \rightarrow^* b$. $\mathcal{A}$ *has unique normal forms* (UN) if for all $a, b \in A$ with $a \leftrightarrow^* b$ and $a, b \in$ NF($\mathcal{A}$) we have $a = b$. $\mathcal{A}$ *has unique normal forms w.r.t. reduction* (UN$^{\rightarrow}$) if for all $a, b, c \in A$ with $a \leftarrow^* b \rightarrow^* c$ and $a, c \in$ NF($\mathcal{A}$) we have $a = c$.

If an ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ has a certain property P (denoted by $P(\mathcal{A})$), we also say that $\rightarrow$ has the property P (and also write $P(\rightarrow)$).

1.1.2 Term Rewriting

Terms are built over a signature $\mathcal{F}$ of function symbols (with fixed arities) and a countably infinite set $\mathcal{V}$ of variables. The set of all terms is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$,

A *context* $C[, \ldots,]$ is a term with 'holes', i.e. a term in $\mathcal{T}(\mathcal{F} \uplus \{\Box\}, \mathcal{V})$ (the symbol '$\uplus$' denotes disjoint set union) where $\Box$ is a new special constant symbol. If $C[, \ldots,]$ is a context with n occurrences of $\Box$ and $t_1, \ldots, t_n$ are terms then $C[t_1, \ldots, t_n]$ is the term obtained from $C[, \ldots,]$ by replacing from left to right the occurrences of $\Box$ by $t_1, \ldots, t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[]$.

A *term rewriting system (TRS)* is a pair $\mathcal{R} = (\mathcal{F}, R)$ consisting of a signature $\mathcal{F}$ and a set $R \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$ of (rewrite) rules (l, r), denoted by $l \rightarrow r$, with $l \notin \mathcal{V}$ and $V(r) \subseteq V(l)$. The rewrite or reduction relation induced by a TRS $\mathcal{R} = (\mathcal{F}, R)$ is denoted by $\rightarrow_{\mathcal{R}}$ or just $\rightarrow$ if $\mathcal{R}$ is clear from the context. We say that a TRS $\mathcal{R}$ is *(inter)reduced* (IR) if (a) r is $\mathcal{R}$-irreducible for every rule $l \rightarrow r \in \mathcal{R}$, and (b) no lhs l, for some $l \rightarrow r \in \mathcal{R}$, is $\mathcal{R} \setminus \{l \rightarrow r\}$-reducible.[2]

For common well-known syntactic properties of (rewrite rules and) TRSs we use the following abbreviations: left-linear (LL), right-linear (RL), *non-collapsing* (NCOL) – i.e., no rhs of a rule is a variable, *non-duplicating* (NDUP) – i.e., no variable occurs (strictly) more often in a rhs side than in the lhs of a rule, *non-erasing* (or *variable-preserving*) (NE),

[1] or *has the Church-Rosser property*

[2] Of course, equality of rules is meant modulo renaming of variables.

overlaying or being an *overlay system* (OS) – i.e., all critical pairs are critical *overlays* (in other words, there is no critical overlap below the root of rules).

A *rewrite ordering* (on $\mathcal{T}(\mathcal{F},\mathcal{V})$) is a strict partial ordering on terms closed under contexts and substitutions. A *reduction ordering* is a well-founded rewrite ordering. A rewrite ordering $>$ is a *simplification ordering* if it possesses the subterm property $C[s] > s$ for any s and any non-empty context $C[]$. A TRS $\mathcal{R}^{\mathcal{F}}$ is *simplifying* if there exists a simplification ordering $>$ with $\rightarrow_{\mathcal{R}} \subseteq >$. It is *simply terminating* if there exists a well-founded simplification ordering $>$ which contains $\rightarrow_{\mathcal{R}}$. The *embedding* TRS $\mathcal{R}^{\mathcal{F}}_{emb} = (\mathcal{F}, R^{\mathcal{F}}_{emb}) = \{f(x_1,\ldots,x_n) \rightarrow x_i \mid 1 \leq i \leq n = ar(f), f \in \mathcal{F}\}$ consists of all projection rules for all $f \in \mathcal{F}$. $\mathcal{R}^{\mathcal{F}}$ is $\mathcal{C}_{\mathcal{E}}$-terminating ($\mathcal{C}_{\mathcal{E}}$-SN) if $\mathcal{R}^{\mathcal{F}} \uplus \{G(x,y) \rightarrow x, G(x,y) \rightarrow y)\}$ is terminating. $\mathcal{R}^{\mathcal{F}}$ is said to be *consistent* (CONS) if $x \leftrightarrow^*_{\mathcal{R}} y$ for distinct variables x, y does not hold, and *consistent w.r.t. reduction* (CONS$^{\rightarrow}$) if there is no term s with $x \leftarrow^*_{\mathcal{R}} s \rightarrow^* y$ for two distinct variables x, y.

1.1.3 Modularity

Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be TRSs with disjoint signatures $\mathcal{F}_1$, $\mathcal{F}_2$. Their *disjoint union* $\mathcal{R}^{\mathcal{F}}$ is the TRS $(\mathcal{F}, R)$ with $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$, $R = R_1 \uplus R_2$. A property $\mathcal{P}$ of TRSs is said to be *modular* if for all disjoint TRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ the following holds: $\mathcal{R}^{\mathcal{F}}$ has property $\mathcal{P}$ iff both $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ have property $\mathcal{P}$. Let $t = C[t_1,\ldots,t_n]$, $n \geq 1$, with $C[,\ldots,] \neq \Box$. We write $t = C[\![t_1,\ldots,t_n]\!]$ if $C[,\ldots,]$ is a context over the signature $\mathcal{F}_a$ and $root(t_1),\ldots,root(t_n) \in \mathcal{F}_b$ for some $a, b \in \{1,2\}$ with $a \neq b$. In this case the t_i's are the *principal subterms* or *principal aliens* of t. Note that every $t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V}) \setminus (\mathcal{T}(\mathcal{F}_1,\mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2,\mathcal{V}))$ has a unique representation of the form $t = C[\![t_1,\ldots,t_n]\!]$. The set of all *aliens* (or *special subterms*) of t can be recursively defined in an obvious manner.

The *rank* of a term $t \in \mathcal{T}(\mathcal{F}_1 \uplus \mathcal{F}_2, \mathcal{V})$ is defined by

$$rank(t) = \begin{cases} 1 & \text{if} \quad t \in \mathcal{T}(\mathcal{F}_1,\mathcal{V}) \cup \mathcal{T}(\mathcal{F}_2,\mathcal{V}) \\ 1 + max\{rank(t_i) | 1 \leq i \leq n\} & \text{if} \quad t = C[\![t_1,\ldots,t_n]\!] \end{cases}$$

Reduction is always rank-decreasing. A reduction step can only be strictly rank-decreasing if it uses a *collapsing* rule $l \rightarrow r$, i.e. with $r \in \mathcal{V}$. Subsequently, when we speak about modularity, it is always meant w.r.t. to disjoint unions of TRSs.

2 Non-Modularity Results, Counterexamples and Sufficient Modularity Criteria

The following (incomplete) table summarizes some of the most important known modularity results concerning termination and confluence properties. Note that it does not mention at all many related asymmetric modularity criteria

property	is modular?	reason/reference
SN	$-$	[17, 18], Example 1
CR	$+$	[17, 18]
NF	$-$	[10]
NF $\wedge$ LL	$+$	[11]
UN	$+$	[10]
$\text{UN}^{\rightarrow}$	$-$	[10]
$\text{UN}^{\rightarrow}$ $\wedge$ LL	$+$	[9]
CONS	$+$	[15]
$\text{CONS}^{\rightarrow}$	$-$	[9]
$\text{CONS}^{\rightarrow}$ $\wedge$ LL	$+$	[9, 16]
SN $\wedge$ CR	$-$	[17, 3], Example 2
SN $\wedge$ CR $\wedge$ IR	$-$	[17]
SN $\wedge$ CR $\wedge$ NE	$-$	[13]
SN $\wedge$ CR $\wedge$ NE$\wedge$ IR	$-$	[13]
SN $\wedge$ CR $\wedge$ LL	$+$	[19, 20, 8]
SN $\wedge$ CR $\wedge$ OS	$+$	[4, 6]
SN $\wedge$ $\text{CONS}^{\rightarrow}$	$-$	[17, 3]
SN $\wedge$ $\text{CONS}^{\rightarrow}$ $\wedge$ LL	$+$	[16]
SN $\wedge$ NE $\wedge$ LL	$+$	[16]
SN $\wedge$ NCOL	$+$	[14]
SN $\wedge$ NDUP	$+$	[14]
simply SN	$+$	[7]
$\mathcal{C}_{\mathcal{E}}$-SN	$+$	[5, 12]

Toyama's famous counterexample to modularity of termination is the following.

Example 1 (SN is not modular, [18, 17])**.**

$$\mathcal{R}_1 = \left\{ \; f(a,b,x) \to f(x,x,x) \; \right\} \qquad\qquad \mathcal{R}_2 = \left\{ \begin{array}{c} H(x,y) \to x \\ H(x,y) \to y \end{array} \right\}$$

In the disjoint union we have the cyclic derivation

$$f(a,b,H(a,b)) \to f(H(a,b),H(a,b),H(a,b)) \to^{+} f(a,b,H(A,b))\,.$$

Note that $\mathcal{R}_2$ above is not confluent. But even (SN $\wedge$ CR) is not modular, as shown in [17]. The following counterexample is due to [3].

Example 2 ((SN $\wedge$ CR) is not modular, [17, 3])**.**

$$\mathcal{R}_1 = \left\{ \begin{array}{c} f(a,b,x) \to f(x,x,x) \\ a \to c \\ b \to c \\ f(x,y,z) \to c \end{array} \right\} \qquad\qquad \mathcal{R}_2 = \left\{ \begin{array}{c} K(x,y,y) \to x \\ K(y,x,y) \to x \end{array} \right\}$$

In the disjoint union we have (with $s = K(a,b,c)$) the cyclic derivation

$$f(a,b,s) \to f(s,s,s) \to^{+} f(K(a,c,c),K(c,b,c),K(a,b,c)) \to^{+} f(a,b,K(a,b,c)) = f(a,b,s)\,.$$

3 Minimal Counterexamples

When trying to verify some property in the disjoint union, the minimal rank of potentially existing counterexamples may be of interest. Suppose it is some (small) natural number

n, then in indirect proofs of sufficient modularity criteria via minimal counterexamples one could exploit this knowledge by deriving more concrete knowledge, e.g. about the shape of derivations in minimal counterexamples, thus possibly leading to a substantial simplification of the overall proof.

For all non-modular termination properties mentioned above the rank of counterexamples must be at least 3. Here, a counterexample is just an infinite derivation $s_1 \to s_2 \to \ldots$ in the disjoint union, with its rank being defined as $min\{rank(s_i) \mid 1 \leq i\}$. Terms of rank 1 are trivially terminating. Terms of rank 2 are also terminating, by an easy abstraction argument. For terms of rank 3 the collapsing behaviour may be more complex, but is by far not as complex as for terms with rank > 3.

It seems worth to note that almost all counterexamples (to modularity of termination) in the literature (cf. e.g. [17, 3, 13]) have rank 3. But this need not always be the case, cf. [5]. A *family of counterexamples* is just a family $(\mathcal{R}_1^n, \mathcal{R}_2^n)_{n \in \mathbb{N}}$ of parameterized pairs of disjoint TRSs. In the disjoint union of two terminating TRSs a *minimal counterexample* (to the modularity of termination) is an infinite derivation $D\colon s_1 \to s_2 \to \ldots$ in the union such that $rank(D)$ is minimal among all non-terminating derivations.

Theorem 3 ([5])**.** *Minimal counterexamples to modularity of termination may have arbitrarily high rank.*

Proof. In Example 4 below we give a family of counterexamples to modularity of termination such that for every $k \in \mathbb{N}$ there exists a member of the family, i.e., a pair of disjoint terminating TRSs whose union is non-terminating and whose minimal rank of corresponding counterexamples is at least k. □

Example 4 (arbitrarily high rank of minimal counterexamples, [5])**.**

$$\mathcal{R}_1^n = \{\ f(x, g(x), \ldots, g^n(x), y) \to f(y, \ldots, y)\ \} \qquad \mathcal{R}_2^n = \left\{ \begin{array}{l} H(x) \to x \\ H(x) \to A \end{array} \right\}$$

Note that $\mathcal{R}_2^n$ is fixed and $\mathcal{R}_1^n$ is parameterized by $n \geq 1$.

- *Non-termination witness of minimal rank* $2n{+}2$*:* $s = f(\phi^n(A), \ldots, \phi^n(A))$ *where* $\phi(x) = H(g(x))$
- *Non-terminating reduction:*

$$s = f((Hg)^n(A), (Hg)^n(A), \ldots, (Hg)^n(A)) \to^+ f(A, g(A), \ldots, g^n(A)) \to s$$

- *Minimality:* t *needed with* $t \to^* u$, $t \to^* g(u)$, $\ldots t \to^* g^n(u)$ *implies* $t = H^+(g(H^+, g(\ldots g(H^+(u))\ldots)))$

In the above Example 4 the family of disjoint combinations has an arbitrarily high minimal rank of counterexamples to termination. But note that one of the systems is always non-left-linear. This means that in positive modularity criteria for termination that do not exclude non-left-linear TRSs, no assumption can be made in the proofs about a bound on the minimal rank of potentially existing counterexamples. But how about modularity criteria for termination of (only) left-linear systems? Since such (known) modularity proofs are often rather or sometimes extremely complex [19, 20, 16, 8], it would be nice if the minimal rank of counterexamples could be limited (to always 3). Thus the proofs of these results could be substantially simplified. But it turns out that this assumption is also not true in general as we will show now.

Theorem 5. *Minimal counterexamples to modularity of termination of left-linear TRSs may have arbitrarily high rank.*

Proof. In Example 6 we give a family of counterexamples to modularity of termination of left-linear TRSs such that for every $k \in \mathbb{N}$ there exists a member of the family, i.e., a pair of disjoint terminating left-linear TRSs whose disjoint union is non-terminating and whose minimal rank of corresponding counterexamples is at least k. $\square$

Example 6 (SN $\wedge$ LL is not modular, rank of counterexamples arbitrarily high)**.**

$$\mathcal{R}_1^n = \left\{ \begin{array}{rcl} f_1(g(x), a, y) & \to & f_2(x, x, y) \\ & \vdots & \\ f_{n-1}(g(x), a, y) & \to & f_n(x, x, y) \\ f_n(g(x), a, y) & \to & f_1(y, y, y) \end{array} \right\} \quad \mathcal{R}_2^n = \left\{ \begin{array}{l} H(x, y) \to x \\ H(x, y) \to y \end{array} \right\}$$

Note again that $\mathcal{R}_2^n$ is fixed and $\mathcal{R}_1^n$ is parameterized by $n \geq 1$. We observe that this example is based on a variant of Toyama's Counterexample 1, where the difference in the first argument of f ($g(x)$ instead of the constant a) is exploited to iterate the basic construction more and more often which enforces the initial argument to be built via alternating sequences of H and g, combined with a. Furthermore note that for $n = 1$ we get the variant of Toyama's Counterexample 1: $\mathcal{R}_1 = \{f_1(g(x), a, y) \to f_1(y, y, y)\}$, $\mathcal{R}_2 = \{H(x, y) \to x, H(x, y) \to y\}$.

- *Non-termination witness of minimal rank $2n + 1$: $s = f_1(\phi^n(a), \phi^n(a), \phi^n(a))$ where $\phi(x) = H(g(x), a)$.*
- *Non-terminating reduction:*

$$\begin{array}{rcl} s & = & f_1(\phi^n(a), \phi^n(a), \phi^n(a)) \\ & = & f_1(H(g(\phi^{n-1}(a)), a), H(g(\phi^{n-1}(a)), a), \phi^n(a)) \\ & \to^+ & f_1(g(\phi^{n-1}(a)), a, \phi^n(a)) \\ & \to & f_2(\phi^{n-1}(a), \phi^{n-1}(a), \phi^n(a))) \\ & \to^+ & f_2(g(\phi^{n-2}(a)), a, \phi^n(a)) \\ & \vdots & \vdots \\ & \to^+ & f_n(g(\phi^0(a)), a, \phi^n(a)) \\ & = & f_n(g(a), a, \phi^n(a)) \\ & \to & f_1(\phi^n(a), \phi^n(a), \phi^n(a)) = s \end{array}$$

- *Minimality: $s = f_1(t, t, t)$ needed with $t \to^* g(t_1)$, $t \to^* a$, $t_1 \to^* g(t_2)$, $t_1 \to^* a$, $\ldots$, $t_{n-1} \to^* g(t_n)$, $t_{n-1} \to^* a$. This implies $rank(s) \geq 2n + 1$.*

Note that for every family member in Example 6 the second of the systems is not confluent, as was the case for Example 4. We will now show that we can also get an arbitrarily high rank of minimal counterexamples to modularity of termination of confluent systems, by using the technique of Example 2 and applying it to a modified version of Example 6.

Theorem 7. *Minimal counterexamples to modularity of termination of confluent TRSs may have an arbitrarily high rank.*

Proof. See Example 8. $\square$

Example 8 ((SN $\wedge$ CR) is not modular, with arbitrarily high rank of counterexamples)**.**

$$\mathcal{R}_1^n = \left\{ \begin{array}{rcl} f_1(g(x),a,y) & \to & f_2(x,x,y) \\ & \vdots & \\ f_{n-1}(g(x),a,y) & \to & f_n(x,x,y) \\ f_n(g(x),a,y) & \to & f_1(y,y,y) \\ g(x) & \to & b \\ a & \to & b \\ f_i(x,y,z) & \to & c \ \ (\textit{ for all } 1 \leq i \leq n) \end{array} \right\} \qquad \mathcal{R}_2^n = \left\{ \begin{array}{l} K(x,y,y) \to x \\ K(y,x,y) \to x \end{array} \right\}$$

Note again that $\mathcal{R}_2^n$ is fixed and $\mathcal{R}_1^n$ is parameterized by $n \geq 1$. Furthermore observe that the purpose of the last rule schema of $\mathcal{R}_1^n$ is to make the system confluent. The two preceding rules together with the confluent $\mathcal{R}_2^n$ enable to extract from terms of shape $K(g(s),a,b)$ both $g(s)$ as well as a, via $K(g(s),a,b) \to K(g(s),b,b) \to g(s)$ and $K(g(s),a,b) \to K(b,a,b) \to a$. This is all what we need to get a minimal counterexample of rank $2n+1$ as in Example 8.

- *Non-termination witness of minimal rank $2n+1$: $s = f_1(\phi^n(a), \phi^n(a), \phi^n(a))$ where $\phi(x) = K(g(x),a,b)$*
- *Non-terminating reduction:*

$$\begin{array}{rcl} s & = & f_1(\phi^n(a), \phi^n(a), \phi^n(a)) \\ & = & f_1(K(g(\phi^{n-1}(a)),a,b), K(g(\phi^{n-1}(a)),a,b), \phi^n(a)) \\ & \to^+ & f_1(g(\phi^{n-1}(a)), a, \phi^n(a)) \\ & \to & f_2(\phi^{n-1}(a), \phi^{n-1}(a), \phi^n(a))) \\ & \to^+ & f_2(g(\phi^{n-2}(a)), a, \phi^n(a)) \\ & \vdots & \vdots \\ & \to^+ & f_n(g(\phi^0(a)), a, \phi^n(a)) \\ & = & f_n(g(a), a, \phi^n(a)) \\ & \to & f_1(\phi^n(a), \phi^n(a), \phi^n(a)) = s \end{array}$$

- *Minimality: $s = f_1(t,t,t)$ needed with $t \to^* g(t_1)$, $t \to^* a$, $t_1 \to^* g(t_2)$, $t_1 \to a$, ..., $t_{n-1} \to^* g(t_n)$, $t_{n-1} \to^* a$. This implies rank$(s) \geq 2n+1$.*

Remark 9. *Note that taking Example 4 and making it confluent analogously to Example 2 does not work, in the sense that then the minimal rank of counterexamples becomes 3.*

Finally, we show that proving non-modularity of $\mathrm{UN}^{\to}$ (or $\mathrm{CONS}^{\to}$, respectively) for disjoint TRSs enjoying $\mathrm{UN}^{\to}$ (or $\mathrm{CONS}^{\to}$, respectively) may be arbitrarily complex, in terms of the minimal rank of counterexamples. Here, the rank of a $\mathrm{UN}^{\to}$-counterexample $D\colon x \leftarrow^* s \to^* y$ is the minimal rank of the terms in the derivation D. The rank of a $\mathrm{CONS}^{\to}$-counterexample is defined analogously.

Example 10 (falsifying $\mathrm{UN}^{\to}$ (or $\mathrm{CONS}^{\to}$), in disjoint unions of TRSs satisfying $\mathrm{UN}^{\to}$ (or $\mathrm{CONS}^{\to}$) may be arbitrarily difficult)**.** *Consider*

$$\mathcal{R}_1^n = \left\{ \begin{array}{rcl} f_1(g(x),a,y,u,v) & \rightarrow & f_2(x,x,y,u,v) \\ & \vdots & \\ f_{n-1}(g(x),a,y,u,v) & \rightarrow & f_n(x,x,y,u,v) \\ f_n(g(x),a,y,u,v) & \rightarrow & f_1(y,y,y,v,u) \\ f_1(g(x),a,y,u,v) & \rightarrow & u \\ g(x) & \rightarrow & b \\ a & \rightarrow & b \\ f_i(x,y,z,u,v) & \rightarrow & c \text{ (for all } 1 \leq i \leq n) \\ c & \rightarrow & c \end{array} \right\} \quad \mathcal{R}_2^n = \left\{ \begin{array}{l} K(x,y,y) \rightarrow x \\ K(y,x,y) \rightarrow x \end{array} \right\}$$

- *$\mathcal{R}_1^n$, $\mathcal{R}_2^n$ are $UN^{\rightarrow}$ and hence $CONS^{\rightarrow}$.*
- *In $\mathcal{R}_1^n$ for instance: $f_1(g(x),a,y,u,v) \rightarrow u$ and $f_1(g(x),a,y,u,v) \rightarrow f_2(x,x,y,u,v) \rightarrow^* c \notin NF$*
- *In the disjoint union, with $\phi(x) = K(g(x),a,b)$:*

$$\begin{array}{rcl} s & = & f_1(\phi^n(a),\phi^n(a),\phi^n(a),u,v) \\ & = & f_1(K(g(\phi^{n-1}(a)),a,b),K(g(\phi^{n-1}(a)),a,b),\phi^n(a),u,v) \\ & \rightarrow^+ & f_1(g(\phi^{n-1}(a)),a,\phi^n(a),u,v) = t \rightarrow u \end{array}$$

$$\begin{array}{rcl} t & = & f_1(g(\phi^{n-1}(a)),a,\phi^n(a),u,v) \\ & \rightarrow & f_2(\phi^{n-1}(a),\phi^{n-1}(a),\phi^n(a)),u,v) \\ & \rightarrow^+ & f_2(g(\phi^{n-2}(a)),a,\phi^n(a),u,v) \\ & \vdots & \vdots \\ & \rightarrow^+ & f_n(g(\phi^0(a)),a,\phi^n(a),u,v) \\ & = & f_n(g(a),a,\phi^n(a),u,v) \\ & \rightarrow & f_1(\phi^n(a),\phi^n(a),\phi^n(a),v,u) \\ & \rightarrow^+ & f_1(g(\phi^{n-1}(a)),a,\phi^n(a),v,u) \rightarrow v \end{array}$$

- *Hence: $s \rightarrow^* u$, $s \rightarrow^* v$. If u, v are distinct variables, it follows: $\neg UN^{\rightarrow}$, $\neg CONS^{\rightarrow}$.*
- *Minimality: Analogous to before.*

References

[1] Franz Baader and Tobias Nipkow. *Term rewriting and All That*. Cambridge University Press, 1998.

[2] Marc Bezem, Jan Willem Klop, and Roel de Vrijer, editors. *Term Rewriting Systems*. Cambridge Tracts in Theoretical Computer Science 55. Cambridge University Press, March 2003.

[3] Klaus Drosten. *Termersetzungssysteme*. Informatik-Fachberichte 210. Springer-Verlag, 1989. In German.

[4] Bernhard Gramlich. Relating innermost, weak, uniform and modular termination of term rewriting systems. In A. Voronkov, editor, *International Conference on Logic Programming and Automated Reasoning, St. Petersburg*, volume 624 of *Lecture Notes in Artificial Intelligence*, pages 285–296. Springer-Verlag, 1992.

[5] Bernhard Gramlich. Generalized sufficient conditions for modular termination of rewriting. *Applicable Algebra in Engineering, Communication and Computing*, 5:131–158, 1994.

[6] Bernhard Gramlich. Abstract relations between restricted termination and confluence properties of rewrite systems. *Fundamenta Informaticae*, 24:3–23, 1995.

[7] Masahito Kurihara and Azuma Ohuchi. Modularity of simple termination of term rewriting systems. *Journal of IPS, Japan*, 34:632–642, 1990.

[8] Massimo Marchiori. Modularity of completeness revisited. In Jieh Hsiang, editor, *Proc. 6th Int. Conf. on Rewriting Techniques and Applications (RTA'95)*, volume 914 of *Lecture Notes in Computer Science*, pages 2–10, Kaiserslautern, Germany, April 1995. Springer-Verlag.

[9] Massimo Marchiori. On the modularity of normal forms in rewriting. *Journal of Symbolic Computation*, 22(2):143–154, 1996.

[10] Aart Middeldorp. Modular aspects of properties of term rewriting systems related to normal forms. In N. Dershowitz, editor, *Proc. 3rd Int. Conf. on Rewriting Techniques and Applications*, volume 355 of *Lecture Notes in Computer Science*, pages 263–277. Springer-Verlag, 1989.

[11] Aart Middeldorp. *Modular Properties of Term Rewriting Systems*. PhD thesis, Free University, Amsterdam, 1990.

[12] Enno Ohlebusch. On the modularity of termination of term rewriting systems. *Theoretical Computer Science*, 136:333–360, 1994.

[13] Enno Ohlebusch. Termination is not modular for confluent variable-preserving term rewriting systems. *Information Processing Letters*, 53:223–228, March 1995.

[14] Michaël Rusinowitch. On termination of the direct sum of term rewriting systems. *Information Processing Letters*, 26:65–70, 1987.

[15] Manfred Schmidt-Schauß. Unification in a combination of arbitrary disjoint equational theories. *Journal of Symbolic Computation*, 8(1):51–99, 1989.

[16] Manfred Schmidt-Schauß, Massimo Marchiori, and Sven Eric Panitz. Modular termination of r-consistent and left-linear term rewriting systems. *Theoretical Computer Science*, 149(2):361–374, October 1995.

[17] Yoshihito Toyama. Counterexamples to termination for the direct sum of term rewriting systems. *Information Processing Letters*, 25:141–143, 1987.

[18] Yoshihito Toyama. On the Church-Rosser property for the direct sum of term rewriting systems. *Journal of the ACM*, 34(1):128–143, 1987.

[19] Yoshihito Toyama, Jan Willem Klop, and Henk Pieter Barendregt. Termination for the direct sum of left-linear term rewriting systems. In N. Dershowitz, editor, *Proc. 3rd Int. Conf. on Rewriting Techniques and Applications (RTA'89)*, volume 355 of *Lecture Notes in Computer Science*, pages 477–491. Springer-Verlag, 1989.

[20] Yoshihito Toyama, Jan Willem Klop, and Henk Pieter Barendregt. Termination for direct sums of left-linear complete term rewriting systems. *Journal of the ACM*, 42(6):1275–1304, November 1995.