On Termination and Confluence Properties of Disjoint and Constructor-Sharing Conditional Rewrite Systems ⋆

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Abstract

We investigate the modularity behaviour of termination and confluence properties of (join) conditional term rewriting systems. We give counterexamples showing that the properties weak termination, weak innermost termination and (strong) innermost termination are not preserved in general under signature extensions. Then we develop sufficient conditions for the preservation of these properties under signature extensions, and more generally, for their modularity in the disjoint union case. This leads to new criteria for modularity of termination and completeness generalizing known results for unconditional systems. Finally, combining our analysis with recent related results on the preservation of semi-completeness, we show how to cover the (non-disjoint) case of combined conditional rewrite systems with shared constructors, too.

1 Introduction, Motivation and Overview

Starting with the seminal work of Toyama [31] the investigation of preservation properties of term rewriting systems (TRSs for short) under various forms of combinations has become a very interesting and active area of research. From a practical point of view this field has a great potential in applications

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of rewriting techniques since it provides the theoretical basis for a systematic construction of large systems of rewrite rules with some desired properties from smaller ones with corresponding properties. Vice versa, it is also crucial for analyzing properties of large systems by decomposing them into smaller units where corresponding properties are often easier to verify. From a theoretical point of view, the problems to be dealt with have turned out to be very fruitful, non-trivial and sometimes even challenging. Even for the simplest conceivable case of combination mechanisms, namely that of disjoint unions, the analysis of rewriting in the combined system is quite involved.

In [31] it was shown that confluence is indeed preserved under disjoint unions of (unconditional) TRSs whereas termination is not (cf. also [32]). This phenomenon was the starting point of a couple of investigations about how to obtain sufficient criteria for the preservation of termination, completeness (i.e., termination plus confluence) and of other interesting properties of TRSs under disjoint combinations (cf. e.g. [30], [19], [33], [20], [14], [7], [8], [25]). Non-disjoint unions of TRSs with common constructors have been considered e.g. in [23], [15], [7], [8], [25], [27]. More general hierarchical combinations of TRSs have recently been dealt with in [17], [2], [18], [8], [6]. Some preservation results for (disjoint and non-disjoint, but non-hierarchical) combinations of conditional TRSs (CTRSs for short) finally have been obtained in [20], [22]), [21], [9], [28].

As shown in [33] (by a very involved proof), completeness is preserved under disjoint union of left-linear (unconditional) TRSs. Instead of left-linearity one may also require a stronger confluence property. More precisely, termination (and hence also completeness) is preserved under disjoint unions of non-overlapping TRSs as well as of locally confluent overlay systems ([8]). The crucial point is that for such TRSs (strong) innermost termination implies already (strong) termination. Recently we have been able to show that this latter property does indeed also hold for CTRSs ([10]). In the present paper we shall exploit this property and show how to obtain corresponding preservation results for disjoint and constructor-sharing unions of CTRSs. However, this generalization of [8] to the conditional case turned out to be pretty more complicated than expected, due to reasons which will be explained later on.

Before going into details let us give a summary of our main results (for join CTRSs admitting extra variables in the conditions):

- We give counterexamples showing that weak termination, weak innermost termination and (strong) innermost termination are not preserved under signature extensions in general (cf. Example 21 which contradicts Theorem 5.2 in [22]) whereas (strong) termination is indeed preserved under signature extensions (cf. Lemma 29).
- We give abstract sufficient conditions for the preservation of weak termina-

tion, weak innermost termination and (strong) innermost termination under signature extensions and more generally, for their modularity (cf. Theorem 31, Lemmas 32-39). In particular we show that these restricted termination properties are modular for confluent CTRSs and for CTRSs without extra variables (cf. Theorem 40).

- We show that (strong) termination and completeness are modular properties for non-overlapping CTRSs as well as for conditional overlay systems with joinable critical pairs (cf. Theorems 45, 46).
- We show how to extend these modularity results to the (non-disjoint) case of combined CTRSs with shared constructors by combining our analysis with recent results of [27] (cf. Theorems 62, 66–69).

The rest of the paper is structured as follows. In the next section we shall introduce some necessary terminology. Then, in Section 3, we shall summarize results of [8], [10] on restricted and general termination and confluence properties of (C)TRSs. In the main Section 4, we shall develop a couple of new preservation results for termination and confluence properties of CTRSs in the disjoint union case. In Section 5 we show how to extend these results to combined CTRSs with shared constructors. Finally we summarize known and new results and discuss some open problems and related work.

2 Preliminaries

We assume familiarity with the basic theory for term rewriting (cf. e.g. [3], [12]). For brevity we shall make use of the following abbreviations (cf. e.g. [8], [12]) which apply to TRSs (and, if sensible, also to terms; in this case we shall also use notations like WN$(s, \mathcal{R})$ in order to indicate the respective rewrite system $\mathcal{R}$): SN (strongly) terminating (strongly normalizing), WN weakly terminating (weakly normalizing), WIN weakly innermost terminating (weakly innermost normalizing), SIN (strongly) innermost terminating (strongly innermost normalizing), JCP (all) critical pairs are joinable, CR confluent (or equivalently, having the Church-Rosser property), WCR locally confluent, NE non-erasing (or variable-preserving), NO non-overlapping, OS overlay system, NF set of normal forms. Completeness (COMP) means termination plus confluence, semi-completeness stands for weak termination plus confluence. *Innermost* reduction steps $s \to t$ are denoted by $s \underset{i}{\to} t$. We recall that $s \underset{i}{\to} t$ means that s reduces (rewrites) to t by contracting some redex in s all proper subterms of which are irreducible. In order to indicate the position p of the contracted redex in $s \to t$, the applied rule $l \to r$ and the matching substitution σ we shall also use the notation $s \to_{p,\sigma,l\to r} t$. If $p = \lambda$ (the root position) then we speak of a *root reduction* step. The *domain* of a substitution σ is given by $dom(\sigma) = \{x \in \mathcal{V} \mid \sigma(x) \neq x\}$. The identity substitution is denoted by id.

2.1 Basic Notions and Notations for Disjoint Unions

We adopt usual basic notions and notations for disjoint unions of rewrite systems as described e.g. in [31], [20], [22], but repeat some conventions here for the sake of readability. When considering a disjoint union $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$ we say that $\mathcal{R}_1^{\mathcal{F}_1}$ ($\mathcal{R}_2^{\mathcal{F}_2}$) is the *black* (*white*) system. Symbols from $\mathcal{F}_1$ ($\mathcal{F}_2$) are *black* (*white*). Variables have no colour. A *black* (*white*) term does not contain white (black) symbols. A term is *top black* (*top white*) if its top symbol is black (white). The *special subterms* (or *aliens*), *principal subterms* and the *rank* of terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$ are defined as usual. If in a reduction step $s \to_{\mathcal{R}^{\mathcal{F}}} t$ the contracted redex is inside some principal alien of s, we speak of an *inner* step and write $s \xrightarrow{i}_{\mathcal{R}^{\mathcal{F}}} t$. Otherwise it is an *outer* step, which is denoted by $s \xrightarrow{o}_{\mathcal{R}^{\mathcal{F}}} t$. If s is a term with $rank(s) > 1$ then it has the form $s = C[\![s_1,\ldots,s_n]\!]$ where the s_i, $1 \le i \le n$ are the principal aliens of s and $C[,\ldots,]$ is a *context*, i.e. a term with 'holes' (represented by some fresh constant $\Box$). By $C[\,]_p$ we mean a context with one occurrence of $\Box$ at position p. Similarly (but slightly abusing notation), if Π is some set of mutually disjoint positions of $C[,\ldots,]$ then $C[\,]_\Pi$ denotes a context with $\Box$ at all positions from Π. A *non-empty* context is a context different from $\Box$. For terms $s_1,\ldots,s_n,t_1,\ldots,t_n$ we write $\langle s_1,\ldots,s_n\rangle \propto \langle t_1,\ldots,t_n\rangle$ if $t_i = t_j$ whenever $s_i = s_j$, for all $1 \le i,j \le n$. If both $\langle s_1,\ldots,s_n\rangle \propto \langle t_1,\ldots,t_n\rangle$ and $\langle t_1,\ldots,t_n\rangle \propto \langle s_1,\ldots,s_n\rangle$ hold, this is denoted by $\langle s_1,\ldots,s_n\rangle \infty \langle t_1,\ldots,t_n\rangle$. This notion as well as those of being black (white), top black (top white) extend in a straightforward way to substitutions. Following [20], [22] we introduce some special notations in order to enable a compact treatment of 'degenerate' cases of '$t = C[\![t_1,\ldots,t_n]\!]$'. First the notion of context is extended. We write $C\langle,\ldots,\rangle$ for a term containing zero or more occurrences of $\Box$ and $C\{,\ldots,\}$ denotes a term different from $\Box$ itself, containing zero or more occurrences of $\Box$. If $t_1,\ldots,t_n$ are the (possibly zero) principal subterms of some term t (from left to right), then we write $t = C\{\!\{t_1,\ldots,t_n\}\!\}$ provided $t = C\{t_1,\ldots,t_n\}$. We write $t = C\langle\!\langle t_1,\ldots,t_n\rangle\!\rangle$ if $t = C\langle t_1,\ldots,t_n\rangle$ and either $C\langle,\ldots,\rangle \ne \Box$ and $t_1,\ldots,t_n$ are the principal subterms of t, or $C\langle,\ldots,\rangle = \Box$ and $t \in \{t_1,\ldots,t_n\}$.

2.2 Conditional Term Rewriting Systems

Moreover, we need some basic terminology about conditional term rewriting systems (CTRSs) (cf. e.g. [3], [12], [22]).

Definition 1 A *CTRS* is a pair $(\mathcal{F},\mathcal{R})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R}$ of *conditional rewrite rules* of the form

$$s_1 = t_1 \wedge \ldots \wedge s_n = t_n \implies l \to r$$

with $s_1, \ldots, s_n, t_1, \ldots, t_n, l, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Moreover, we require $l \notin \mathcal{V}$ and $V(r) \subseteq V(l)$ as for unconditional TRSs, i.e., no variable left hand sides and no extra variables on the right hand side. Extra variables in conditions are allowed if not stated otherwise. If the condition is empty, i.e., $n = 0$, we simply write $l \to r$. Instead of $(\mathcal{F}, \mathcal{R})$ we also write $\mathcal{R}^{\mathcal{F}}$ or simply $\mathcal{R}$ when $\mathcal{F}$ is clear from the context or irrelevant. If $\mathcal{R}$ has no extra variables (in the conditions) this is denoted by NEV($\mathcal{R}$).

Depending on the interpretation of the equality sign in the conditions of rewrite rules, different reduction relations may be associated with a given CTRS as usual (yielding a *join*, *normal* or *semi-equational* CTRS).

Definition 2

- In a *join CTRS* $\mathcal{R}$ the equality sign in the conditions of rewrite rules is interpreted as joinability. Formally this means: $s \to_{\mathcal{R}} t$ if there exists a rewrite rule $s_1 = t_1 \wedge \ldots \wedge s_n = t_n \implies l \to r \in \mathcal{R}$, a substitution σ and a context $C[\,]$ such that $s = C[\sigma l]$, $t = C[\sigma r]$ and $\sigma s_i \downarrow_{\mathcal{R}} \sigma t_i$ for all $i \in \{1, \ldots, n\}$. For rewrite rules of a join CTRS we shall also use the notation $s_1 \downarrow t_1 \wedge \ldots \wedge s_n \downarrow t_n \implies l \to r$.
- *Semi-equational* CTRSs are obtained by interpreting the equality sign in the conditions as convertibility, i.e., as $\leftrightarrow^*$.
- *Normal* CTRSs have rules of the form $s_1 \to^* t_1 \wedge \ldots \wedge s_n \to^* t_n \implies l \to r$ such that all t_i's are ground terms which are irreducible w.r.t. the unconditional version of the CTRS $\mathcal{R}$ considered (which is obtained from $\mathcal{R}$ by removing all conditions).

Definition 3 The reduction relation corresponding to a given (join, semi-equational or normal) CTRS $\mathcal{R}$ can also be inductively defined as follows ($\Box$ denotes $\downarrow$, $\leftrightarrow^*$ or $\to^*$, respectively):

$$
\begin{aligned}
\mathcal{R}_0 &= \emptyset \,, \\
\mathcal{R}_{i+1} &= \{\sigma l \to \sigma r \mid s_1 \Box t_1 \wedge \ldots \wedge s_n \Box t_n \implies l \to r \in \mathcal{R}, \\
&\qquad \sigma s_j \Box_{\mathcal{R}_i} \sigma t_j \text{ for } j = 1, \ldots, n\} \,, \\
s \to_{\mathcal{R}} t :\iff & \; s \to_{\mathcal{R}_i} t \text{ for some } i \geq 0, \text{ i.e., } \to_{\mathcal{R}} = \bigcup_{i \geq 0} \to_{\mathcal{R}_i} .
\end{aligned}
$$

Note in particular that all unconditional rules of $\mathcal{R}$ are contained in $\mathcal{R}_1$ (because the empty conditions are vacuously satisfied) as well as all conditional rules with trivial conditions only, i.e., conditions of the form $s \Box s$. In fact, rules of the latter class can be considered to be essentially unconditional.

Definition 4 If $s \to_{\mathcal{R}} t$ then the *depth* of $s \to_{\mathcal{R}} t$ is defined to be the minimal n with $s \to_{\mathcal{R}_n} t$. If $s \to^*_{\mathcal{R}} t$ then the *depth* of $s \to^*_{\mathcal{R}} t$ is defined to be the minimal

n with $s \rightarrow^*_{\mathcal{R}_n} t$. If $s \downarrow_{\mathcal{R}} t$, then its *depth* is the minimal n with $s \downarrow_{\mathcal{R}_n} t$. If the depth of $s \rightarrow^*_{\mathcal{R}} t$ is less than or equal to n we denote this by $s \xrightarrow{n}{}^{*}_{\mathcal{R}} t$.

In this paper we shall mainly deal with join CTRSs which – from a practical and operational point of view – is the most important type of CTRSs. Hence, if not stated otherwise, we shall tacitly assume in the sequel that all considered CTRSs are join ones.

For the sake of readability we shall use in the following some compact notations for conditional rules and conjunctions of conditions. When writing $P \implies l \rightarrow r$ for some conditional rewrite rule then P stands for the conjunction of all conditions. Similarly, if P is $s_1 \downarrow t_1 \wedge \ldots \wedge s_n \downarrow t_n$ and σ is a substitution, then $\sigma(P) \downarrow$ means $\sigma(s_1) \downarrow_{\mathcal{R}} \sigma(t_1) \wedge \ldots \wedge \sigma(s_n) \downarrow_{\mathcal{R}} \sigma(t_n)$.

Definition 5 Let $\mathcal{R}$ be a CTRS, and let $P_1 \implies l_1 \rightarrow r_1$ and $P_2 \implies l_2 \rightarrow r_2$ be two rewrite rules of $\mathcal{R}$ which have no variables in common (of course, this variable disjointness can always be achieved by appropriately renaming rules). Suppose $l_1 = C[t]_p$ with $t \notin \mathcal{V}$ for some (possibly empty) context $C[\,]_p$ such that t and l_2 are unifiable with most general unifier σ. Then $\sigma(P_1) \wedge \sigma(P_2) \implies \langle \sigma(C[r_2]) = \sigma(r_1) \rangle$ is said to be a *(conditional) critical pair* of $\mathcal{R}$. If the two rules are renamed versions of the same rule of $\mathcal{R}$, we do not consider the case $C[\,] = \Box$, i.e., we do not overlap a rule with itself at root position (since this gives only rise to improper (conditional) divergences). A (conditional) critical pair $P \implies \langle s = t \rangle$ is said to be *joinable* if $\sigma(s) \downarrow_{\mathcal{R}} \sigma(t)$ for every substitution σ with $\sigma(P) \downarrow$. A substitution σ which satisfies the conditions, i.e., for which $\sigma(P) \downarrow$ holds, is said to be *feasible*. Otherwise, σ is *infeasible*. Analogously, a (conditional) critical pair is said to be *feasible (infeasible)* if there exists some (no) feasible substitution for it.

Note that testing joinability of conditional critical pairs is in general much more difficult than in the unconditional case since one has to consider all substitutions which satisfy the correspondingly instantiated conditions. Moreover, the critical pair lemma does not hold for CTRSs in general as shown e.g. by the following example.

Example 6 *([1]) Consider the CTRS*

$$\mathcal{R}_1 = \begin{cases} x \downarrow f(x) \implies f(x) \rightarrow a \\ \qquad\qquad\qquad\quad b \rightarrow f(b) \end{cases}$$

Here we get $f(b) \rightarrow a$ due to $b \downarrow f(b)$ and hence $f(f(b)) \rightarrow f(a)$. We also have $f(f(b)) \rightarrow a$ because of $f(b) \downarrow f(f(b))$. But a and $f(a)$ do not have a common reduct which is easily shown. Thus $\mathcal{R}$ is not locally confluent despite the lack of critical pairs. Note moreover that $\mathcal{R}$ is even orthogonal when considered as *unconditional TRS, i.e., when omitting the condition in the first rule.*

Definition 7 (cf. [1], [12]) Let $\mathcal{R}$ be a CTRS and let $\mathcal{R}_u$ be its unconditional version, i.e., $\mathcal{R}_u := \{l \to r \mid P \Longrightarrow l \to r \in \mathcal{R}\}$. Then $\mathcal{R}$ is said to be *non-overlapping (*NO*) / orthogonal /* a *(conditional) overlay system (*OS*)* if $\mathcal{R}_u$ is non-overlapping / orthogonal / an (unconditional) overlay system.

According to this definition Example 6 above shows that orthogonal CTRSs need not be confluent. But note that the CTRS $\mathcal{R}$ defined in Example 6 is not (strongly) innermost terminating. This indicates that their might be some hope for generalizing (some of) our results for the unconditional case to the conditional one, in particular those involving the (strong) innermost termination property.

Remark 8 The careful reader may have observed that the definition of being non-overlapping above is somehow rather restrictive. Namely, the case that there exist conditional critical pairs all of which are infeasible (and hence should not be 'properly critical') is not covered. Analogously, for a CTRS to be an overlay system one might allow critical pairs which are not overlays but require that all of them are infeasible, hence not 'properly critical'). These slightly generalized *semantic* versions of the properties of being non-overlapping and being an overlay system are treated in Section 4.4.

The other basic notions for unconditional TRSs introduced above generalize in a straightforward manner to CTRSs.

In general, conditional rewriting is much more complicated than unconditional rewriting. For instance, the rewrite relation may be undecidable even for complete CTRSs without extra variables in the conditions (cf. [11]). For some positive results see e.g. [4].

3 Restricted Termination and Confluence Properties of Conditional Term Rewriting Systems

Here we shall summarize some known and recently obtained new results on restricted termination and confluence properties of unconditional and conditional TRSs.

Theorem 9 *(cf. e.g. [29], [24], [12]) Let $\mathcal{R}$ be an orthogonal, i.e., non-overlapping and left-linear (unconditional) TRS. Then $\mathcal{R}$ is confluent, and the following properties hold:*

(1a) $\forall t : [\,\mathrm{WIN}(t) \Longrightarrow \mathrm{SIN}(t)\,]$.
(1b) $\mathrm{WIN}(\mathcal{R}) \Longrightarrow \mathrm{SIN}(\mathcal{R})$.

(2a) $\forall t : [\,\mathrm{SIN}(t) \implies \mathrm{SN}(t)\,]$.
(2b) $\mathrm{SIN}(\mathcal{R}) \implies \mathrm{SN}(\mathcal{R})$.
(3) $\forall s, t : [\,s \underset{i}{\rightarrow} t \wedge \mathrm{SN}(t) \implies \mathrm{SN}(s)\,]$.
(4a) $\mathrm{NE}(\mathcal{R}) \implies [\forall t : [\,\mathrm{WN}(t) \implies \mathrm{SN}(t)\,]]$.
(4b) $\mathrm{NE}(\mathcal{R}) \implies [\,\mathrm{WN}(\mathcal{R}) \implies \mathrm{SN}(\mathcal{R})\,]$.

For CTRSs the following technical result, which is a stronger local version of a result from [5], is crucial in order to obtain analogous relations between restricted and general termination and confluence properties of CTRSs.

Theorem 10 *([10]) Let $\mathcal{R}$ be a CTRS with* $\mathrm{OS}(\mathcal{R})$ *and* $\mathrm{JCP}(\mathcal{R})$ *and let s be a term with* $\mathrm{SN}(s)$. *Furthermore let $C[\,]_\Pi$ be a context, and t, u, v be terms. Then we have:*

$$u = C[s]_\Pi \rightarrow^* v \wedge s \rightarrow^* t \implies C[t]_\Pi \downarrow v\,.$$

Straightforward consequences of (the proof of) Theorem 10 are the following.

Lemma 11 *([10]) Let $\mathcal{R}$ be a CTRS with* $\mathrm{OS}(\mathcal{R})$ *and* $\mathrm{JCP}(\mathcal{R})$, *and let s, t be terms with $s \rightarrow_{p,\sigma,P \Longrightarrow l \rightarrow r} t$. Furthermore let σ' be given with $\sigma \rightarrow^* \sigma'$, i.e., $\sigma(x) \rightarrow^* \sigma'(x)$ for all $x \in dom(\sigma)$, such that* $\mathrm{SN}(\sigma(x))$ *holds for all $x \in dom(\sigma)$. Then we have: $s = C[\sigma(l)]_p \rightarrow^* C[\sigma'(l)]_p \rightarrow_{p,\sigma',P \Longrightarrow l \rightarrow r} C[\sigma'(r)]_p$ (due to $\sigma'(P)\downarrow$) and $t = C[\sigma(r)]_p \rightarrow^* C[\sigma'(r)]_p$ for some context $C[\,]_p$.*

Corollary 12 *([10]) Let $\mathcal{R}$ be a CTRS with* $\mathrm{OS}(\mathcal{R})$ *and* $\mathrm{JCP}(\mathcal{R})$, *and let s be a term with* $\mathrm{SN}(s)$. *Then we have* $\mathrm{CR}(s)$ *and hence* $\mathrm{COMP}(s)$, *too.*

The termination assumption concerning s in this result is crucial as demonstrated by the following example.

Example 13 *(Example 6 continued) Here*

$$\mathcal{R}_1 = \begin{cases} x \downarrow f(x) \implies & f(x) \rightarrow a \\ & b \rightarrow f(b) \end{cases}$$

clearly is an overlay system with joinable critical pairs (it is even non-overlapping). Moreover we have $f(f(b)) \rightarrow a$ and $f(f(b)) \rightarrow f(a)$ but not $a \downarrow f(a)$. Obviously, $\mathrm{SN}(f(f(b)))$ *does not hold due to the presence of the rule $b \rightarrow f(b)$ in $\mathcal{R}$. Note that $f(f(b))$ is not even (strongly) innermost terminating (*$\mathrm{SIN}(f(f(b)))$*) and also not weakly innermost terminating (*$\mathrm{WIN}(f(f(b)))$*).*

Under the stronger assumption of global termination we get from Corollary 12 the following known critical pair criterion for confluence of conditional overlay systems.

Theorem 14 *([5]) A terminating CTRS which is an overlay system such that all its conditional critical pairs are joinable is confluent, hence complete.*

Concerning termination properties of CTRSs we have obtained the following results which generalize Theorem 9 to the conditional case and moreover, do not require left-linearity.

Theorem 15 *([10]) Let $\mathcal{R}$ be a non-overlapping CTRS. Then the following properties hold:*

(1a) $\forall t : [\,\mathrm{WIN}(t) \Longrightarrow \mathrm{SIN}(t)\,]$.
(1b) $\mathrm{WIN}(\mathcal{R}) \Longrightarrow \mathrm{SIN}(\mathcal{R})$.
(2a) $\forall t : [\,\mathrm{SIN}(t) \Longrightarrow \mathrm{SN}(t)\,]$.
(2b) $\mathrm{SIN}(\mathcal{R}) \Longrightarrow \mathrm{SN}(\mathcal{R})$.
(3) $\forall\, s, t : [\, s \xrightarrow[i]{} t \wedge \mathrm{SN}(t) \Longrightarrow \mathrm{SN}(s)\,]$.
(4a) $\mathrm{NE}(\mathcal{R}) \Longrightarrow [\forall t : [\,\mathrm{WN}(t) \Longrightarrow \mathrm{SN}(t)]]$.
(4b) $\mathrm{NE}(\mathcal{R}) \Longrightarrow [\,\mathrm{WN}(\mathcal{R}) \Longrightarrow \mathrm{SN}(\mathcal{R})]$.

Moreover, it is possible to generalize Theorem 15(2) by allowing root overlaps but guaranteeing joinability of critical pairs.

Theorem 16 *([10]) For any CTRS $\mathcal{R}$ we have:*

(a) $\mathrm{OS}(\mathcal{R}) \wedge \mathrm{JCP}(\mathcal{R}) \wedge \mathrm{SIN}(\mathcal{R}) \Longrightarrow \mathrm{SN}(\mathcal{R}) \wedge \mathrm{CR}(\mathcal{R})$, *and*
(b) $\mathrm{OS}(\mathcal{R}) \wedge \mathrm{JCP}(\mathcal{R}) \quad \Longrightarrow \quad [\forall s : [\mathrm{SIN}(s) \Longrightarrow \mathrm{SN}(s) \wedge \mathrm{CR}(s)]]$,

i.e., any (strongly) innermost terminating overlay system with joinable critical pairs is terminating and confluent, hence complete (part (a)), which also holds in the stronger local version (b).

Theorem 16 states that any (strongly) innermost terminating (conditional) overlay system with joinable critical pairs is (strongly) terminating and confluent, hence complete, which even holds in a stronger local version. In other words, for (conditional) overlay systems it suffices to verify (strong) innermost termination and joinability of all critical pairs in order to infer general termination and confluence, i.e., completeness. The non-triviality of this result is obvious taking into account the fact that for CTRSs the critical pair lemma does not hold in general and almost all known sufficient criteria for confluence presume even stronger properties than termination plus joinability of (conditional) critical pairs.

4 Modularity Results for Disjoint Unions of CTRSs

In this section, after a brief general discussion, we shall investigate in 4.1 the preservation behaviour of the properties termination (SN), weak termination (WN), (strong) innermost termination (SIN), weak innermost termination (WIN) under signature extensions. Modularity w.r.t. disjoint unions of WN, WIN, SIN is treated in 4.2. These results will be used to establish modularity criteria w.r.t. disjoint unions for termination (SN) and completeness ($SN \wedge CR$) in 4.3. Finally we consider various extensions (of obtained results) and counterexamples in 4.4.

Modular properties of (C)TRSs, i.e., properties which are preserved under disjoint unions, have attracted an increasing attention within the last few years. Two (C)TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ over disjoint signatures $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, are said to be *disjoint* if $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint, i.e. $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ (in that case the rule sets of $\mathcal{R}_1$ and $\mathcal{R}_2$ are necessarily disjoint, too). The (disjoint) union of two disjoint (C)TRSs $\mathcal{R}_1$, $\mathcal{R}_2$ is denoted by $\mathcal{R}_1 \oplus \mathcal{R}_2$. We shall also speak of the disjoint union of $\mathcal{R}_1$ and $\mathcal{R}_2$ using the implicit convention that $\mathcal{R}_1$ and $\mathcal{R}_2$ are assumed to be disjoint (C)TRSs. Formally, a property P of (C)TRSs is said to be *modular* if the following holds for all disjoint (C)TRSs $\mathcal{R}_1$, $\mathcal{R}_2$: $\mathcal{R}_1 \oplus \mathcal{R}_2$ has property P if and only if both $\mathcal{R}_1$ and $\mathcal{R}_2$ have property P.

Toyama [31] has shown that confluence is modular (for TRSs). The termination property, however, is in general not modular for TRSs (cf. [31], [32]) as witnessed by the following famous counterexample of [31]:

Example 17 *Consider the disjoint TRSs*

$$\mathcal{R}_1 = \left\{ f(a,b,x) \to f(x,x,x) \right\} \qquad \textit{and} \qquad \mathcal{R}_2 = \left\{ \begin{array}{l} G(x,y) \to x \\ G(x,y) \to y \end{array} \right\}$$

Clearly, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating, but $\mathcal{R}_1 \oplus \mathcal{R}_2$ admits e.g. the following infinite derivation:

$$\begin{aligned} f(a,b,G(a,b)) &\to_{\mathcal{R}_1} f(G(a,b),G(a,b),G(a,b)) \\ &\to_{\mathcal{R}_2} f(a,G(a,b),G(a,b)) \\ &\to_{\mathcal{R}_2} f(a,b,G(a,b)) \\ &\to_{\mathcal{R}_1} \cdots \end{aligned}$$

When investigating the modularity behaviour of CTRSs the situation is much more complicated than for (unconditional) TRSs. For instance, as exhibited in [20], the fundamental decomposition property of TRSs

$$s \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} t \quad \Longrightarrow \quad s \to_{\mathcal{R}_1} t \ \vee \ s \to_{\mathcal{R}_2} t$$

does not hold any more. This is due to the fact that when a rule of one of the systems is applied, rules of the other system may be needed in order to satisfy the conditions. Nevertheless, confluence has turned out to be a modular property of CTRSs, too, as shown by Middeldorp.

Theorem 18 *([20], [22]) Confluence is a modular property of CTRSs.*

The more interesting direction of this modularity result for CTRSs, namely, that confluence of disjoint CTRSs $\mathcal{R}_1$, $\mathcal{R}_2$ implies confluence of $\mathcal{R}_1 \oplus \mathcal{R}_2$, is proved in [22] by a non-trivial reduction to the corresponding result for the unconditional case. The other direction is trivial for CTRSs without extra variables but less trivial otherwise (see Lemma 43 below).

For some sufficient conditions for modularity of termination of CTRSs we refer to [20], [22], [21], [9], [27]. Here we shall concentrate on CTRSs satisfying some structural restrictions. In [8] we have proved that any (strongly) innermost terminating, locally confluent (unconditional) overlay system is terminating, and hence confluent and complete. Moreover – by exploiting the fact that (strong) innermost termination is modular (for TRSs) in contrast to the non-modularity of (strong) termination – we have shown there how to derive from this result new modularity results and simplified proofs of known ones. Here we would also like to follow the same line of reasoning – exploiting Theorems 15 and 16 – in order to obtain derived modularity results for CTRSs. But again the situation is much more complicated for CTRSs than for TRSs since neither weak termination, nor weak innermost termination nor strong innermost termination are modular in general for CTRSs. This can be seen e.g. from the next example.

Example 19 *(cf. [22] for similar examples) Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$*

$$\mathcal{R}_1 = \left\{ x \downarrow b \wedge x \downarrow c \implies a \to a \right\} \qquad \textit{and} \qquad \mathcal{R}_2 = \left\{ \begin{array}{l} G(x,y) \to x \\ G(x,y) \to y \end{array} \right\}$$

over $\mathcal{F}_1 = \{a, b, c\}$ and $\mathcal{F}_2 = \{G, A\}$, respectively. Here, we have $a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a$ by applying the $\mathcal{R}_1$-rule (x is substituted by $G(b, c)$), but neither $a \to_{\mathcal{R}_1} a$ nor $a \to_{\mathcal{R}_2} a$. Hence, a is an $\mathcal{R}_i$-normal form (for $i = 1, 2$) but not a normal form w.r.t. $\mathcal{R}_1 \oplus \mathcal{R}_2$. Obviously, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are strongly (hence also weakly and innermost) terminating but their disjoint union is not. For instance, $a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \to_{\mathcal{R}_1 \oplus \mathcal{R}_2} \ldots$ is an infinite innermost derivation, and a does not have a normal from w.r.t. $(\mathcal{R}_1 \oplus \mathcal{R}_2)$.

From the observation in Example 19 above one might be tempted to conjecture (as it is done in [22]) that the preservation of normal forms, defined by (with $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2$, $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$)

$$\mathrm{NFP}(\mathcal{R}_1, \mathcal{R}_2): \quad \mathrm{NF}(\mathcal{R}^{\mathcal{F}}) = \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}}) \cap \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}}),$$

where $\mathrm{NF}(\mathcal{R}^{\mathcal{F}}) = \{s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid s \text{ irreducible w.r.t. } \mathcal{R}^{\mathcal{F}}\}$, should be a sufficient condition for the modularity of weak termination. But this is also not true in general, thus contradicting Theorem 5.2 in [22] (cf. also Theorem 4.3.20 in [20]) the proof of which implicitly relies on the incorrect assumption $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1}) \iff \mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1 \uplus \mathcal{F}_2})$. The situation is even worse, since – surprisingly – it may happen that a weakly terminating CTRS may become not weakly terminating under the disjoint union with another 'empty' CTRS, i.e., simply by extending the signature without adding new rules.

Example 20 *Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ given by*

$$\mathcal{R}_1 = \left\{ \begin{array}{lcr} x \downarrow z \wedge z \downarrow y & \Longrightarrow & g(x,y) \to g(x,y) \\ & & g(x,x) \to x \\ & & g(x,a) \to c \\ & & g(x,b) \to c \\ & & g(x,g(y,z)) \to c \\ & & g(a,x) \to c \\ & & g(b,x) \to c \\ & & g(g(y,z),x) \to c \\ & & c \to a \\ & & c \to b \end{array} \right.$$

over $\mathcal{F}_1 = \{g, a, b, c\}$ and $\mathcal{R}_2 = \emptyset$ over $\mathcal{F}_2 = \{G\}$ (with G unary). It is straightforward to verify that $\mathcal{R}_1^{\mathcal{F}_1}$ is weakly terminating. The only potential reason for non-existence of a normal form of some given term from $\mathcal{T}(\mathcal{F}_1, \mathcal{V})$ is the first $\mathcal{R}_1$-rule. But whenever this rule is applicable another rule is applicable, too, which may be preferred (note that without the first rule the system $\mathcal{R}_1$ is even strongly terminating). Now consider the combined system $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1 \uplus \{G\}}$ and the term $g(G(a), G(b))$. In $\mathcal{R}^{\mathcal{F}}$ we get the following cyclic derivation:

$$g(G(a), G(b)) \to_{\mathcal{R}^{\mathcal{F}}} g(G(a), G(b)) \to_{\mathcal{R}^{\mathcal{F}}} g(G(a), G(b)) \to_{\mathcal{R}^{\mathcal{F}}} \ldots$$

by applying the first rule (which is indeed applicable since instantiating the extra variable z by $G(c)$ we easily obtain $G(a) \downarrow_{\mathcal{R}^{\mathcal{F}}} G(c)$ and $G(c) \downarrow_{\mathcal{R}^{\mathcal{F}}} G(b)$ as desired). Note moreover that there is no other way of reducing $g(G(a), G(b))$ (all its proper subterms are in normal form w.r.t. $\mathcal{R}^{\mathcal{F}}$, and the second rule is clearly not applicable). Hence, $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1 \uplus \{G\}}$ is not weakly terminating.

Note that Example 20 also shows that WIN, the weak innermost termination property, need not be preserved under signature extensions (since we have $\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1})$ but not $\mathrm{WIN}(f(G(a), G(b)), \mathcal{R}_1^{\mathcal{F}_1 \uplus \mathcal{F}_2})$, hence not $\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1 \uplus \mathcal{F}_2})$.

By slightly modifying Example 20 we can also show that SIN is not preserved under signature extensions in general.

Example 21 *Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ given by*

$$\mathcal{R}_1 = \begin{cases} x \downarrow z \wedge z \downarrow y \quad \Longrightarrow & f(g(x,y)) \to f(g(x,y)) \\ & g(x,x) \to x \\ & g(x,a) \to c \\ & g(x,b) \to c \\ & g(x,f(y)) \to c \\ & g(x,g(y,z)) \to c \\ & g(a,x) \to c \\ & g(b,x) \to c \\ & g(f(y),x) \to c \\ & g(g(y,z),x) \to c \\ & c \to a \\ & c \to b \end{cases}$$

over $\mathcal{F}_1 = \{f, g, a, b, c\}$ and $\mathcal{R}_2 = \emptyset$ over $\mathcal{F}_2 = \{G\}$ (with G unary). It is straightforward to verify that $\mathcal{R}_1^{\mathcal{F}_1}$ is weakly terminating, weakly innermost terminating and even strongly innermost terminating. The crucial point is that any arbitrary infinite $\mathcal{R}_1^{\mathcal{F}_1}$- derivation must contain rewrite steps using the first rule. But whenever this rule is applicable, the contracted redex cannot be innermost, since some proper subterm must then be reducible by the remaining rules which constitute a (strongly) terminating CTRS. Nevertheless, in the extended system $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1 \uplus \{G\}}$ we get the cyclic (hence infinite) innermost $\mathcal{R}^{\mathcal{F}}$-derivation:

$$f(g(G(a), G(b))) \xrightarrow[i]{}_{\mathcal{R}^{\mathcal{F}}} f(g(G(a), G(b))) \xrightarrow[i]{}_{\mathcal{R}^{\mathcal{F}}} f(g(G(a), G(b))) \xrightarrow[i]{}_{\mathcal{R}^{\mathcal{F}}} \cdots$$

by applying the first rule (instantiating the extra variable z by $G(c)$). Note moreover that there is no other way of reducing $f(g(G(a),G(b)))$ (all its proper subterms are in normal form w.r.t. $\mathcal{R}^{\mathcal{F}}$, and the second rule is clearly not applicable). Hence, $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1 \uplus \{G\}}$ is neither strongly innermost terminating nor weakly innermost terminating nor weakly terminating.

By a thorough analysis of abstraction and innermost reduction properties which allow to project reduction sequences on mixed terms to certain reduction sequences on pure terms we shall show below how the monotonicity of WN, WIN and SIN under signature extensions can be restored. More generally, we also develop sufficient criteria for the modularity of these properties. This analysis heavily relies on some very useful terminology and technical results from [22].

Lemma 22 *(cf. [22, Prop. 2.23]) Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs. Then every substitution σ can be decomposed into $\sigma_2 \circ \sigma_1$ such that σ_1 is black (white), σ_2 is top white (top black), and $\sigma_2 \propto \epsilon$.*

Definition 23 (cf. [22, Definition 3.2]) Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs with $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$. Then the rewrite relation $\to_1$ is defined as follows: $s \to_1 t$ if there exists a rule $s_1 \downarrow t_1 \wedge \ldots \wedge s_n \downarrow t_n \implies l \to r$ in $\mathcal{R}_1$, a context $C[\,]$ and a substitution σ such that $s = C[\sigma(l)]$, $t = C[\sigma(r)]$ and $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ for $i = 1, \ldots, n$, where $\sigma(s_i) \downarrow_1^o \sigma(t_i)$ means that s_i and t_i are joinable using only $\xrightarrow{o}_1$-reduction steps. The relation $\to_2$ is defined analogously. The union of $\to_1$ and $\to_2$ is denoted by $\to_{1,2}$.

Lemma 24 *(cf. [22], Prop. 3.5) ('injective abstraction' is possible for $\xrightarrow{o}_1$-steps)*
Let $\mathcal{R}_1$, $\mathcal{R}_2$ be disjoint CTRSs, s, t be black terms and σ be a top white substitution with $\sigma(s) \xrightarrow{o}_1 \sigma(t)$. Then, for any substitution τ with $\sigma \propto \tau$, we have $\tau(s) \xrightarrow{o}_1 \tau(t)$.

Note that this result implies in particular that a step $\sigma(s) \xrightarrow{o}_1 \sigma(t)$ on mixed terms can be injectively abstracted into a 'pure' step $\tau(s) \to_{\mathcal{R}_1} \tau(t)$ by injectively replacing the maximal top white aliens of $\sigma(s)$ and $\sigma(t)$ by fresh variables. One may wonder whether such an injective abstraction is also possible for an arbitrary outer step $\sigma(s) \xrightarrow{o}_{\mathcal{R}_1} \sigma(t)$, where for verifying the conditions of applied $\mathcal{R}_1$-rules also inner $\mathcal{R}_1$-steps are allowed. This is not the case as shown next.

Example 25 *Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ given by*

$$\mathcal{R}_1 = \begin{cases} x \downarrow y \implies f(x,y) \to x \\ \qquad\qquad\qquad a \to b \end{cases}$$

over $\mathcal{F}_1 = \{f, a, b\}$ and $\mathcal{R}_2 = \emptyset$ over $\mathcal{F}_2 = \{G\}$ (with G unary). Here we have $f(G(a), G(b)) \xrightarrow{o}_{\mathcal{R}_1} G(a)$ (due to $G(a) \downarrow_{\mathcal{R}_1} G(b)$), but not $f(G(a), G(b)) \xrightarrow{o}_1 G(a)$ since for satisfying $G(a) \downarrow_{\mathcal{R}_1} G(b)$ we need an inner $\mathcal{R}_1$-step. Note that after injective abstraction of $s = f(G(a), G(b))$ and $t = G(a)$ into $f(x, y)$ and x, respectively, the reduction $f(x, y) \to_{\mathcal{R}_1} x$ is not possible any more.

The above example somehow suggests that by forbidding the possibility of inner reduction steps injective abstraction of reduction steps might still be possible. But even if all maximal top white aliens are irreducible, this is not possible in general.

Example 26 *Consider the CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ given by*

$$\mathcal{R}_1 = \begin{cases} x \downarrow z \wedge z \downarrow y \implies & f(x, y) \to x \\ & c \to a \\ & c \to b \end{cases}$$

with $\mathcal{F}_1 = \{f, a, b, c\}$, $\mathcal{R}_2 = \emptyset$, $\mathcal{F}_2 = \{G\}$ (with G unary). Here all proper subterms of $f(G(a), G(b))$ are ($\mathcal{R}^{\mathcal{F}}$-) irreducible and $f(G(a), G(b)) \xrightarrow{o}_{\mathcal{R}_1} G(a)$ but not $f(G(a), G(b)) \xrightarrow{o}_1 G(a)$ since for satisfying $G(a) \downarrow_{\mathcal{R}_1} z \wedge z \downarrow_{\mathcal{R}_1} G(b)$ we have to instantiate the extra variable z in the condition of the first rule by a mixed term of the form $G(u)$, e.g. $G(c)$, and to use inner $\mathcal{R}_1$-steps for establishing $G(a) \downarrow_{\mathcal{R}_1} G(u)$, $G(u) \downarrow_{\mathcal{R}_1} G(b)$. Note again that after injective abstraction of $s = f(G(a), G(b))$ and $t = G(a)$ into $f(x, y)$ and x, respectively, the reduction $f(x, y) \to_{\mathcal{R}_1} x$ is not possible any more.

Whereas in the above examples an injective abstraction of certain reduction steps on mixed terms to a corresponding reduction step on pure terms is not possible, a non-injective 'identifying' one, which replaces all maximal top white aliens by the *same* fresh variable, is indeed possible. This is shown next.

Lemma 27 *('identifying abstraction' is always possible)*
Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs, s and t be black terms, σ be a top white substitution, and $\widetilde{\sigma}$ be defined by $\widetilde{\sigma}(x) := z$ for all $x \in dom(\sigma)$ where z is a 'fresh' variable, i.e., not occurring in s. Then we have:

(a) $\sigma(s) \xrightarrow{o}_{\mathcal{R}_1} \sigma(t) \implies \widetilde{\sigma}(s) \xrightarrow{o}_{\mathcal{R}_1} \widetilde{\sigma}(t)$, *and*
(b) $\sigma(s) \xrightarrow{i}_{\mathcal{R}_1} \sigma(t) \implies \widetilde{\sigma}(s) = \widetilde{\sigma}(t)$.

Proof. (b) is trivially satisfied by definition of $\xrightarrow{i}$ and of $\widetilde{\sigma}$. We show (a) by induction on n, the depth of rewriting (using the same proof structure as in [22] for Prop. 3.5, see Lemma 24 above). In the base case, i.e., for $n \leq 1$, (a) is trivially satisfied. For the induction step, assume $\sigma(s) \stackrel{o}{\rightarrow}_{\mathcal{R}_1} \sigma(t)$ is of depth $n+1$. Hence we have $\sigma(s) = C[\rho(l)] \stackrel{o}{\rightarrow}_{\mathcal{R}_1} C[\rho(r)]$ for some context $C[\,]$, some substitution ρ and some rule $s_1 \downarrow t_1 \wedge \ldots \wedge s_m \downarrow t_n \implies l \rightarrow r$ from $\mathcal{R}_1$ such that $\rho(s_i) \downarrow_{\mathcal{R}_1} \rho(t_i)$ for $i = 1, \ldots, m$ with depth less than or equal to n (note that ρ may also instantiate extra variables in the conditions of the applied $\mathcal{R}_1$-rule). Lemma 22 yields a decomposition $\rho_1 \circ \rho_2$ of ρ such that ρ_1 is black, ρ_2 top white and $\rho_2 \propto \mathrm{id}$. We observe that whenever $\rho_2(u_1) \rightarrow_{\mathcal{R}_1} u_2$ for some black term u_1, then there is some black term u_3 with $u_2 = \rho_2(u_3)$. Hence, for any i, $1 \leq i \leq m$, the conversion $\rho(s_i) \downarrow_{\mathcal{R}_1} \rho(t_i)$ has the form $\rho_2(u_{i,1}) \rightarrow_{\mathcal{R}_1} \rho_2(u_{i,2}) \rightarrow_{\mathcal{R}_1} \ldots \rightarrow_{\mathcal{R}_1} \rho_2(u_{i,k_i}) = \rho_2(v_{i,l_i}) \leftarrow_{\mathcal{R}_1} \ldots \leftarrow_{\mathcal{R}_1} \rho_2(v_{i,2}) \leftarrow_{\mathcal{R}_1} \rho_2(v_{i,1})$ (with $u_{i,1} = \rho_1(s_1)$, $v_{i,1} = \rho_1(t_1)$) for black terms $u_{i,1} \ldots u_{i,k_i}$, $v_{i,1}, \ldots, v_{l_i}$. Thus, by using the induction hypothesis and (b) for any single step in $\rho_2(\rho_1(s_i)) \downarrow_{\mathcal{R}_1} \rho_2(\rho_1(t_i))$ it is straightforward to show $\widetilde{\rho_2}(\rho_1(s_i)) \downarrow_{\mathcal{R}_1} \widetilde{\rho_2}(\rho_1(t_i))$ by an additional induction on the length of the conversion $\rho_2(\rho_1(s_i)) \downarrow_{\mathcal{R}_1} \rho_2(\rho_1(t_i))$, for $i = 1, \ldots, m$. $\square$

A straightforward consequence of this 'identifying abstraction' lemma is the fact that for rewriting some black term with a black CTRS, considered as CTRS (with the same set of rules) over an extended black-white signature, it is not necessary to instantiate extra variables in the conditions of the (black) rules with non-black terms.

Corollary 28 *Let $\mathcal{R}^{\mathcal{F}}$ be a CTRS, $\mathcal{F}'$ be a signature with $\mathcal{F} \subseteq \mathcal{F}'$, and $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Then we have: $s \rightarrow_{\mathcal{R}^{\mathcal{F}'}} t \implies s \rightarrow_{\mathcal{R}^{\mathcal{F}}} t$.*

Subsequently, we shall sometimes tacitly make use of this basic property.

Using Lemma 27 we are now able to show that at least termination (SN) is preserved under signature extensions.

Lemma 29 *Let $\mathcal{R}^{\mathcal{F}}$ be a CTRS and $\mathcal{F}'$ be a signature with $\mathcal{F} \subseteq \mathcal{F}'$. Then the following holds:* $\mathrm{SN}(\mathcal{R}^{\mathcal{F}}) \iff \mathrm{SN}(\mathcal{R}^{\mathcal{F}'})$.

Proof. It clearly suffices to show: $\mathrm{SN}(\mathcal{R}^{\mathcal{F}}) \implies \mathrm{SN}(\mathcal{R}^{\mathcal{F}'})$. Hence, assuming $\mathrm{SN}(\mathcal{R}^{\mathcal{F}})$, we prove $\mathrm{SN}(s)$ for all $s \in \mathcal{T}(\mathcal{F}', \mathcal{V})$ by induction on $n = rank(s)$ (note that we may consider here $\mathcal{R}^{\mathcal{F}'}$ as disjoint union $\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$ with $\mathcal{R}_1^{\mathcal{F}_1} = \mathcal{R}^{\mathcal{F}}$, $\mathcal{R}_2 = \emptyset$ and $\mathcal{F}_2 = \mathcal{F}' \setminus \mathcal{F}$). For $rank(s) = 1$ we get $\mathrm{SN}(s)$ by assumption and Corollary 28. Let $rank(s) > 1$. If the root symbol of s is a new one, i.e. from $(\mathcal{F}' \setminus \mathcal{F})$, then $\mathrm{SN}(s)$ follows by induction hypothesis, since rewrite steps in the top layer of s are impossible. If $root(s) \in \mathcal{F}$ then – again by induction hypothesis and the fact that $(\mathcal{F}' \setminus \mathcal{F})$-layers cannot collapse – any infinite $\mathcal{R}^{\mathcal{F}'}$-derivation starting with s would have to contain infinitely many outer $\mathcal{R}^{\mathcal{F}'}$-steps. But then identifying abstraction using Lemma 27 would yield an infinite $\mathcal{R}^{\mathcal{F}}$-derivation contradicting the assumption $\mathrm{SN}(\mathcal{R}^{\mathcal{F}})$. $\square$

In order to present sufficient criteria for the preservation of restricted termination properties under signature extensions and – more generally – under disjoint unions, we will introduce now some more notations, in particular for certain innermost reductions steps.

Definition 30 Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs and $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2} = (\mathcal{R}_1 \uplus \mathcal{R}_2)^{\mathcal{F}_1 \uplus \mathcal{F}_2}$ be their disjoint union. If (for $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$) $s \to_{\mathcal{R}^{\mathcal{F}}} t$ by applying some $\mathcal{R}_1$-rule (where for satisfying the conditions also $\mathcal{R}_2$-rules are allowed) we denote this by $s \to_{\mathcal{R}_1^{\mathcal{F}}/\mathcal{R}^{\mathcal{F}}} t$ or simply $s \to_{\mathcal{R}_1/\mathcal{R}} t$. Furthermore, for $j = 1, 2$ we define the innermost reduction properties $\mathrm{IRP}_1(\mathcal{R}_j, \mathcal{R})$, $\mathrm{IRP}_2(\mathcal{R}_j, \mathcal{R})$ and $\mathrm{IRP}_3(\mathcal{R}_j, \mathcal{R})$ by

$$\begin{cases} \mathrm{IRP}_1(\mathcal{R}_j, \mathcal{R}) : \forall s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) : \quad s \underset{i}{\to}_{\mathcal{R}_j} t \implies s \underset{i}{\to}_j t\,, \\ \mathrm{IRP}_2(\mathcal{R}_j, \mathcal{R}) : \forall s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) : s \underset{i}{\to}_{\mathcal{R}_j/\mathcal{R}} t \implies \exists t' \in \mathcal{T}(\mathcal{F}, \mathcal{V}) : s \underset{i}{\to}_{\mathcal{R}_j} t' \\ \mathrm{IRP}_3(\mathcal{R}_j, \mathcal{R}) : \forall s \in \mathcal{T}(\mathcal{F}, \mathcal{V}) : s \underset{i}{\to}_{\mathcal{R}_j/\mathcal{R}} t \implies s \underset{i}{\to}_{\mathcal{R}_j} t\,. \end{cases}$$

We observe that the notations used here for innermost reduction are slightly ambiguous (for the sake of readability). When writing $s \underset{i}{\to}_{\mathcal{R}_j/\mathcal{R}} t$, $s \underset{i}{\to}_{\mathcal{R}_j} t$ or $s \underset{i}{\to}_j t$, we always mean that the contracted subterm is an innermost redex of s w.r.t. $\to_{\mathcal{R}}$ (then it is also an innermost redex of s w.r.t. $\to_{\mathcal{R}_j/\mathcal{R}}$, $\to_{\mathcal{R}_j}$ or $\to_j$, respectively, since $\to_{\mathcal{R}_j/\mathcal{R}}$, $\to_{\mathcal{R}_j}$ and $\to_j$ are subsets of $\to_{\mathcal{R}}$).

Note that IRP_1 enables injective abstraction (via Lemma 24) which will be useful for establishing preservation results under signature extensions. Combined with IRP_2 or the stronger property IRP_3 it will turn out to capture the essence for obtaining modularity results later on.

Using the first innermost reduction property IRP_1 defined above we obtain a sufficient criterion for the preservation of WN, WIN and SIN under signature extensions as follows.

Theorem 31 *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs with $\mathcal{R}_2 = \emptyset$ (and $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2} = \mathcal{R}_1^{\mathcal{F}_1 \uplus \mathcal{F}_2}$) such that $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_1^{\mathcal{F}})$ holds. Then we have the following equivalences:*

(a) $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1}) \iff \mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1 \cup \mathcal{F}_2})$.
(b) $\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1}) \iff \mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1 \cup \mathcal{F}_2})$.
(c) $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1}) \iff \mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1 \cup \mathcal{F}_2})$.

Proof. Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$, $\mathcal{R}^{\mathcal{F}}$ be given as above satisfying $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_1^{\mathcal{F}})$. The '$\Longleftarrow$'-directions of (a), (b) and (c) are easy by Corollary 28.

(a) For proving $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1}) \implies \mathrm{WN}(\mathcal{R}_1^{\mathcal{F}})$ we proceed by contradiction assuming $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1})$. Let s be counterexample of minimal rank, i.e., $s \in$

$\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $\mathrm{WN}(s, \mathcal{R}_1^{\mathcal{F}})$ does not hold, with $rank(s)$ minimal. The case $rank(s) = 1$ is impossible by the assumption $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}_1})$, Corollary 28 and the fact that $\mathcal{R}_2 = \emptyset$. If $rank(s) > 1$ then s has the form $s = C[\![s_1, \ldots, s_m]\!]$. If the top layer $C[, \ldots,]$ is white then by the minimality assumption and $\mathcal{R}_2 = \emptyset$ we get $\mathrm{WN}(s, \mathcal{R}_1^{\mathcal{F}})$, hence a contradiction. Otherwise, in the interesting case where $C[, \ldots,]$ is black, we know by the minimality assumption that every s_i $(1 \le i \le m)$ has a normal form w.r.t. $\mathcal{R}_1^{\mathcal{F}}$, let's say t_i. Hence, we get

$$s = C[\![s_1, \ldots, s_m]\!] \to^*_{\mathcal{R}_1^{\mathcal{F}}} C[t_1, \ldots, t_m] = C'\,\{\!\{\, u_1, \ldots, u_n\}\!\}$$

for some black context $C'\{, \ldots, \}$ and top white normal forms u_j w.r.t. $\mathcal{R}_1^{\mathcal{F}}$. Choosing fresh variables $x_1, \ldots, x_n$ injectively, i.e., with $\langle u_1, \ldots, u_n\rangle \infty \langle x_1, \ldots, x_n\rangle$, we have $rank(C'\{x_1, \ldots, x_n\}) = 1$, hence again by the minimality assumption $C'\{x_1, \ldots, x_n\}$ can be reduced in $\mathcal{R}_1^{\mathcal{F}}$ to a normal form $C''\langle x_{i_1}, \ldots, x_{i_p}\rangle$. Thus we get

$$s \to^*_{\mathcal{R}_1^{\mathcal{F}}} C'\,\{\!\{\, u_1, \ldots, u_n\}\!\} \;\to^*_{\mathcal{R}_1^{\mathcal{F}}} C''\,\langle\!\langle\, u_{i_1}, \ldots, u_{i_p}\rangle\!\rangle$$

with $u_{i_1}, \ldots, u_{i_p} \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$. Now, denoting u_{i_j} by v_j, it suffices to show $C''\,\langle\!\langle\, v_1, \ldots, v_p\rangle\!\rangle \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$. If this were not the case then there would exist an (outer) innermost $\mathcal{R}_1^{\mathcal{F}}$-reduction step of the form

$$C''\,\langle\!\langle\, v_1, \ldots, v_p\rangle\!\rangle \quad \overset{o}{\underset{i}{\to}}_{\mathcal{R}_1^{\mathcal{F}}} C'''\,\langle\!\langle\, v_{k_1}, \ldots, v_{k_q}\rangle\!\rangle$$

with $1 \le k_l \le p$, $1 \le l \le q$. But then, due to the assumption $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_1^{\mathcal{F}})$, we could apply Lemma 24 which would yield (denoting x_{i_j} by y_j, $1 \le j \le p$)

$$C''\langle y_1, \ldots, y_p\rangle \to_{\mathcal{R}_1^{\mathcal{F}}} C'''\langle y_{k_1}, \ldots, y_{k_q}\rangle\;.$$

But this is a contradiction to $C''\langle y_1, \ldots, y_p\rangle \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$. Hence in all cases we have a contradiction.

(b) For proving $\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1}) \implies \mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}})$ we use the same proof structure as for (a), but the argumentation is slightly different. Namely, (using the notations from above) in the interesting case we get

$$s = C[\![s_1, \ldots, s_m]\!] \;\underset{i}{\to}^*_{\mathcal{R}_1^{\mathcal{F}}} C[t_1, \ldots, t_m] = C'\,\{\!\{\, u_1, \ldots, u_n\}\!\}$$

by innermost normalizing the s_i to t_i, $1 \le i \le n$. Furthermore, from

$$C'\{x_1, \ldots, x_n\} \;\underset{i}{\to}^*_{\mathcal{R}_1^{\mathcal{F}}} C''\langle x_{i_1}, \ldots, x_{i_p}\rangle \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$$

(which we get by the minimality assumption) we obtain, using for any step the same rule at the same position,

$$C' \{\!\{ u_1, \ldots, u_n \}\!\} \rightarrow^*_{\mathcal{R}_1^{\mathcal{F}}} C'' \langle\!\langle u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle .$$

Note that all $\mathcal{R}_1^{\mathcal{F}}$-reductions here are innermost steps, because otherwise we could conclude, using Lemma 24 (which is applicable due to the assumption $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_1^{\mathcal{F}})$), that the corresponding step in

$$C'\{x_1, \ldots, x_n\} \underset{i}{\rightarrow}^*_{\mathcal{R}_1^{\mathcal{F}}} C''\langle x_{i_1}, \ldots, x_{i_p}\rangle$$

is a non-innermost $\mathcal{R}_1^{\mathcal{F}}$-step. Hence we have

$$s \underset{i}{\rightarrow}^*_{\mathcal{R}_1^{\mathcal{F}}} C' \{\!\{ u_1, \ldots, u_n \}\!\} \underset{i}{\rightarrow}^*_{\mathcal{R}_1^{\mathcal{F}}} C'' \langle\!\langle u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle .$$

Now $C'' \langle\!\langle u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle$ must be $\mathcal{R}_1^{\mathcal{F}}$-irreducible by the same argument as in (a) (exploiting again the assumption $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_1^{\mathcal{F}})$).

(c) For proving $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1}) \Longrightarrow \mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}})$ we assume $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1})$ and show by contradiction that for every term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ we have $\mathrm{SIN}(s, \mathcal{R}_1^{\mathcal{F}})$. Consider a counterexample which is minimal w.r.t. the subterm relation, i.e., a term s which has an infinite innermost $\mathcal{R}_1^{\mathcal{F}}$-derivation

$$(D) \quad s =: s_0 \underset{i}{\rightarrow}_{\mathcal{R}_1^{\mathcal{F}}} s_1 \underset{i}{\rightarrow}_{\mathcal{R}_1^{\mathcal{F}}} s_2 \underset{i}{\rightarrow}_{\mathcal{R}_1^{\mathcal{F}}} \cdots$$

such that all proper subterms of s are strongly innermost terminating (w.r.t. $\mathcal{R}_1^{\mathcal{F}}$). Then, necessarily s is a top black term with $rank(s) > 1$. By the minimality assumption we know that there is some (first) innermost $\mathcal{R}_1^{\mathcal{F}}$-step

$$s_k = C_k[\![t_1, \ldots, t_n]\!] \underset{i}{\rightarrow}_{\mathcal{R}_1^{\mathcal{F}}} C' \langle\!\langle t_{i_1}, \ldots, t_{i_p} \rangle\!\rangle = s_{k+1}$$

in (D) which is a root reduction step. This implies in particular that all maximal top white aliens of s_k (as well as of all $s_{k'}$, $k' > k$) are $\mathcal{R}_1^{\mathcal{F}}$-irreducible. Hence all steps in

$$s_k \underset{i}{\rightarrow}_{\mathcal{R}_1^{\mathcal{F}}} s_{k+1} \underset{i}{\rightarrow}_{\mathcal{R}_1^{\mathcal{F}}} s_{k+2} \underset{i}{\rightarrow}_{\mathcal{R}_1^{\mathcal{F}}} \cdots$$

are outer steps. Due to the assumption $\mathrm{IRP}_1(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}_1^{\mathcal{F}})$ we may apply now Lemma 24 and Corollary 28 which yields an infinite innermost $\mathcal{R}_1^{\mathcal{F}_1}$ derivation

$$\widehat{s_k} \underset{i}{\rightarrow}_{\mathcal{R}_1^{\mathcal{F}_1}} \widehat{s_{k+1}} \underset{i}{\rightarrow}_{\mathcal{R}_1^{\mathcal{F}_1}} \widehat{s_{k+2}} \underset{i}{\rightarrow}_{\mathcal{R}_1^{\mathcal{F}_1}} \cdots$$

by injectively abstracting the maximal top white aliens by fresh variables. But this is a contradiction to $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}_1})$. $\square$

4.2 Modularity Results for WN, WIN *and* SIN

Using the second innermost reduction property from Definition 30, we get the following modularity result for weak and weak innermost termination.

Lemma 32 *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs and $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then we have:*

(a) $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R})$
$\Longrightarrow [\,\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{WN}(\mathcal{R}_2^{\mathcal{F}}) \iff \mathrm{WN}(\mathcal{R}^{\mathcal{F}})\,]$.

(b) $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R})$
$\Longrightarrow [\,\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{WIN}(\mathcal{R}_2^{\mathcal{F}}) \iff \mathrm{WIN}(\mathcal{R}^{\mathcal{F}})\,]$.

Proof. Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$, $\mathcal{R}^{\mathcal{F}}$ satisfy $\mathrm{IRP}_2(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}^{\mathcal{F}})$. The '$\Longleftarrow$'-directions of the equivalences in (a) and (b) are easy by Corollary 28.

(a) For proving $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{WN}(\mathcal{R}_2^{\mathcal{F}}) \Longrightarrow \mathrm{WN}(\mathcal{R}^{\mathcal{F}})$ we proceed by contradiction assuming $\mathrm{WN}(\mathcal{R}_1^{\mathcal{F}})$, $\mathrm{WN}(\mathcal{R}_2^{\mathcal{F}})$. Let s be a counterexample of minimal rank, i.e., $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $\mathrm{WN}(s, \mathcal{R}^{\mathcal{F}})$ does not hold, with $rank(s)$ minimal. The case $rank(s) = 1$ is impossible, due to the assumptions $\mathrm{WN}(\mathcal{R}_j^{\mathcal{F}})$, $\mathrm{IRP}_2(\mathcal{R}_j, \mathcal{R})$ (for $j = 1, 2$). If $rank(s) > 1$ then s has the form $s = C[\![s_1, \ldots, s_m]\!]$ with s top black w.l.o.g., and we know by the minimality assumption that every s_i $(1 \leq i \leq m)$ has a normal form w.r.t. $\mathcal{R}^{\mathcal{F}}$, let's say t_i. Hence, we get

$$s = C[\![s_1, \ldots, s_m]\!] \to_{\mathcal{R}^{\mathcal{F}}}^{*} C[t_1, \ldots, t_m] = C'\,\{\!\{\, u_1, \ldots, u_n \}\!\}$$

for some black context $C'\{, \ldots, \}$ and top white normal forms u_j w.r.t. $\mathcal{R}^{\mathcal{F}}$. By assumption, we can reduce $C'\,\{\!\{\, u_1, \ldots, u_n \}\!\}$ to some normal form w.r.t. $\mathcal{R}_1^{\mathcal{F}}$, i.e.

$$s \to_{\mathcal{R}^{\mathcal{F}}}^{*} C'\,\{\!\{\, u_1, \ldots, u_n \}\!\} \to_{\mathcal{R}_1^{\mathcal{F}}}^{*} C''\,\langle\!\langle\, u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle$$

with $C''\,\langle\!\langle\, u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle$ irreducible w.r.t. $\to_{\mathcal{R}_2/\mathcal{R}}$. Now, to obtain a contradiction, it suffices to show $C''\,\langle\!\langle\, u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle \in \mathrm{NF}(\mathcal{R}^{\mathcal{F}})$. If this were not the case then $C''\,\langle\!\langle\, u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle$ would be innermost $\to_{\mathcal{R}_1^{\mathcal{F}}/\mathcal{R}^{\mathcal{F}}}$-reducible, hence by the assumption $\mathrm{IRP}_2(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}^{\mathcal{F}})$ it would also be (innermost) $\to_{\mathcal{R}_1^{\mathcal{F}}}$-reducible, contradicting $C''\,\langle\!\langle\, u_{i_1}, \ldots, u_{i_p} \rangle\!\rangle \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$. Hence we are done.

(b) For proving $\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1}) \Longrightarrow \mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}})$ we use the same proof structure as for (a), but the argumentation is slightly different. Namely, (using the notations from above) in the interesting case we get

$$s = C[\![s_1, \dots, s_m]\!] \underset{i}{\to}^*_{\mathcal{R}_1^{\mathcal{F}}} C[t_1, \dots, t_m] = C' \{\{ u_1, \dots, u_n \}\}$$

by innermost normalizing the s_i to t_i, $1 \le i \le n$. Furthermore, from the assumption $\mathrm{WIN}(\mathcal{R}_1^{\mathcal{F}_1})$ we obtain

$$C' \{\{ u_1, \dots, u_n \}\} \underset{i}{\to}^*_{\mathcal{R}_1^{\mathcal{F}_1}} C'' \langle\!\langle u_{i_1}, \dots, u_{i_p} \rangle\!\rangle =: t$$

with t in $\mathcal{R}_1^{\mathcal{F}}$ normal form. Moreover, t is also $\mathcal{R}^{\mathcal{F}}$-irreducible since otherwise, by the assumption $\mathrm{IRP}_2(\mathcal{R}_1^{\mathcal{F}_1}, \mathcal{R}^{\mathcal{F}})$ we would get a contradiction to $t \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$. $\square$

An equivalent characterization of the precondition in Lemma 32 is given by the normal form property NFP defined above. More precisely, we obtain the following result.

Lemma 33 *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs and $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then we have:*

$$\mathrm{NF}(\mathcal{R}^{\mathcal{F}}) = \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}}) \cap \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}}) \iff \mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R}) .$$

Proof.
"$\Longrightarrow$": Assuming $\mathrm{NF}(\mathcal{R}^{\mathcal{F}}) = \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}}) \cap \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}})$ we have to show $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R})$. Now, w.l.o.g. it suffices to show: $s \underset{i}{\to}_{\mathcal{R}_1/\mathcal{R}} t \Longrightarrow \exists t' : s \underset{i}{\to}_{\mathcal{R}_1} t'$. Focussing on the contracted redex in s, let's say at position p, we get $s/p = C \{\{ s_1, \dots, s_n \}\} \underset{i}{\to}_{\lambda, \mathcal{R}_1/\mathcal{R}} C' \langle\!\langle s_{i_1}, \dots, s_{i_p} \rangle\!\rangle = t/p$ with s/p irreducible w.r.t. $\mathcal{R}_2^{\mathcal{F}}$. Hence, by $\to_{\mathcal{R}_1/\mathcal{R}}$-reducibility of s and the assumption $\mathrm{NFP}(\mathcal{R}_1, \mathcal{R}_2)$ we know that s/p (and thus also s) must be $\mathcal{R}_1^{\mathcal{F}}$-reducible, too, as desired.
"$\Longleftarrow$": Assuming $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R})$ it suffices to show $\mathrm{NF}(\mathcal{R}_1^{\mathcal{F}}) \cap \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}}) \subseteq \mathrm{NF}(\mathcal{R}^{\mathcal{F}})$. Suppose there exists a term $s \in (\mathrm{NF}(\mathcal{R}_1^{\mathcal{F}}) \cap \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}})) \setminus \mathrm{NF}(\mathcal{R}^{\mathcal{F}})$. Hence s is $\mathcal{R}^{\mathcal{F}}$-reducible and also $\mathcal{R}^{\mathcal{F}}$-innermost reducible, let's say with an $\mathcal{R}_1$-rule. Thus we have $s \underset{i}{\to}_{\mathcal{R}_1/\mathcal{R}} t$ which by $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R})$ implies $s \underset{i}{\to}_{\mathcal{R}_1} t'$ for some t'. But this is a contradiction to $s \in \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}})$, hence we are done. $\square$

Now, considering Lemma 32 above, the property $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R})$ does not yet suffice for the equivalence $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{SIN}(\mathcal{R}_2^{\mathcal{F}}) \iff \mathrm{SIN}(\mathcal{R}^{\mathcal{F}})$. To see this, consider the following slightly modified version of Example 19.

Example 34 *Consider the disjoint CTRSs $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ given by*

$$\mathcal{R}_1 = \begin{cases} x \downarrow b \wedge x \downarrow c \implies a \to a \\ \qquad\qquad\qquad\qquad a \to d \end{cases} \qquad \mathcal{R}_2 = \begin{cases} G(x,y) \to x \\ G(x,y) \to y \end{cases}$$

over $\mathcal{F}_1 = \{a,b,c,d\}$ and $\mathcal{F}_2 = \{G, A\}$, respectively. Here, both $\mathcal{R}_1^{\mathcal{F}_1 \uplus \mathcal{F}_2}$ and $\mathcal{R}_2^{\mathcal{F}_1 \uplus \mathcal{F}_2}$ are (strongly) innermost terminating – and even (strongly) terminating (it is easily shown that the first $\mathcal{R}_1$-rule is never applicable) – but their disjoint union $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \uplus \mathcal{R}_2)^{\mathcal{F}_1 \uplus \mathcal{F}_2}$ is not (strongly) innermost terminating due to $a \underset{i}{\to}_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \underset{i}{\to}_{\mathcal{R}_1 \oplus \mathcal{R}_2} a \underset{i}{\to}_{\mathcal{R}_1 \oplus \mathcal{R}_2} \ldots$. Nevertheless $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfy the normal form property $\mathrm{NFP}(\mathcal{R}_1, \mathcal{R}_2)$*:* $\mathrm{NF}(\mathcal{R}^{\mathcal{F}}) = \mathrm{NF}(\mathcal{R}_1^{\mathcal{F}}) \cap \mathrm{NF}(\mathcal{R}_2^{\mathcal{F}})$ *or equivalently* $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R})$ *(since a is now both $\mathcal{R}_1^{\mathcal{F}}$- and $\mathcal{R}^{\mathcal{F}}$-reducible due the presence of the rule $a \to d$ in $\mathcal{R}_1$).*

Requiring instead of $\mathrm{IRP}_2(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_2(\mathcal{R}_2, \mathcal{R})$ the stronger property $\mathrm{IRP}_3(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_3(\mathcal{R}_2, \mathcal{R})$ accounts for this fact.

Lemma 35 *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs and $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then we have:*

$$\mathrm{IRP}_3(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_3(\mathcal{R}_2, \mathcal{R}) \implies [\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{SIN}(\mathcal{R}_2^{\mathcal{F}}) \iff \mathrm{SIN}(\mathcal{R}^{\mathcal{F}})].$$

Proof. Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ and $\mathcal{R}^{\mathcal{F}}$ satisfy $\mathrm{IRP}_3(\mathcal{R}_1, \mathcal{R}) \wedge \mathrm{IRP}_3(\mathcal{R}_2, \mathcal{R})$. Now, the "$\Longleftarrow$"-direction is straightforward using Corollary 28. Vice versa, for proving $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}}) \wedge \mathrm{SIN}(\mathcal{R}_2^{\mathcal{F}}) \implies \mathrm{SIN}(\mathcal{R}^{\mathcal{F}})$ we proceed by contradiction assuming $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}})$ and $\mathrm{SIN}(\mathcal{R}_2^{\mathcal{F}})$. Consider a minimal counterexample, i.e., an infinite innermost $\mathcal{R}^{\mathcal{F}}$-derivation $s_0 \underset{i}{\to}_{\mathcal{R}^{\mathcal{F}}} s_1 \underset{i}{\to}_{\mathcal{R}^{\mathcal{F}}} s_2 \underset{i}{\to}_{\mathcal{R}^{\mathcal{F}}} \ldots$ such that no proper subterm of s_0 admits infinite innermost $\mathcal{R}^{\mathcal{F}}$-derivations. By the minimality assumption some step $s_k \underset{i}{\to}_{\mathcal{R}^{\mathcal{F}}} s_{k+1}$ in the above derivation must be a root reduction step, let's say using a rule from $\mathcal{R}_1$. But then we know that all subsequent steps must also be $\to_{\mathcal{R}_1/\mathcal{R}}$-steps. Thus, by the assumption $\mathrm{IRP}_3(\mathcal{R}_1, \mathcal{R})$ we can conclude that $s_k \underset{i}{\to}_{\mathcal{R}_1^{\mathcal{F}}} s_{k+1} \underset{i}{\to}_{\mathcal{R}_1^{\mathcal{F}}} s_{k+2} \underset{i}{\to}_{\mathcal{R}_1^{\mathcal{F}}} \ldots$ is an infinite innermost $\mathcal{R}_1^{\mathcal{F}}$-derivation contradicting $\mathrm{SIN}(\mathcal{R}_1^{\mathcal{F}})$. $\square$

Next we shall provide sufficient conditions for the innermost reduction properties IRP_1, IRP_2 and IRP_3. Having again a look at Example 21 we see that the system $\mathcal{R}_1$ there is non-confluent and has a rule with extra variables which seems to be essential. And indeed, forbidding extra variables or requiring confluence turns out to be crucial as will be shown next.

Lemma 36 *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs without extra variables, with $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then the innermost reduction properties* $\mathrm{IRP}_k(\mathcal{R}_j, \mathcal{R})$

hold for $j = 1,2$ and $k = 1,2,3$.

Proof. It suffices to show the following for $j = 1,2$:

$$\forall s \in \mathcal{T}(\mathcal{F},\mathcal{V}) : s \underset{i}{\rightarrow}_{\mathcal{R}_j/\mathcal{R}} t \implies s \underset{i}{\rightarrow}_{\mathcal{R}_j} t \implies s \underset{i}{\rightarrow}_j t\,.$$

This is straightforward by induction on the depth of rewriting, and exploiting the absence of extra variables (since the step $s \underset{i}{\rightarrow}_{\mathcal{R}_j/\mathcal{R}} t$ is innermost and due to the absence of extra variables we know that for verifying the conditions of the applied $\mathcal{R}_j$-rule only outer $\rightarrow_j$-steps are possible). $\square$

For the case of confluent CTRSs we need two more technical lemmas from [22].

Lemma 37 *(cf. Propositions 3.6 and 3.8 in [22]) Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two disjoint confluent CTRSs and $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$. Then $\rightarrow_{1,2}$ is confluent and $\downarrow_{1,2}$ coincides with $\leftrightarrow^*_{\mathcal{R}}$.*

Lemma 38 *(cf. Prop. 3.13 in [22]) Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two disjoint, confluent CTRSs, $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ and $s_1, \ldots, s_n, t_1, \ldots, t_n$ be black terms. Then, for every substitution σ with $\sigma(s_i) \downarrow_{1,2} \sigma(t_i)$ for $i = 1, \ldots, n$ there exists a substitution τ such that $\sigma \rightarrow^*_{1,2} \tau$ and $\tau(s_i) \downarrow^o_1 \tau(t_i)$ for $i = 1, \ldots, n$.*

Lemma 39 *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint confluent CTRSs, with $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2})$. Then the innermost reduction properties* $\mathrm{IRP}_k(\mathcal{R}_j, \mathcal{R})$ *hold for $j = 1,2$ and $k = 1,2,3$.*

Proof. It suffices to show the following for $j = 1,2$:

$$\forall s \in \mathcal{T}(\mathcal{F},\mathcal{V}) : s \underset{i}{\rightarrow}_{\mathcal{R}_j/\mathcal{R}} t \implies s \underset{i}{\rightarrow}_{\mathcal{R}_j} t \implies s \underset{i}{\rightarrow}_j t$$

Now consider a step $s \underset{i}{\rightarrow}_{\mathcal{R}_j/\mathcal{R}} t$ using some $\mathcal{R}_j$-rule $P \implies l \rightarrow r$ with matching substitution σ (which may also instantiate the extra variables in the conditions P). Hence, we have $\sigma(u) \downarrow_{\mathcal{R}} \sigma(v)$ for all conditions $u \downarrow v$ in P. By Lemma 37 and Lemma 38 we obtain the existence of some substitution τ with $\sigma \rightarrow^*_{1,2} \tau$ and $\tau(u) \downarrow^o_j \tau(v)$ for all $u \downarrow v$ in P. Since the step $s \underset{i}{\rightarrow}_{\mathcal{R}_j/\mathcal{R}} t$ is innermost we know that $\sigma(x)$ is irreducible for all $x \in dom(\sigma) \cap \mathcal{V}(l)$, hence σ and τ coincide on $\mathcal{V}(l)$. Thus we get $\sigma(l) = \tau(l)$, $\sigma(r) = \tau(r)$ which implies $s \underset{i}{\rightarrow}_j t$ as desired. $\square$

Note in particular that Theorem 31 together with Lemmas 32, 35, 36 and 39 shows that WN, WIN and SIN are preserved under signature extensions for confluent CTRSs as well as for CTRSs without extra variables.

By combining Theorem 31 and the Lemmas 32, 35, 36 and 39 we now obtain the following modularity results.

Theorem 40 *Weak termination, weak innermost termination and (strong) innermost termination are modular properties for CTRSs without extra variables (in the conditions) and for confluent CTRSs.*

Together with Theorem 18 this implies in particular the following.

Corollary 41 *Semi-completeness is modular for CTRSs.*

4.3 Modularity Results for Termination and Completeness

Before turning to further modularity results concerning termination and completeness let us now have a look upon the fact that for disjoint CTRSs $\mathcal{R}_1$, $\mathcal{R}_2$ confluence of $\mathcal{R}_1 \oplus \mathcal{R}_2$ implies confluence of both $\mathcal{R}_1$ and $\mathcal{R}_2$. As already mentioned this is straightforward for systems without extra variables but less obvious if extra variables are allowed. The problem is the following. Let the disjoint union $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ of $\mathcal{R}_1$ and $\mathcal{R}_2$ be confluent. Then we would like to show that for any black term s, whenever we have $s \to_{\mathcal{R}_1}^* t_1$ and $s \to_{\mathcal{R}_1}^* t_2$ (with t_1, t_2 necessarily black, too) then $t_1 \to_{\mathcal{R}_1}^* u$, $t_2 \to_{\mathcal{R}_1}^* u$ for some black term u. By confluence of $\mathcal{R}_1 \oplus \mathcal{R}_2$ we know that there exists (a black term) u with $t_1 \to_{\mathcal{R}}^* u$, $t_2 \to_{\mathcal{R}}^* u$. Now we would like to conclude that all the steps in $t_1 \to_{\mathcal{R}}^* u$, $t_2 \to_{\mathcal{R}}^* u$ are $\mathcal{R}_1$-steps. So, in a sense we would like to be sure that it is not necessary to substitute non-black terms for the extra variables of the black rules (in order to verify their associated, correspondingly instantiated conditions) applied in $t_1 \to_{\mathcal{R}}^* u$, $t_2 \to_{\mathcal{R}}^* u$. This can indeed be proved as will be sketched now. First we need another technical result adapted from [22].

Lemma 42 *Let $\mathcal{R}_1$, $\mathcal{R}_2$ be disjoint CTRSs such that $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is confluent, and let $s_1, \ldots, s_n$, $t_1, \ldots, t_n$ be black terms. Then, for every substitution σ with $\sigma(s_i) \downarrow_{\mathcal{R}} \sigma(t_i)$ for $i = 1, \ldots, n$ there exists a substitution τ such that $\sigma \to_{\mathcal{R}}^* \tau$ and $\tau(s_i) \downarrow_1^o \tau(t_i)$ for $i = 1, \ldots, n$.*

Proof. We omit a detailed proof here, because it is quite lengthy and completely analogous to the proof of [22, Proposition 3.13], cf. Lemma 38 above (more precisely, Lemma 42 is obtained from [22, Proposition 3.13] by replacing $\to_{1,2}$ by $\to_{\mathcal{R}}$ and accordingly, $\downarrow_{1,2}$ by $\downarrow_{\mathcal{R}}$, and assuming confluence of $\mathcal{R}$ instead of confluence of $\to_{1,2}$). $\square$

Lemma 43 *Let $\mathcal{R}_1$, $\mathcal{R}_2$ be disjoint CTRSs such that $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is confluent, and let s be a black term with $s \to_{\mathcal{R}_1/\mathcal{R}} t$. Then $s \to_{\mathcal{R}_1} t$ holds, too (by applying the same $\mathcal{R}_1$-rule).*

Proof. Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$, s and t as above. Moreover assume that s_1,...,s_n, t_1,...,t_n are the (black) condition terms of the applied $\mathcal{R}_1$-rule $P \Longrightarrow l \to r$ with matching substitution σ and $\sigma(s_i) \downarrow_{\mathcal{R}} \sigma(t_i)$ for $i = 1, ..., n$. Note that – due to the possibility of extra variables in P – we have to take into account that σ may substitute non-black terms for some extra variables. Using Lemma 22 we can decompose σ into $\sigma_1 \circ \sigma_2$ with σ_1 black, σ_2 top white, $dom(\sigma_1) \supseteq V(l)$ and such that σ_2 does not affect the σ_1-instances of the left hand side variables, i.e. $\sigma_2(\sigma_1(x)) = \sigma_1(x)$ for all $x \in V(l)$. Then we would like to conclude that there exists a black substitution σ' with $\sigma'(s_i) \downarrow_{\mathcal{R}_1} \sigma'(t_i)$ and $\sigma'(x) = \sigma(x) = \sigma_1(x)$ for all $x \in V(l)$, i.e. the substitutions σ and σ' may differ but only on the extra variables. Now, from $\sigma_2(\sigma_1(s_i)) \downarrow_{\mathcal{R}} \sigma_2(\sigma_1(t_i))$ we know that $\sigma_1(s_i)$, $\sigma_1(t_i)$ are black, hence Lemma 42 yields a substitution τ with $\sigma_2 \to^*_{\mathcal{R}} \tau$ and $\tau(\sigma_1(s_i)) \downarrow^{\circ}_1 \tau(\sigma_1(t_i))$ for $i = 1, \ldots, n$ such that by applying again Lemma 22 we obtain a decomposition of $\sigma_1 \circ \tau$ into $\sigma'_1 \circ \tau'$ with σ'_1 black and τ' top white and moreover $\sigma(x) = \sigma_1(x) = \sigma'_1(x) = \tau'(\sigma'_1(x))$ for all $x \in V(l)$. Applying Lemma 24 now yields $\tau''(\sigma'_1(s_i)) \downarrow^{\circ}_1 \tau''(\sigma'_1(t_i))$ for every τ'' with $\tau' \propto \tau''$. We choose τ'' to be defined by $\tau''(x) = z$ for all $x \in dom(\tau')$ with z some new variable. Hence we get $\tau''(\sigma'_1(s_i)) \downarrow^{\circ}_1 \tau''(\sigma'_1(t_i))$ with $(\sigma'_1 \circ \tau'')$ a black substitution satisfying $\sigma(x) = \sigma_1(x) = \sigma'_1(x) = \tau''(\sigma'_1(x))$ for all $x \in V(l)$. Since $\stackrel{\circ}{\to}_1$ is a subset of $\to_{\mathcal{R}_1}$ we are done. This means that there exists a black substitution satisfying the conditions w.r.t. $\mathcal{R}_1$ which coincides with the original (possibly mixed) substitution σ on $V(l)$. □

As straightforward consequence of Lemma 43 we obtain the following.

Corollary 44 *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs (with extra variables in the conditions allowed) such that $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$ is confluent. Then both $\mathcal{R}_1^{\mathcal{F}_1}$ and $\mathcal{R}_2^{\mathcal{F}_2}$ are confluent, too.*

Let us come back now to the question under which conditions termination and completeness are modular for CTRSs. Combining different results enables us to prove the following.

Theorem 45 *Termination and completeness are modular for non-overlapping CTRSs.*

Proof. Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two disjoint non-overlapping and terminating CTRSs. Applying Theorem 14 yields confluence of $\mathcal{R}_i$ for $i = 1, 2$. By assumption we know in particular that both systems are (strongly) innermost terminating. Hence, by Theorem 40 we get that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is (strongly) innermost terminating, too. The property of being non-overlapping is obviously modular for CTRSs. Hence $\mathcal{R}_1 \oplus \mathcal{R}_2$ is (strongly) innermost terminating and non-overlapping. Finally, applying Theorem 15 yields (strong) termination of $\mathcal{R}_1 \oplus \mathcal{R}_2$ which – again by Theorem 14 – implies confluence of $\mathcal{R}_1 \oplus \mathcal{R}_2$. Hence, $\mathcal{R}_1 \oplus \mathcal{R}_2$ is a confluent, (strongly) terminating and non-overlapping CTRS. Vice versa, assume that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is non-overlapping and terminating, hence complete. Then we know that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-overlapping and terminating, hence complete (by Theorem 14). □

Note that in the above proof we did not make use of the modularity of confluence (Theorem 18). However, this is necessary for the case of conditional overlay systems.

Theorem 46 *Termination and completeness are modular for conditional overlay systems with joinable critical pairs.*

Proof. Let $\mathcal{R}_1$, $\mathcal{R}_2$ be two disjoint terminating, conditional overlay systems with joinable critical pairs. Applying Theorem 14 yields confluence of $\mathcal{R}_i$ for $i = 1, 2$. By assumption we know in particular that both systems are (strongly) innermost terminating. Hence, by Theorem 40 we get that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is (strongly) innermost terminating, too. The property of being a conditional overlay system is obviously modular for CTRSs. Hence $\mathcal{R}_1 \oplus \mathcal{R}_2$ is a (strongly) innermost terminating, conditional overlay system. Now, in order to be able to apply Theorem 16 for inferring (strong) termination of $\mathcal{R}_1 \oplus \mathcal{R}_2$ we need to establish joinability of all (conditional) critical pairs of $\mathcal{R}_1 \oplus \mathcal{R}_2$. Since both $\mathcal{R}_1$ and $\mathcal{R}_2$ are confluent we know by Theorem 18 that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is confluent, too. Hence, in particular, all critical pairs of $\mathcal{R}_1 \oplus \mathcal{R}_2$ must be joinable. Applying Theorem 16 now yields that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is a (strongly) terminating and confluent conditional overlay system (with joinable critical pairs, of course). Vice versa, assume that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is a conditional overlay system with joinable critical pairs which is terminating, hence confluent and complete. Then we know that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating conditional overlay systems. By Corollary 44 confluence of $\mathcal{R}_1 \oplus \mathcal{R}_2$ implies confluence of both $\mathcal{R}_1$ and $\mathcal{R}_2$ (hence in particular also joinability of critical pairs). □

4.4 Further Extensions and Counterexamples

Note that – compared to the unconditional case – the proofs of Theorem 45 and Theorem 46 are more complicated. As pointed out above, one additional complication is due to the fact that some basic properties which are trivial for unconditional TRSs become non-trivial for CTRSs.

4.4.1 On Non-Modularity of WCR *and* JCP

Another reason consists in the fact that, in contrast to the unconditional case, both local confluence and joinability of all critical pairs are not modular for CTRSs in general.

Example 47 *([22]) Consider the disjoint CTRSs*

$$\mathcal{R}_1 = \begin{cases} x \downarrow z \wedge z \downarrow y & \Longrightarrow \quad f(x,y) \to x \\ x \downarrow z \wedge z \downarrow y & \Longrightarrow \quad f(x,y) \to y \end{cases} \qquad \mathcal{R}_2 = \begin{cases} b \to a \\ b \to c \\ c \to b \\ c \to d \end{cases}$$

It is easy to show that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are locally confluent (cf. [22]) – $\mathcal{R}_1$ is even confluent – and hence have joinable critical pairs. But the only (conditional) critical pair of $\mathcal{R}_1$, namely

$$x \downarrow z \wedge z \downarrow y \quad \Longrightarrow \quad \langle x, y \rangle$$

is not joinable in $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ since we have $f(a,d) \to_{\mathcal{R}} a$ and $f(a,d) \to_{\mathcal{R}} d$, due to $a \downarrow_{\mathcal{R}} b \downarrow_{\mathcal{R}} d$ (the extra variable z here is substituted by b), but not $a \downarrow_{\mathcal{R}} d$. This means that $\mathcal{R}$ is neither locally confluent nor has joinable critical pairs.

Note that extra variables are not essential for the existence of such counterexamples. To see this consider the following modified version of Example 47.

Example 48 *Let $\mathcal{R}_2$ be as above and $\mathcal{R}_1$ be given by*

$$\mathcal{R}_1 = \begin{cases} x \downarrow z \wedge z \downarrow y & \Longrightarrow \quad f(x,z,y) \to x \\ x \downarrow z \wedge z \downarrow y & \Longrightarrow \quad f(x,z,y) \to y \end{cases} .$$

Again it is easy to show that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are locally confluent and hence have joinable critical pairs. But the only (conditional) critical pair of $\mathcal{R}_1$, namely

$$x \downarrow z \wedge z \downarrow y \quad \Longrightarrow \quad \langle x, y \rangle$$

is not joinable in $\mathcal{R} := \mathcal{R}_1 \oplus \mathcal{R}_2$ *since we have* $f(a,b,d) \to_{\mathcal{R}} a$ *and* $f(a,b,d) \to_{\mathcal{R}} d$*, due to* $a \downarrow_{\mathcal{R}} b \downarrow_{\mathcal{R}} d$*, but not* $a \downarrow_{\mathcal{R}} d$*.*

Note that in Examples 47 and 48 above the CTRS $\mathcal{R}_2$ is not (strongly) innermost terminating. But requiring SIN is still not sufficient for guaranteeing modularity of JCP as can be seen from the following example.

Example 49 *Let* $\mathcal{R}_1^{\mathcal{F}_1}$*,* $\mathcal{R}_2^{\mathcal{F}_2}$ *be given by*

$$\mathcal{R}_1 = \begin{cases} x \downarrow z \wedge z \downarrow y & \Longrightarrow & f(x,z,y) \to x \\ x \downarrow z \wedge z \downarrow y & \Longrightarrow & f(x,z,y) \to y \end{cases} \qquad \mathcal{R}_2 = \begin{cases} g(b) \to g(c) \\ g(c) \to g(b) \\ g(b) \to a \\ g(c) \to d \\ b \to e \\ c \to e' \\ g(e) \to a \\ g(e') \to d \end{cases}$$

over $\mathcal{F}_1 = \{f\}$ *and* $\mathcal{F}_2 = \{g, a, b, c, d, e, e'\}$*, respectively. Here, both* $\mathcal{R}_1^{\mathcal{F}_1}$ *and* $\mathcal{R}_2^{\mathcal{F}_2}$ *are locally confluent, have joinable critical pairs, no extra variables and are (strongly) innermost terminating (note that the critical steps in* $\mathcal{R}_2$*, namely* $g(b) \to g(c)$ *and* $g(c) \to g(b)$*, are non-innermost). But their disjoint union* $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$ *does neither satisfy* JCP$(\mathcal{R}^{\mathcal{F}})$ *nor* WCR$(\mathcal{R}^{\mathcal{F}})$*, since we have e.g.* $a \leftarrow_{\mathcal{R}^{\mathcal{F}}} f(a, g(b), d) \to_{\mathcal{R}^{\mathcal{F}}} d$ *(due to* $a \downarrow_{\mathcal{R}^{\mathcal{F}}} g(b) \downarrow_{\mathcal{R}^{\mathcal{F}}} d$*) but not* $a \downarrow_{\mathcal{R}^{\mathcal{F}}} d$*.*

These counterexamples indicate that it is not easy to find reasonable sufficient conditions for the modularity of JCP and WCR (which do not imply modularity of CR). This corresponds somehow to the well-known fact that – in contrast to the situation for unconditional TRSs – joinability of (all) critical pairs, local confluence and confluence cannot be neatly disentangled from each other in the conditional case (cf. [5]).

4.4.2 Semantic Versions of Non-Overlapping and Overlay CTRSs

An obvious question arising from Theorems 45 and 46 is whether these results do also hold for the more general semantic versions of the properties of being non-overlapping and being an overlay system (see Remark 8). This is indeed the case as we will show below.

Definition 50 A CTRS $\mathcal{R}$ is said to be *semantically non-overlapping (*SEM-NO*)* if all its critical pairs are infeasible. $\mathcal{R}$ is said to be a *(conditional) semantic overlay system (*SEM-OS*)* if all its feasible critical pairs are critical overlays, i.e., are obtained by overlapping rules at root position.

Clearly, a (syntactically) non-overlapping CTRS is semantically non-overlapping, too, but not vice versa in general. Analogously, a (syntactical) overlay CTRS is a semantic overlay system but not vice versa in general.

Note that the syntactical versions of the properties of being non-overlapping and of being an overlay system can be easily tested (for finite systems) whereas establishing their semantic versions may be very difficult. The reason is that e.g. for proving SEM-NO for some CTRS $\mathcal{R}$ one has to show that all (conditional) critical pairs of $\mathcal{R}$ are infeasible. But this is undecidable in general.

Furthermore note that, considering single CTRSs, it is fairly obvious to see that most results which depend on the property of being non-overlapping or being an overlay system do not only hold for the usual syntactical but also for the more general semantic versions of these properties. In particular, this applies to Theorems 15 and 16.

But concerning preservation properties of combined systems the situation is quite different and considerably more complicated as we shall see.

First let us note that – in contrast to OS – the property SEM-OS is not modular as witnessed by the following example.

Example 51 *Consider the disjoint CTRSs*

$$\mathcal{R}_1 = \begin{cases} x \downarrow b \wedge x \downarrow c \implies a \to a \\ x \downarrow b \wedge x \downarrow c \implies f(a) \to a \end{cases} \qquad \mathcal{R}_2 = \begin{cases} G(x,y) \to x \\ G(x,y) \to y \end{cases}.$$

Here we have SEM-OS$(\mathcal{R}_j)$ *for $j = 1,2$ as is easily shown (note that $\mathcal{R}_1$ has no feasible critical pair). But the disjoint union $\mathcal{R}_1 \oplus \mathcal{R}_2$ has the critical pair $x \downarrow b \wedge x \downarrow c \implies \langle f(a) = a \rangle$ (between the first and the second $\mathcal{R}_1$-rule) which is feasible (by replacing the extra variable x by $G(b,c)$) but not an overlay. Hence,* SEM-OS$(\mathcal{R}_1 \oplus \mathcal{R}_2)$ *does not hold.*

Note that Example 48 above shows that SEM-OS $\wedge$ JCP is also not modular for CTRSs (even without extra variables). The corresponding problem for the non-overlapping property remains open. Actually, we do don't know whether SEM-NO and SEM-NO $\wedge$ JCP are modular or not. But SEM-OS $\wedge$ CR turns out to be modular as well as SEM-NO $\wedge$ CR.

Lemma 52 SEM-OS $\wedge$ CR *is a modular property of CTRSs.*

Proof. Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be disjoint CTRSs with SEM-OS($\mathcal{R}_j^{\mathcal{F}_j}$) and CR($\mathcal{R}_j^{\mathcal{F}_j}$) for $j = 1, 2$. Let $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$. By Theorem 18 we know CR($\mathcal{R}^{\mathcal{F}}$). Now, for a proof by contradiction assume that SEM-OS($\mathcal{R}^{\mathcal{F}}$) does not hold. Thus there exists a critical pair $P \implies \langle s = t \rangle$ in $\mathcal{R}^{\mathcal{F}}$ which is not an overlay and feasible w.r.t. $\mathcal{R}^{\mathcal{F}}$. Assume further w.l.o.g. that $P \implies \langle s = t \rangle$ is a critical pair between two $\mathcal{R}_1$-rules. By SEM-OS($\mathcal{R}_1^{\mathcal{F}_1}$) we know that $P \implies \langle s = t \rangle$ is infeasible w.r.t. $\mathcal{R}_1^{\mathcal{F}_1}$. Hence, for all $\mathcal{F}_1$-substitutionsσ (i.e., with $\sigma(x) \in \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ for every $x \in dom(\sigma)$) we have $\neg(\sigma(P) \downarrow_{\mathcal{R}_1})$. But there exists some mixed $(\mathcal{F}_1 \uplus \mathcal{F}_2)$-substitution σ with $\sigma(P) \downarrow_{\mathcal{R}}$. Applying Lemma 37 yields $\sigma(u) \downarrow_{1,2} \sigma(v)$ for all $u \downarrow v$ in P. Since u and v are black terms we conclude by Lemma 42 that there exists some substitution τ with $\sigma \to_{1,2}^* \tau$ such that $\tau(u) \downarrow_1^o \tau(v)$ holds for all $u \downarrow v$ in P. Lemma 22 yields a decomposition of τ into $\tau_2 \circ \tau_1$ with τ_1 black, τ_2 top white and $\tau_2(\tau_1(u)) \downarrow_1^o \tau_2(\tau_1(v))$ for all $u \downarrow v$ in P. From Lemma 24 we get $\tau'(\tau_1(u)) \downarrow_1^o \tau'(\tau_1(v))$ for every τ' satisfying $\tau_2 \propto \tau'$ and every $u \downarrow v$ in P. We choose now τ' to be defined by $\tau'(x) = z$ for every $x \in dom(\tau_2)$ with z some new variable. Thus $\tau'' = \tau' \circ \tau_1$ is a black substitution with $\tau''(u) \downarrow_1^o \tau''(v)$ for all $u \downarrow v$ in P. But this means that the critical pair $P \implies \langle s = t \rangle$ is feasible w.r.t. $\mathcal{R}_1^{\mathcal{F}_1}$. Hence we have a contradiction.

Vice versa, for $\mathcal{R}^{\mathcal{F}} = \mathcal{R}_1^{\mathcal{F}_1} \oplus \mathcal{R}_2^{\mathcal{F}_2}$ with SEM-OS($\mathcal{R}^{\mathcal{F}}$) and CR($\mathcal{R}^{\mathcal{F}}$) we obtain CR($\mathcal{R}_j^{\mathcal{F}_j}$) for $j = 1, 2$ by Corollary 44. Finally, SEM-OS($\mathcal{R}_j^{\mathcal{F}_j}$) for $j = 1, 2$ follows from SEM-OS($\mathcal{R}$) (since feasibility is monotonic w.r.t. signature extensions). $\square$

Similarly we obtain the following result.

Lemma 53 SEM-NO $\wedge$ CR *is a modular property of CTRSs.*

Proof. The proof is analogous to the proof of Lemma 52. $\square$

Combining Lemmas 53 and 52 with (the generalized semantic version of) Theorem 16 we obtain the following results.

Theorem 54 *Both* SEM-NO $\wedge$ SIN *and* SEM-OS $\wedge$ CR $\wedge$ SIN *are modular properties of CTRSs.*

Theorem 55 *Completeness is modular for semantically non-overlapping CTRSs as well as for semantic (conditional) overlay systems.*

The following tables summarize the preservation behaviour of the discussed properties of CTRSs under signature extensions and under disjoint unions. We recall that the abbreviation NEV stands for the property of CTRSs to have no extra variables in the conditions (remember that extra variables in right-hand sides are forbidden by definition).

Preservation Properties of CTRSs under Signature Extensions		
Property	is preserved	reason/reference
SN	+	Lemma 29
CR	+	[22]
WN	–	Example 20
WN $\wedge$ NEV	+	Theorem 40
WN $\wedge$ CR	+	Theorem 40
WIN	–	Example 20
WIN $\wedge$ NEV	+	Theorem 40
WIN $\wedge$ CR	+	Theorem 40
SIN	–	Example 21
SIN $\wedge$ NEV	+	Theorem 40
SIN $\wedge$ CR	+	Theorem 40

Modularity Properties of CTRSs		
Property	is modular	reason/reference
SN	−	[31]
CR	+	[22]
WN ∧ NEV	+	Theorem 40
WN ∧ CR	+	Theorem 40
WIN ∧ NEV	+	Theorem 40
WIN ∧ CR	+	Theorem 40
SIN ∧ NEV	+	Theorem 40
SIN ∧ CR	+	Theorem 40
WCR	−	Example 47 ([22])
WCR ∧ OS	−	Example 47 ([22])
JCP	−	Example 47 ([22])
JCP ∧ OS	−	Example 47 ([22])
JCP ∧ OS ∧ NEV	−	Example 48
JCP ∧ SIN	−	Example 49
JCP ∧ SIN ∧ NEV	−	Example 49
SN ∧ CR	−	[32]
SN ∧ CR ∧ NO	+	Theorem 45
SN ∧ CR ∧ OS	+	Theorem 46
NO	+	trivial
OS	+	trivial
SEM-NO	?	
SEM-NO ∧ JCP	?	
SEM-OS	−	Example 51
SEM-NO ∧ CR	+	Lemma 53
SEM-OS ∧ CR	+	Lemma 52
SN ∧ CR ∧ SEM-NO	+	Theorem 55
SN ∧ CR ∧ SEM-OS	+	Theorem 55

5 Combined CTRSs with Shared Constructors

Let us now investigate whether the obtained results can be generalized to non-disjoint unions of CTRSs which is important from a practical point of view. More precisely, we consider *constructor sharing combinations* (combined systems which share at most constructors), i.e., CTRSs where the respective sets of left-hand side root symbols (the *defined* symbols) are disjoint. Such combinations have been introduced by Kurihara & Ohuchi [15] and studied subsequently by many authors. Slightly less general are *combinations of constructor systems* (with disjoint sets of defined symbols) considered in Middeldorp & Toyama [23] (for unconditional TRSs) and Middeldorp [21] (for CTRSs).

Since the proofs of many modularity results for the disjoint union case mainly rely on the *layered* structure of terms and on the *rank decreasingness* property of reduction in the combined system, most of these results also carry over to constructor sharing combinations when taking into account the additional phenomenon that *layer collapses* are not only caused by *collapsing* rules (i.e., rules with a variable as right-hand side) but also by *constructor-lifting* rules (i.e., rules with a constructor as right-hand side root symbol).

We show now that the main results of the previous section indeed carry over to constructor sharing combinations. For the preservation results under signature extensions there is nothing to do. The abstract criteria for the preservation of the restricted termination properties WN, WIN and SIN (cf. Theorem 31) also easily carry over. Modularity of WN, WIN and SIN for constructor sharing combinations without extra variables is easy, too. But, unfortunately, the other modularity results in the disjoint union case, in particular for termination and completeness, make essential use of the preservation of confluence. And it is well-known that for constructor sharing combinations confluence may get lost, even in the unconditional case.

Example 56 *([15]) Consider the disjoint TRSs*

$$\mathcal{R}_1 = \left\{ \begin{array}{c} f(x,x) \to a \\ f(x,c(x)) \to b \end{array} \right\} \qquad \textit{and} \qquad \mathcal{R}_2 = \left\{ d \to c(d) \right\}$$

which share the constructor c. Both systems are clearly confluent but their combination is not, since e.g. the term $f(d,d)$ has two distinct normal forms a and b.

This problem is due to the fact that in the combined system a term need not have a *preserved reduct*, i.e., a reduct with a stable layer structure. This property turned out to be crucial (and was easily verified) in the simplified proof of

Toyama's theorem ([13]) stating that confluence is modular for disjoint unions of (unconditional) TRSs. By guaranteeing the existence of preserved reducts, the proof of [13] essentially carries over to constructor sharing combinations of TRSs as recognized by Ohlebusch ([26]). One straightforward criterion to ensure the crucial preservation property property is to require weak normalization of the involved TRSs which implies the modularity of semi-completeness for constructor sharing TRSs ([26]). The preservation property was already implicitly used by Middeldorp ([20], [22]) for proving modularity of confluence for disjoint unions of CTRSs. By combining this proof structure of [20] with [26] Ohlebusch recently succeeded in proving the modularity of semi-completeness for constructor sharing CTRSs ([27]). The latter result now enables us to extend our modularity criteria for termination and completeness of CTRSs from disjoint unions to constructor sharing combinations, too. This will be exhibited now. First let us fix the extended setting.

Definition 57 Let $\mathcal{R}^{\mathcal{F}}$, $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be CTRSs.

(1) The set $\mathcal{D}$ of *defined* function symbols of $\mathcal{R}^{\mathcal{F}}$ is given by $\mathcal{D} = \{root(l) \mid s_1 \downarrow t_1 \wedge \ldots \wedge s_n \downarrow t_n \implies l \to r \in \mathcal{R}^{\mathcal{F}}\}$, the set $\mathcal{C}$ of *constructors* (or *constructor* function symbols) by $\mathcal{C} = \mathcal{F} \setminus \mathcal{D}$. Terms containing no defined function symbols are *constructor terms*.

(2) $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ are called *constructor-sharing* if they share at most constructors, i.e., if $\mathcal{D}_1 \cap \mathcal{F}_2 = \mathcal{D}_2 \cap \mathcal{F}_1 = \emptyset$. In this case, their union $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$ is called the *combined CTRS with shared constructors* $(\mathcal{C}_1 \cap \mathcal{C}_2)$. A property $\mathcal{P}$ of CTRSs is called *modular for constructor-sharing CTRSs* if for all constructor-sharing CTRSs $\mathcal{R}_1$, $\mathcal{R}_2$ their union $\mathcal{R}_1 \cup \mathcal{R}_2$ has the property $\mathcal{P}$ if and only if both $\mathcal{R}_1$ and $\mathcal{R}_2$ have the property $\mathcal{P}$.

(3) $\mathcal{R}^{\mathcal{F}}$ is a *conditional constructor system* if all arguments of left-hand sides of rules in $\mathcal{R}$ are constructor terms.

Subsequently, we tacitly assume that $\mathcal{R}_i^{\mathcal{F}_i}$, $i = 1, 2$, are constructor-sharing CTRSs with corresponding sets $\mathcal{C}_i$ and $\mathcal{D}_i$ of constructors and defined function symbols, respectively. Reduction in the combined system $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \cup \mathcal{R}_2)^{\mathcal{F}_1 \cup \mathcal{F}_2}$ is denoted by $\to_{\mathcal{R}}$ or simply $\to$.

The following colour scheme has turned out to be useful for enhancing readability of statements about the combined system.

Definition 58 Function symbols from $\mathcal{F}_1 \setminus (\mathcal{C}_1 \cap \mathcal{C}_2)$ are called *black* , those from $\mathcal{F}_2 \setminus (\mathcal{C}_1 \cap \mathcal{C}_2)$. Shared constructors (i.e., the function symbols of $\mathcal{C}_1 \cap \mathcal{C}_2$) and variables are considered to be *transparent*. A *black (white) term* does not contain white (black) symbols. A *transparent term* contains only shared constructors and variables. A term is called *top black (top white, top-transparent)* if its root symbol is black (white, transparent).

Note that according to this colour scheme black (white) terms are *pure* $\mathcal{T}(\mathcal{F}_1, \mathcal{V})$-terms ($\mathcal{T}(\mathcal{F}_2, \mathcal{V})$-terms), i.e., without function symbols from $\mathcal{F}_2$ ($\mathcal{F}_1$). This is only slightly different from defining a function symbol to be *black (white, transparent)* if it is from $\mathcal{D}_1 = \mathcal{F}_1 \setminus \mathcal{C}_1$ ($\mathcal{D}_2 = \mathcal{F}_2 \setminus \mathcal{D}_2$, $\mathcal{C}_1 \cup \mathcal{C}_2$) but has some subtle advantages when generalizing constructor sharing systems to *composable* ones where a common rule part is allowed, too (cf. [28]).

For representing terms in the combined system we adapt the notations from the disjoint union case.

Definition 59 Let $s = C[s_1, \ldots, s_n]$ for some (non-empty) context $C[, \ldots,]$. If s is top transparent, and s_i top black or top white, for $i = 1, \ldots, n$, we write $s = C^t[\![s_1, \ldots, s_n]\!]$. If s is top black (top white), and s_i top white (top black), for $i = 1, \ldots, n$, we write $s = C^b[\![s_1, \ldots, s_n]\!]$ ($s = C^w[\![s_1, \ldots, s_n]\!]$). These notations generalize in a straightforward way to 'degenerate' cases of context representations (cf. Section 2.1).

Definition 60 The *rank* of a top black term s is defined by

$$rank(s) = \begin{cases} 1 & \text{if } s \text{ is black} \\ 1 + max\{rank(s_i) \mid 1 \leq i \leq n\} & \text{if } s = C^b[\![s_1, \ldots, s_n]\!] \end{cases}$$

(and analogously for s top white). If s is top transparent then its *rank* is defined by

$$rank(s) = \begin{cases} 0 & \text{if } s \text{ is transparent} \\ max\{rank(s_i) \mid 1 \leq i \leq n\} & \text{if } s = C^t[\![s_1, \ldots, s_n]\!] \end{cases}$$

Definition 61 For a top black term $s = C^b[\![s_1, \ldots, s_n]\!]$, if $s \to t$ by reducing one of the s_j's then this step is called an *inner* step which is denoted by $s \xrightarrow{i} t$ (analogously for s top white). For a top transparent term $s = C^t[\![s_1, \ldots, s_n]\!]$, if $s \to t$ then t must be of the form $t = C^t[s_1, \ldots, s_{j-1}, t_j, s_{j+1}, \ldots, s_n]$ for some t_j. In this case the step $s \to t$ is *inner* (again denoted by $s \xrightarrow{i} t$) if $s_j \xrightarrow{i} t_j$, and *outer* (denoted by $s \xrightarrow{o} t$) if $s_j \xrightarrow{o} t_j$. Using these conventions the relations $\to_1$, $\to_2$ and $\to_{1,2}$ of Definition 23 extend to the constructor-sharing case in the obvious way.

After these preliminary considerations and adaptations of the terminology for constructor-sharing combinations we are prepared to carry over the main results from the disjoint union case. As already mentioned, for the preservation results under signature extensions there is nothing to do. First we observe that Lemmas 22, 24 and 27 carry over easily using the same proof structure as formerly. Concerning the abstract modularity criteria for weak normalization

(WN), weak innermost normalization (WIN) and strong innermost normalization (SIN) we remark that Definition 30 carries over to constructor-sharing CTRSs directly. Now it is straightforward to verify that the Lemmas 32, 33, 35 and 36 extend to the constructor-sharing case, too, using the same proof structures and essentially the same arguments, respectively. In particular we note that for the additional case of a top transparent term $s = C^t[\![s_1, \ldots, s_n]\!]$, s is WN (WIN, SIN) w.r.t. the combined system if and only if all its maximal aliens $s_1, \ldots, s_n$ are so. This is simply due to the fact there are no rules applicable in s in the topmost transparent layer (which consists of constructors and variables, only). And moreover, any innermost step in some s_i is also innermost in s. Hence, as for disjoint unions, combining these extended results we obtain the following first modularity result for constructor-sharing CTRSs.

Theorem 62 *Weak termination, weak innermost termination and strong innermost termination are modular for constructor-sharing CTRS without extra variables (in the conditions of the rules).*

For extending the other results, however, which rely on the modularity of confluence for disjoint unions of CTRSs, we need the following recent results of Ohlebusch ([27]).

Lemma 63 *(extended version of Lemma 37, [27]) If $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ are semi-complete constructor-sharing CTRSs then $\rightarrow_{1,2}$ is semi-complete and $\downarrow_{1,2}$ coincides with $\leftrightarrow_{\mathcal{R}}^*$.*

Lemma 64 *(extended version of Lemma 38, [27]) Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be constructor-sharing CTRSs such that $\rightarrow_{1,2}$ is semi-complete. Let $s_1, \ldots, s_n, t_1, \ldots, t_n$ be black terms. If σ is a substitution with σ with $\sigma(s_i) \downarrow_{1,2} \sigma(t_i)$ for $i = 1, \ldots, n$ then $\sigma^{\rightarrow}(s_i) \rightarrow_1^o \sigma^{\rightarrow}(t_i)$ for $i = 1, \ldots, n$ (where $\sigma^{\rightarrow}$ denotes the corresponding normalized substitution w.r.t. $\rightarrow_{1,2}$, i.e., for all $x \in dom(\sigma)$, $\sigma^{\rightarrow}(x)$ is the unique normal form of x w.r.t. $\rightarrow_{1,2}$ which exists due to semi-completeness of $\rightarrow_{1,2}$).*

This enables us to prove the following adapted version of Lemma 39.

Lemma 65 *Let $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ be semi-complete constructor-sharing CTRSs. Then the (adapted) innermost reduction properties* $\mathrm{IRP}_k(\mathcal{R}_j, \mathcal{R})$ *hold for $j = 1, 2$ and $k = 1, 2, 3$.*

Proof. The proof is analogous to the proof of Lemma 39 using Lemmas 63, 64 instead of Lemmas 37, 38. □

Now the adapted second part of Theorem 40 reads as follows.

Theorem 66 *Weak termination, weak innermost termination and (strong) innermost termination are modular for constructor-sharing semi-complete CTRSs.*

Proof. Straightforward by combining the adapted versions of Theorem 31 and of the Lemmas 32, 35 with Lemma 65 above (note that for the less interesting direction of these modularity results we also tacitly make use of Corollary 28). □

The 'semi-completeness part' of Theorem 66 is already subsumed by the following main result of Ohlebusch [27].

Theorem 67 *([27]) Semi-completeness is modular for constructor-sharing CTRSs.*

This latter result now permits to generalize our modularity criteria for termination and completeness in Section 4 from the disjoint union case to constructor-sharing combinations.

Theorem 68 *(generalized version of Theorem 45) Termination and completeness are modular for constructor-sharing non-overlapping CTRSs.*

Proof. Straightforward by combining Theorems 14, 15, 66 and 67 (as well as Corollary 28). □

Theorem 69 *Termination and completeness are modular for constructor-sharing conditional overlay systems with joinable critical pairs.*

Proof. Straightforward by combining Theorems 14, 16, 66 and 67. □

By a similar reasoning as in the proof of Lemma 52 (which itself does not extend to the constructor-sharing case, cf. Example 56 above), but exploiting additionally the modularity of semi-completeness and the mentioned related technical results, it is easy to show that Theorems 68 and 69 also hold for the semantic versions of the properties of being non-overlapping and being an overlay system.

6 Discussion, Related Work and Open Problems

Our results above and the counterexamples falsifying a couple of quite obvious and tempting conjectures demonstrate once more the inherent complexity and intricacy of CTRSs. Allowing extra variables in the conditions gives rise to some additional complications and phenomena. We have also seen that termination and confluence properties of CTRSs depend on each other in a very subtle manner. Compared to unconditional TRSs the compositional behaviour of CTRSs under disjoint unions and even under signature extensions has turned out to be much more complicated (cf. e.g. Example 21). In particular, the properties of having (only) joinable critical pairs (JCP), being locally confluent (WCR) and being weakly, weakly innermost and (strongly) innermost terminating (WN, WIN and SIN, respectively) are in general not preserved under disjoint unions, in contrast to the unconditional case. An additional complication for the constructor-sharing case is the well-known fact that confluence is no longer modular in general, but fortunately semi-completeness. The latter result enabled us to generalize our main results, namely modularity of completeness for non-overlapping and overlay CTRSs, to the constructor-sharing case, too.

In the presentation we have focussed on join CTRSs which – from an operational point of view – is the most interesting type. It needs some further investigations (but should not be too difficult) to find out which of our results do also hold for semi-equational CTRSs. Note for instance, that the counterexample 21 does not work anymore for the semi-equational case.

Finally we would like to mention that Middeldorp [21] has obtained some closely related results. He showed in [21] that semi-completeness and completeness are preserved under combinations of conditional constructor systems (with disjoint sets of defined function symbols) without extra variables, and conjectured that this should also hold for systems with extra variables allowed. Theorems 67 and 69 above show that this is indeed the case, even in a slightly more general form (the systems need not be constructor systems). But, in order to be more precise, the main results of [21] do even hold for *composable* conditional constructor systems where the rules for some defined symbols may be shared. Hence, it still remains to be investigated whether semi-completeness and completeness are modular for the corresponding slightly more general case of *composable* CTRSs which may share constructors and must share all defining rules for some defined function symbol whenever that symbol is shared (cf. [28] and [16] for some recent results on composable (C)TRSs). This seems to be plausible but may be technically tedious to achieve. From a practical applicability point of view it seems to be more useful to investigate which properties are preserved under certain hierarchical combinations of CTRSs where one system may refer to defined symbols of the other one but not vice versa. Various results in this direction have been obtained in [17], [2], [18], [8], [6], but mainly for unconditional TRSs.

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