# Confluence of Terminating Conditional Rewrite Systems Revisited*

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**Abstract.** We present a new and powerful criterion for confluence of terminating (terms in) join conditional term rewriting systems. This criterion is based on a certain joinability property for *shared parallel critical peaks* and does neither require the systems to be decreasing nor left-linear nor normal, but only terminating.

## 1 Introduction and Overview

We investigate sufficient criteria for confluence of terminating join conditional term rewriting systems (CTRSs for short). It is well-known that several basic results from unconditional term rewriting do not hold for the conditional case. In particular, in contrast to the unconditional case, local confluence of CTRSs is not equivalent to joinability of all (conditional) critical pairs. In other words, this means that variable overlaps may be critical, too. Even under the assumption of termination this may happen ([5]), i.e., there are terminating CTRSs which have joinable critical pairs but are not (locally) confluent. That means joinability of all critical pairs does not suffice any more for inferring confluence of terminating CTRSs via Newman's Lemma. The problem is reflected somehow in the fact that reasonable proof attempts for showing confluence of terminating systems with joinable critical pairs either have to require

- a strengthened termination assumption (which enables a justified inductive reasoning also when recursing into the conditions), or
- some syntactical restrictions on the rules and / or strengthened forms of joinability for critical pairs (which are exploited in an inductive proof based on the ordinary termination assumption).

Our approach falls into the second category. The basically new idea is to consider as critical divergences not only (feasible instances of) ordinary (conditional) critical peaks consisting of two conflicting ordinary rewrite steps, but also and more generally (feasible instances of) *shared parallel* (conditional) critical peaks consisting of an ordinary root reduction step and a *shared parallel* reduction

* This research was supported in part by 'DFG, SFB 314 D4'.

step which simultaneously contracts several parallel critical occurrences of the same redex all of which are critical w.r.t. to the ordinary root reduction step. Roughly speaking, this idea is motivated by the fact that (the recursive analysis of) a reduction step in the variable part of a rule instance naturally gives rise to a corresponding shared parallel reduction step in the corresponding variable parts of the conditions, due to multiple occurrences of the respective variable in the conditions. Yet, the definition of appropriate joinability conditions for shared parallel critical peaks requires a thorough analysis of the proof-technical requirements when trying to prove confluence of terminating systems. The (inductive) proof of the technical key result, Theorem 6, is fairly involved. Therefore we have tried to present it in a carefully structured and modular way, exhibiting the crucial arguments in sufficient detail. This clear proof structure then enables us to derive several consequences and extensions in a relatively straightforward manner.

The structure of the paper is as follows. After recalling some basics for CTRSs and introducing the crucial notion of shared parallel critical peaks in section 2 we review in section 3 the most important results on confluence of terminating CTRSs, and discuss a motivating example. In the main section 4 we first define the needed shared parallel critical peak property and prove the central Theorem 6. Subsequently, we develop some consequences and extensions, and present illustrative examples. Finally, in section 5 we briefly summarize the main contributions of our approach in view of other known results and related work.

## 2 Preliminaries

We assume familiarity with the basic no(ta)tions, terminology and theory of term rewriting (cf. e.g. [4], [10]).

The set of terms over some given signature $\mathcal{F}$ and some (disjoint) infinite set $\mathcal{V}$ of variables is denoted by $\mathcal{T}(\mathcal{F},\mathcal{V})$.

Positions (in terms) are defined as usual, and denoted by $p, q, \pi, \ldots$. They are ordered by the prefix ordering $\leq$ as usual. The 'empty' position is denoted by $\lambda$. If $p \leq q$ we say that $p$ is *above* $q$ (or $q$ is *below* $p$). Two positions $p$ and $q$ are said to be *parallel* (or *independent, disjoint*), denoted by $p \parallel q$, if neither $p \leq q$ nor $q \leq p$. These notations extend in a straightforward way to sets of positions denoted by $\Pi, \Pi', \ldots$. Furthermore we also use mixed notations like $p \leq \Pi$ with the obvious meaning (i.e., $\forall \pi \in \Pi : p \leq \pi$).

The set of positions of a term $s$ is denoted by $Pos(s)$. The sets of variable positions and of non-variable, i.e., function symbol, positions of $s$ are denoted by $\mathcal{V}Pos(s)$ and $\mathcal{F}Pos(s)$, respectively. The subterm of $s$ at some position $p \in Pos(s)$ is denoted by $s/p$. The result of replacing in $s$ the subterm at position $p \in Pos(s)$ by $t$ is denoted by $s[p \leftarrow t]$. If the same term $t$ is to substituted for all $p \in P$, $P \subseteq Pos(s)$ (with all $p \in P$ mutually parallel), then we write $s[p \leftarrow t \,|\, p \in P]$.

**Definition 1. (conditional term rewriting system)**
A *join conditional term rewriting system* (CTRS for short) over some signature $\mathcal{F}$ consists of a set $\mathcal{R}$ of *conditional rewrite rules* of the form $l \to r \Longleftarrow P$

where $P$ is a conjunction of equational conditions (where equality is interpreted as joinability) $s_i \downarrow t_i$ $(1 \leq i \leq n)$, with $l, r, s_i, t_i, \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. To simplify the discussion we require $l \notin \mathcal{V}$ and $Var(r) \subseteq Var(l)$, i.e., no variables as left-hand sides and no extra variables on right-hand sides. Rules without conditions (i.e., with $n = 0$) will be written as $l \to r$. The rewrite relation $\to_{\mathcal{R}}$ induced by $\mathcal{R}$ is inductively defined as follows: Let $\mathcal{R}_0 = \emptyset$, and $\mathcal{R}_{i+1} = \{l\sigma \to r\sigma \,|\, (l \to r \Longleftarrow P) \in \mathcal{R}, u\sigma \downarrow_{\mathcal{R}_i} v\sigma$ for all $(u \downarrow v)$ in $P\}$. Note that $\mathcal{R}_i \subseteq \mathcal{R}_{i+1}$, for all $i \geq 0$. Then: $s \to_{\mathcal{R}} t$ if $s \to_{\mathcal{R}_i} t$ for some $i \geq 0$, hence $\to_{\mathcal{R}} = \bigcup_{i \geq 0} \to_{\mathcal{R}_i}$. Instead of $\to_{\mathcal{R}}$ we shall also simply use $\to$.

If $s \to_{\mathcal{R}} t$ then the *depth* of $s \to_{\mathcal{R}} t$ is defined to be the minimal $n$ with $s \to_{\mathcal{R}_n} t$. If the depth of $s \to_{\mathcal{R}} t$ is less than or equal to $n$ we denote this by $s \overset{n}{\to}_{\mathcal{R}} t$.

$P\downarrow$ means joinability of all conditions in $P$, and $P\sigma$ means that all conditions in $P$ are instantiated by $\sigma$. Furthermore, we sometimes subscript reduction steps by additional information as in $s \to_{p,\sigma,l \to r \Longleftarrow P} t$. We shall also freely use mirrored versions of arrows with subscripts like $t \;{}_{p,\sigma,l \to r \Longleftarrow P}{\leftarrow}\; s$ with the obvious meaning.

*Remark 2.* **(encoding of conditions via an equality predicate)**
Note that instead of a CTRS $\mathcal{R}$ over some signature $\mathcal{F}$ one may — within a many-sorted framework — consider the extended system $\mathcal{R}' := \mathcal{R} \uplus \{eq(x,x) \to true\}$ over the extended signature $\mathcal{F}' := \mathcal{F} \uplus \{eq, true\}$. Here, $eq$ is a fresh binary function symbol of a new sort, and $true$ a fresh constant of this new sort (with $x$ a variable of the 'old' sort). Then it is easily shown that $\mathcal{R}'$ is a conservative extension of $\mathcal{R}$ in the following sense: for all 'old' terms $s$, $t$, i.e., with $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, we have:[3]

- $s \overset{n}{\to}_{\mathcal{R}} t \iff s \overset{n}{\to}_{\mathcal{R}'} t$,
- $s \to_{\mathcal{R}} t \iff s \to_{\mathcal{R}'} t$,
- $s \downarrow^n_{\mathcal{R}} t \iff eq(s,t) \downarrow^n_{\mathcal{R}'} true$ (for $n \geq 1$),
- $eq(s,t) \overset{n}{\to}{}^*_{\mathcal{R}'} eq(u,v) \iff s \overset{n}{\to}{}^*_{\mathcal{R}} u \wedge t \overset{n}{\to}{}^*_{\mathcal{R}} v$,
- $eq(s,t) \overset{n}{\to}{}^*_{\mathcal{R}'} true \iff \exists w : eq(s,t) \overset{n}{\to}{}^*_{\mathcal{R}} eq(w,w) \to_{\mathcal{R}'} true$ (for $n \geq 1$),

From these properties it is straightforward to infer that properties like termination, confluence, local confluence and joinability of critical pairs as well as the set of critical pairs / peaks are not affected by considering $\mathcal{R}'$ instead of $\mathcal{R}$, or vice versa. Note in particular, that the equivalence $s \downarrow^n_{\mathcal{R}} t \iff eq(s,t) \downarrow^n_{\mathcal{R}'} true$ (for $n \geq 1$) means: $s \downarrow^n_{\mathcal{R}} t \iff eq(s,t) \overset{n}{\to}{}^*_{\mathcal{R}'} true$, since $true$ is irreducible.

Conditional rewriting is inherently much more complicated than unconditional rewriting. Intuitively, the main reason is that for applying some rule $l \to r \Longleftarrow P$ the appropriately instantiated conditions must be verified recursively using the reduction relation again. This may lead to a non-terminating evaluation process for the conditions, even for terminating systems. In fact, the rewrite relation (and reducibility) may be undecidable even for (finite) CTRSs without extra variables which are terminating and confluent ([8], cf. also [13]).

[3] Note that $s \downarrow^n_{\mathcal{R}} t$ is to denote that the depth of $s \downarrow_{\mathcal{R}} t$ is at most $n$.

In order to ensure decidability of the basic notions like reducibility one has to impose a stronger condition than termination: A CTRS $\mathcal{R}$ is said to be *decreasing* ([5]) if there exists a well-founded partial ordering $>$ extending the reduction relation and the proper subterm ordering such that $l\sigma > s_i\sigma$ and $l\sigma > t_i\sigma$, for every rule $l \to r \Longleftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathcal{R}$ and every substitution $\sigma$.

We need to be a bit more precise with the notion of *critical peak / pair* than it is usually done: Our critical peak conditions to be developed crucially rely on some additional information (namely, the positions of the 'inner' shared redexes contracted) which gets lost if only the resulting *(conditional) critical pair* is considered. The *parallel reduction relation* on terms (induced by $\to_{\mathcal{R}} = \to$ for some CTRS $\mathcal{R}$) is defined as usual, and denoted by $\twoheadrightarrow\!\!\!\!\!\!\!\!\;\|\;$ ($-\!\!\!\Vert\!\!\!\to$).

**Definition 3. (shared parallel critical peaks and related notions)** Let $\mathcal{R}$ be a CTRS, and let $l_1 \to r_1 \Longleftarrow P_1$, $l_2 \to r_2 \Longleftarrow P_2$ be two rewrite rules of $\mathcal{R}$ which have no variables in common. Let $\Pi \subseteq \mathcal{F}Pos(l_2)$ be a non-empty set of parallel non-variable positions of $l_2$ such that $l_1$ and all $l_2/\pi$, $\pi \in \Pi$, are (simultaneously) unifiable with mgu $\sigma$. Then the 6-tuple $\langle t_1, Q_1, t_2, Q_2, u, \Pi\rangle = \langle (l_2[\pi \leftarrow r_1 \mid \pi \in \Pi])\sigma, P_1\sigma, r_2\sigma, P_2\sigma, l_2\sigma, \Pi\rangle$ is said to be a *shared parallel (conditional) critical peak* of $\mathcal{R}$ (obtained by overlapping the rule $l_1 \to r_1 \Longleftarrow P_1$ into $l_2 \to r_2 \Longleftarrow P_2$ simultaneously at all positions from $\Pi$). The corresponding *shared parallel (conditional) critical pair* is obtained by ignoring the overlap $u$ and the set $\Pi$ of parallel overlap positions, written as $\langle t_1 = t_2\rangle \Longleftarrow Q_1, Q_2$. However, we exclude the case where $l_1 \to r_1 \Longleftarrow P_1$, $l_2 \to r_2 \Longleftarrow P_2$ are renamed versions of the same rule of $\mathcal{R}$ and $\Pi = \{\lambda\}$, i.e., we do not overlap a rule with itself at root position (since this gives only rise to trivial (conditional) divergences, due to $Var(r_i) \subseteq Var(l_i)$). For any substitution $\tau$, $\langle t'_1, Q'_1, t'_2, Q'_2, u', \Pi\rangle = \langle t_1\tau, Q_1\tau, t_2\tau, Q_2\tau, u\tau, \Pi\rangle$ is said to be an instance of $\langle t_1, Q_1, t_2, Q_2, u, \Pi\rangle$. If, for an instance $I = \langle t'_1, Q'_1, t'_2, Q'_2, u', \Pi\rangle$, $Q'_1$ and $Q'_2$ are satisfied, i.e., if $Q'_1 \downarrow_{\mathcal{R}}$ and $Q'_2 \downarrow_{\mathcal{R}}$ hold, then $I$ is called *feasible*, and *infeasible* otherwise. If $I$ is feasible, its corresponding *critical divergence* is given by the quadruple $\langle t'_1, t'_2, u', \Pi\rangle$ which – for the sake of readability – is also denoted by $t'_1 \;{}_{\Pi}\!\!\leftarrow\!\!\Vert\!\!- u' \to_\lambda t'_2$. A critical divergence $\langle t'_1, t'_2, u', \Pi\rangle$ is *joinable*, if $t'_1 \downarrow_{\mathcal{R}} t'_2$. The instance $I = \langle t'_1, Q'_1, t'_2, Q'_2, u', \Pi\rangle$ is *joinable* if it is infeasible or satisfies $t'_1 \downarrow_{\mathcal{R}} t'_2$. A shared parallel (conditional) critical peak $\langle t_1, Q_1, t_2, Q_2, u, \Pi\rangle$ is said to be *outside* (or a *(conditional) critical overlay*) if $\Pi = \{\lambda\}$. Otherwise, i.e., if some $\pi \in \Pi$ is strictly below the root ($\pi > \lambda$) (which implies $\pi > \lambda$ for all $\pi \in \Pi$), it is said to be *inside*. It is said to be *joinable* if all its instances are joinable. Ordinary *(conditional) critical peaks* are obtained as a special case in the above definition, namely by requiring that $\Pi$ is a singleton set. If every shared parallel critical peak of $\mathcal{R}$ is a critical overlay, $\mathcal{R}$ is said to be an *overlay CTRS*. $\mathcal{R}$ *has joinable critical peaks* (or *pairs*) if all (feasible) instances of all critical peaks (pairs) are joinable. $\mathcal{R}$ is *shallow confluent* ([5]) if for all $m, n \geq 0$ and all terms $s, t, u$ with $s \stackrel{m}{\to}{}^* t$, $s \stackrel{n}{\to}{}^* u$ there exists a term $v$ such that $t \stackrel{n}{\to}{}^* v$ and $u \stackrel{m}{\to}{}^* v$. An (ordinary) critical peak of $\mathcal{R}$ is *shallow joinable* if for each feasible instance of it the corresponding critical divergence $t_1 \leftarrow u \to t_2$ is shallow joinable, i.e., if $u \stackrel{m}{\to} t_1$ and $u \stackrel{n}{\to} t_2$, then there exists $t_3$ with $t_1 \stackrel{n}{\to}{}^* t_3$ and $t_2 \stackrel{m}{\to}{}^* t_3$. $\mathcal{R}$ is said to be *normal* if for every rule $(\, l \to r \Longleftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n\,) \in \mathcal{R}$ and $1 \leq i \leq n$, $s_i$ or $t_i$ is a ground term that is irreducible w.r.t. $\mathcal{R}_u = \{l \to r \mid \exists (l \to r \Longleftarrow P) \in \mathcal{R}\}$.

Note that testing joinability of (shared parallel) conditional critical peaks is in general much more difficult than in the unconditional case since one has to consider all feasible instances.

## 3 Known Results and Motivation

As already mentioned, joinability of critical pairs is not sufficient for (local) confluence of terminating CTRSs ([5]). On the positive side the following results are known.

**Theorem 4.** *([5]) Let $\mathcal{R}$ be a CTRS. Then the following properties hold:*

*(1) If $\mathcal{R}$ is decreasing and has joinable critical peaks, then it is confluent.*
*(2) If $\mathcal{R}$ is terminating, left-linear, normal and has shallow joinable critical peaks, then it is (shallow) confluent.*
*(3) If $\mathcal{R}$ is a terminating overlay system with joinable critical peaks, then it is confluent (***overlay confluence criterion***).*

According to our introductory classification the underlying approach for result (1) above essentially works by enabling the classical inductive proof technique via a strengthened termination property: The decreasingness assumption justifies the necessary application(s) of the induction hypothesis when recursing into conditions of applied rules (for verifying that variable overlaps are not critical).

Result (2) imposes (quite severe) syntactical restrictions on the form of the rules, and requires a particular form of joinability for (ordinary) critical peaks. (3) is syntactically less restrictive concerning the form of the rules, but only admits critical overlays.

In essence, our main result will be a substantially generalized (local and global) version of the overlay confluence criterion (3). In order to provide some intuition for the basic problems and motivation for our approach let us give a simple, but instructive example.

*Example 1.* The CTRS

$$\mathcal{R} = \begin{cases} f(a,a) \to b & \\ a \to b & \\ f(b,x) \to b & \Longleftarrow \quad f(x,x) \downarrow b \\ f(x,b) \to b & \Longleftarrow \quad f(x,x) \downarrow b \end{cases}$$

is terminating (but obviously not decreasing). It has two (symmetric) outside critical peaks (between the last two rules) with no feasible instances at all, and the two inside critical peaks $f(b,a)\ {}_1\!\leftarrow f(a,a) \to_\lambda b$ and $f(a,b)\ {}_2\!\leftarrow f(a,a) \to_\lambda b$ both of which are joinable from left to right: $f(b,a) \to b$, $f(a,b) \to b$, because in both cases the third or fourth rule, respectively, is applicable due to $f(a,a) \to b$ (using the first rule), hence $f(a,a) \downarrow b$. However, $\mathcal{R}$ is not (locally) confluent. For instance, we have

$$l\sigma' = f(b,b)\ {}_2\!\leftarrow f(b,a) = l\sigma \to_\lambda r\sigma = b\,,$$

where the rule $l \to r \Longleftarrow P$ applied in the right step is $f(b,x) \to b \Longleftarrow f(x,x) \downarrow b$, with matching substitution $\sigma = \{x \mapsto a\}$, and where $\sigma' = \{x \mapsto b\}$. But the two reducts $f(b, b)$ and $b$ are both irreducible. In particular, $l\sigma' \rightarrow_{\lambda,\sigma',l\rightarrow r \Leftarrow P} r\sigma'$ does not hold. Intuitively, the reason for that is that recursively checking the condition $(P\sigma')\downarrow$ leads to the *shared parallel critical peak*

$$f(b,b)\ {}_{\{1,2\}}\!\leftarrow\!\!\Vdash f(a,a) \rightarrow_\lambda b$$

which has a duplicated critical redex and which is not joinable. In other words, a variable overlap has become critical by duplication of the contracted (terminating) redex $a$ in the condition. This suggests to prevent such situations in advance by incorporating a priori such critical situations into the definition of the relevant critical divergences to be analyzed, and naturally leads to the notion of shared parallel critical peaks.

## 4 Main Results

Now we define the property of shared parallel critical peaks which will be shown to suffice for inferring confluence for terminating (terms in given) CTRSs. Here, given a CTRS $\mathcal{R}$, a term $s$ is said to be *terminating* or *strongly normalizing* (SN($s$) for short) if there is no infinite ($\mathcal{R}$-) reduction sequence $s \rightarrow s_1 \rightarrow s_2 \rightarrow \ldots$. Consequently, $\mathcal{R}$ (and $\rightarrow$) is terminating if every term is terminating.

**Definition 5. (quasi-overlay joinability)**
Let $\mathcal{R}$ be a CTRS. Let $I = \langle t'_1, Q'_1, t'_2, Q'_2, u', \Pi\rangle$ be a feasible instance of a shared parallel critical peak of $\mathcal{R}$. Then we say that $I$ (and its corresponding critical divergence $t'_1\ {}_{\Pi}\!\leftarrow\!\!\Vdash u' \rightarrow_\lambda t'_2$) is *quasi-overlay joinable* if it is joinable as follows: There exists some $t'_3$ with $t'_1 \rightarrow^* t'_3$ and $t'_2 \rightarrow^*_{<s} t'_3$, where $s := u'/\pi$ for any $\pi \in \Pi$ (note that $s$ does not depend on which $\pi \in \Pi$ we choose). Here the subscript $_{<s}$ means that in the reduction sequence $t'_2 \rightarrow^* t'_3$ only redexes are allowed to be contracted which are strictly smaller than $s$ w.r.t. $> := (\rightarrow \cup \rhd_{st})^+$ (i.e., if $s(\rightarrow \cup \rhd_{st})^+ w$ for any redex $w$ contracted in $t'_2 \rightarrow^* t'_3$). $I$ is said to be *innermost* if no proper subterm of $s = u'/\pi$ (for $\pi \in \Pi$) is reducible (w.r.t. $\mathcal{R}$). A shared parallel critical peak is said to be *quasi-overlay joinable* if all its feasible instances are quasi-overlay joinable. $\mathcal{R}$ *has quasi-overlay joinable shared parallel critical peaks* if all its shared parallel critical peaks are quasi-overlay joinable.

Let us introduce the following abbreviation (*).

| | |
|---|---|
| all feasible innermost instances of all shared parallel critical peaks of $\mathcal{R}$ are quasi-overlay joinable | **(*)** |

Now we are prepared to state and prove the following central result, which in essence is a substantially generalized local version of [5, Lemma 2] (but with a proof that is different in important details).

**Theorem 6. (technical key result)**
*Let $\mathcal{R}$ be a CTRS satisfying (*). Let $\mathcal{R}' = \mathcal{R} \uplus \{eq(x,x) \rightarrow true\}$ (as in Remark 2). Then the following localized confluence property holds:*

$\forall s \in \mathcal{T}(\mathcal{F},\mathcal{V}), SN(s)\ \ \forall t \in \mathcal{T}(\mathcal{F},\mathcal{V})\ \ \forall u, v \in \mathcal{T}(\mathcal{F}',\mathcal{V})\ \ \forall \Pi, [\forall \pi \in \Pi : u/\pi = s]$ *:*

$$\left.\begin{array}{l} u \to^*_{\mathcal{R}'} v \\ s \to^*_{\mathcal{R}} t \end{array}\right\} \quad \Longrightarrow \quad u[\pi \leftarrow t \,|\, \pi \in \Pi] \downarrow_{\mathcal{R}'} v$$

*Proof.* Let $\bar{s} \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ be a terminating term, i.e., with $\mathrm{SN}(\bar{s})$. We have to show:

$(I)$ $\forall t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ $\forall u, v \in \mathcal{T}(\mathcal{F}', \mathcal{V})$ $\forall \Pi, [\forall \pi \in \Pi : u/\pi = \bar{s}]$:

$$\left.\begin{array}{l} u \to^*_{\mathcal{R}'} v \\ \bar{s} \to^*_{\mathcal{R}} t \end{array}\right\} \quad \Longrightarrow \quad u[\pi \leftarrow t \,|\, \pi \in \Pi] \downarrow_{\mathcal{R}'} v$$

Now, instead of $(I)$ we shall prove the following strengthened version

$(II)$ $\forall s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V}), \bar{s} \geq_1 s$ $\forall u, v \in \mathcal{T}(\mathcal{F}', \mathcal{V})$ $\forall \Pi, [\forall \pi \in \Pi : u/\pi = s]$:

$$\left.\begin{array}{l} u \to^*_{\mathcal{R}'} v \\ s \to^*_{\mathcal{R}} t \end{array}\right\} \quad \Longrightarrow \quad u[\pi \leftarrow t \,|\, \pi \in \Pi] \downarrow_{\mathcal{R}'} v$$

by induction. Here the partial ordering $>_1$ (which depends on $\bar{s}$!) is defined by $>_1 := (\to \cup \rhd_{st})^+{}_{\text{below } \bar{s}}$, where — for some binary (ordering) relation $R$ — $R_{\text{below } a}$ is defined by $R \cap (\{b \,|\, aR^*b\}^2)$. Note that $>_1$ is well-founded due to $\mathrm{SN}(\bar{s})$ which is easily seen by standard arguments. We shall prove $(II)$ by induction, using the following complexity measure $M(s, t, u, v, \Pi)$ for $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ with $\bar{s} \geq_1 s$, $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $u, v \in \mathcal{T}(\mathcal{F}', \mathcal{V})$ and $\Pi$, such that $u/\pi = s$ for all $\pi \in \Pi$, and $u \to^*_{\mathcal{R}'} v$:

$$M(s, t, u, v, \Pi) = \langle s, n, k \rangle \,,$$

where

$$n = n(u, v) = \min\{m \,|\, u \overset{m}{\to}{}^*_{\mathcal{R}'} v\}$$

and

$$k = k(u, v, n) = \min\{l \,|\, u \overset{n}{\to}{}^l_{\mathcal{R}'} v\} \,.$$

These triples $\langle s, n, k \rangle$ are compared using the lexicographic combination $\succ$ of the orderings $>_1$, $>_2$ and $>_3$, where $>_2$ and $>_3$ are the usual ordering $>$ on natural numbers. Now, by well-foundedness of $>_1$ (and of $>$) $\succ$ is well-founded, too. Subsequently, for the sake of readability we shall often omit the subscripts $\mathcal{R}$ and $\mathcal{R}'$, respectively, in notations like $s \to^*_{\mathcal{R}} t$ and $u \to^*_{\mathcal{R}'} v$ (when they are clear from the context).

The rest of the proof is by a complete case analysis using induction w.r.t. the well-founded ordering $\succ$. We distinguish the following cases:

**(1)** If $u = v$ (which implies $n = k = 0$) or $s = t$, then we are trivially done.

**(2)** Otherwise, we may suppose

$$u \overset{n}{\to}_{p,\sigma,l\to r \Leftarrow P} u' \overset{n}{\to}{}^{k-1} v$$

and

$$s \to_{q,\tau,l_1 \to r_1 \Leftarrow P_1} s' \to^* t$$

for $n \geq 1$ minimal with $u \overset{n}{\to}{}^* v$, and $k \geq 1$ minimal with $u \overset{n}{\to}{}^k v$.[4] For the remaining cases it suffices to show

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[4] Note that $n' = \min\{m \,|\, u' \overset{m}{\to}{}^* v\}$, $k' = \min\{l \,|\, u' \overset{n'}{\to}{}^l v\}$ and $n$, $k$ are related as follows: either $n' < n$ or else $n' = n$ and $k' < k$. For that reason, one may apply the induction hypothesis in situations of the form $u' \to^* v$, $s \to^* t$ such that $u'/\pi' = s$ for all $\pi' \in \Pi'$ (for some other set $\Pi'$ of mutually disjoint positions) for inferring $u'[\pi' \leftarrow t \,|\, \pi' \in \Pi'] \downarrow v$, because of a decrease of the measure in the second or third component.

$$u[\pi \leftarrow s' \mid \pi \in \Pi] \downarrow v$$

because then, induction via a decrease in the first component ($s >_1 s'$ because of $s \to s'$) yields

$$u \mathrel{\Vert\!\!\!\!\rightarrow} u[\pi \leftarrow s' \mid \pi \in \Pi] \mathrel{\Vert\!\!\!\!\rightarrow}^* u[\pi \leftarrow t \mid \pi \in \Pi] \downarrow v$$

as desired (this is illustrated in Fig. 1).

$$\begin{tikzcd}
u = u[\pi \leftarrow s \mid \pi \in \Pi] \arrow[r] \arrow[d, "\Vert"'] \arrow[drr, phantom, "\text{by assumption}"] & u' \arrow[r, "k-1"] & v \arrow[d, dashed, "*"] \\
u_1 = u[\pi \leftarrow s' \mid \pi \in \Pi] \arrow[rr, dashed, "*"] \arrow[d, "\Vert *"'] \arrow[drr, phantom, "\text{by induction } (s >_1 s')"] & & v_1 \arrow[d, dashed, "*"] \\
u_2 = u[\pi \leftarrow t \mid \pi \in \Pi] \arrow[rr, dashed, "*"] & & v_2
\end{tikzcd}$$

**Fig. 1.** Case **(2)**: $u[\pi \leftarrow s' \mid \pi \in \Pi] \downarrow v$ suffices for showing $u[\pi \leftarrow t \mid \pi \in \Pi] \downarrow v$

**(2.1)** Next we consider the case that a proper subterm of $s$ is reducible (this case is illustrated in Fig. 2). Hence, let $s/q' = \hat{s} \to \hat{s}'$ for some $q' > \lambda$. Then we get (cf. the upper part of the diagram)

$$u = u[\pi \leftarrow s \mid \pi \in \Pi] = u[\pi q' \leftarrow \hat{s} \mid \pi \in \Pi] \mathrel{\Vert\!\!\!\!\rightarrow} u[\pi q' \leftarrow \hat{s}' \mid \pi \in \Pi] =: u_2$$

by a shared parallel contraction of the redexes $\hat{s}$ in $u$ into $\hat{s}'$. By induction (due to $s \rhd_{st} \hat{s}$, hence $s >_1 \hat{s}$) we conclude that there exists some $v_2$ with

$$u_2 = u[\pi q' \leftarrow \hat{s}' \mid \pi \in \Pi] \to^* v_2 \; {}^*\!\!\leftarrow v \; .$$

For the left part of the diagram in Fig. 2, where we have

$$u = u[\pi \leftarrow s \mid \pi \in \Pi] \mathrel{\Vert\!\!\!\!\rightarrow} u[\pi \leftarrow s' \mid \pi \in \Pi] = u_1$$

by a shared parallel contraction of the redex $s$ into $s'$, we consider the extracted divergence

$$s[q' \leftarrow \hat{s}'] \leftarrow s[q' \leftarrow \hat{s}] = s \to s' \; .$$

By induction (taking $\hat{s}$, $\hat{s}'$, $s$, $s'$, and $\{q'\}$ instead of $s, t, u, v$ and $\Pi$, respectively), which is justified due to $s \rhd_{st} \hat{s}$ and hence $s >_1 \hat{s}$, we obtain $s[q' \leftarrow \hat{s}'] \downarrow s'$, i.e., there exists some $s''$ with

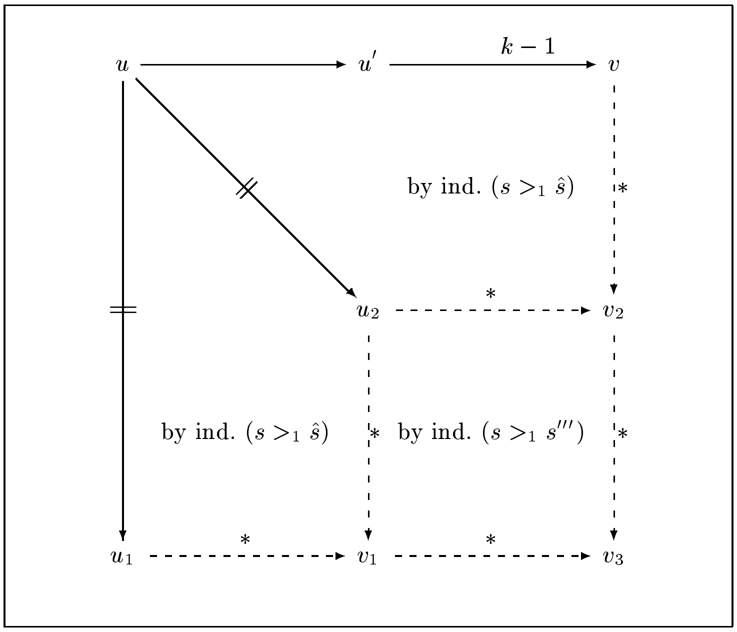


**Fig. 2.** Case **(2.1)** A proper subterm of $s$ is reducible

$$s' \to^* s'' \,{}^*\!\!\leftarrow s[q' \leftarrow \hat{s}'] \ .$$

Consequently, we also get

$$u_1 = u[\pi \leftarrow s' \mid \pi \in \Pi] \to^* u[\pi \leftarrow s'' \mid \pi \in \Pi] =: v_1$$

and

$$u_2 = u[\pi \leftarrow s[q' \leftarrow \hat{s}'] \mid \pi \in \Pi] \to^* u[\pi \leftarrow s'' \mid \pi \in \Pi] = v_1 \ .$$

Finally, for closing the diagram (cf. its lower right part) we appeal again to the induction hypothesis (taking $s''' := s[q' \leftarrow \hat{s}']$, $s''$, $u_2$, $v_2$ and $\Pi$ instead of $s$, $t$, $u$, $v$ and $\Pi$, respectively). This is justified because of $s \to s''' \to^* s''$ (i.e., $s \to_{\mathcal{R}} s''' \to^*_{\mathcal{R}} s''$ due to $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$, $s''' \in \mathcal{T}(\mathcal{F},\mathcal{V})$) and thus $s >_1 s'''$, and yields some $v_3$ with $v_1 \to^* v_3 \,{}^*\!\!\leftarrow v_2$ as desired.

**(2.2)** For the remaining case we may assume that no proper subterm of $s$ is reducible. This implies $q = \lambda$, i.e., $s = l_1\tau$ is an innermost redex which is contracted to $r_1\tau = s'$ using the rule $l_1 \to r_1 \Longleftarrow P_1$. Consequently, the redex contraction in $u \overset{n}{\to}_{\sigma,p,l\to r\Longleftarrow P} u'$ at position $p$ is either above some positions $\pi' \in \Pi' \subseteq \Pi$ of the shared redexes contracted in the shared parallel step $u \mathrel{+\!\!\!\!+\!\!\!\!\to}_{\Pi} u[\pi \leftarrow s' \mid \pi \in \Pi]$, or disjoint from (all of) them. The latter case is no problem (and is also subsumed by the proof of the former), whereas in the first case the overlapping positions may be further partitioned into a "shared parallel variable overlap part" and a "shared parallel critical overlap part". In order to cope with the "shared parallel critical overlap part" the global joinability assumption for shared parallel critical

peaks of $\mathcal{R}$ will be essential. Now we have the following situation:
$p \le \Pi' \neq \emptyset,\ \Pi' \subseteq \Pi,\ p \parallel (\Pi \backslash \Pi')$ (**some occurrences of the shared parallel redex are below the single redex**, cf. Fig. 3): For a precise argumentation

$$
\begin{tikzcd}
u \arrow[rr, "n", "{p,\sigma,l \to r \Longleftarrow P}"'] \arrow[d, "\Pi_d \uplus \Pi_v"] & & u' \arrow[r, "n \quad k-1"] \arrow[d] \arrow[dr, phantom, "\text{by ind. (B)}"] & v \arrow[d, dashed, "*"'] \\
u_1' \arrow[r, "\Pi'"'] \arrow[d, "\Pi_c"] & u_2' \arrow[r, dashed, "\text{by ind. (A)}", "(P\sigma')\downarrow"'] \arrow[d, "\Pi_c"'] \arrow[dr, phantom, "\text{by ass. (C)}"] & u_4' \arrow[r, dashed, "*"] \arrow[d, dashed, "<_1 s,\, *"'] \arrow[dr, phantom, "\text{by ind. (D)}"] & v_1 \arrow[d, dashed, "*"'] \\
u_1 \arrow[r, "\Pi'"] & u_3' \arrow[r, dashed, "*"] & v_2 \arrow[r, dashed, "*"] & v_3
\end{tikzcd}
$$

**Fig. 3.** Case **(2.2)** Some occurrences of the shared parallel redex are strictly below the single redex

we introduce the following definitions:

- $\Pi_x := \{pp'p'' \in \Pi \mid l/p' = x\}$, for $x \in Var(l)$
- $\Pi_v := \{\pi \in \Pi \mid \exists x \in Var(l) : \pi \in \Pi_x\}$
- $\Pi_d := \{\pi \in \Pi \mid \pi \parallel p\}$
- $\Pi_c := \{\pi \in \Pi \mid \exists p' \in \mathcal{F}Pos(l) : \pi = pp'\}$
- $\Pi = \Pi_d \uplus \Pi_v \uplus \Pi_c$
- $\Pi' := \{p\bar{p}p'' \mid \exists p' : \exists x \in Var(l) : (\, l/p' = x = l/\bar{p} \wedge pp'p'' \in \Pi\,)\} \setminus \Pi_v$
- $u = u[\pi \leftarrow s \mid \pi \in \Pi]$
- $u' = u[p \leftarrow r\sigma]$
- $x\sigma' := (x\sigma)[p'' \leftarrow s' \mid \exists p' : (l/p' = x \wedge pp'p'' \in \Pi)]$
- $u_1 = u[\pi \leftarrow s' \mid \pi \in \Pi]$
- $u_1' := u[\pi \leftarrow s' \mid \pi \in \Pi_d \uplus \Pi_v]$
- $u_2' := u[\pi \leftarrow s' \mid \pi \in \Pi_d][p \leftarrow l\sigma']$
- $u_3' := u_2'[\pi \leftarrow s' \mid \pi \in \Pi_c]$
- $u_4' := u[\pi \leftarrow s' \mid \pi \in \Pi_d][p \leftarrow r\sigma']$
- $v_2 := u_3'[p \leftarrow w] = u_4'[p \leftarrow w]$ (here $w$ is yet to be determined)

Now, according to Fig. 3 there are four non-trivial parts to be verified, namely (A)-(D). The other reductions are easily checked using the above definitions. In particular, we observe that the shared parallel reductions $u_1' \mathrel{+\!\!\!\!\!\!\longrightarrow} u_2'$ and $u_1 \mathrel{+\!\!\!\!\!\!\longrightarrow} u_3'$ cope with a potential non-left-linearity of the rule $l \to r \Longleftarrow P$, i.e., these reductions restore an instance of the left-hand side $l$ at position $p$ which may have been destroyed by the shared parallel reduction from $u$ to $u'_1$.

(A) We have to verify the step $u'_2 \rightarrow_{p,\sigma',l\rightarrow r\Leftarrow P} u'_4$. By definition we know $u'_2/p = l\sigma'$, $u'_4/p = r\sigma'$. Hence it suffices to show $(P\sigma') \downarrow$, i.e., $w_1\sigma' \downarrow w_2\sigma'$ for all conditions $w_1 \downarrow w_2$ in $P$. From $u \xrightarrow{n}_{p,\sigma,l\rightarrow r\Leftarrow P} u'$ we know $w_1\sigma \downarrow^{n-1} w_2\sigma$, or, equivalently, $eq(w_1,w_2)\sigma \overset{n-1^*}{\rightarrow}_{\mathcal{R}'} true$ (with the case of $n-1=0$ being trivial). Since we have $eq(w_1,w_2)\sigma \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow} eq(w_1,w_2)\sigma'$ by a shared parallel contraction of occurrences of $s$ in $eq(w_1,w_2)\sigma$ into $s'$, this implies by induction (due to a decrease in the second component in the measure): $eq(w_1,w_2)\sigma' \downarrow_{\mathcal{R}'} true$, hence $eq(w_1,w_2)\sigma' \rightarrow^*_{\mathcal{R}} eq(w_3,w_3) \rightarrow_{\mathcal{R}'} true$, for some $w_3 \in \mathcal{T}(\mathcal{F},\mathcal{V})$, by irreducibility of $true$. Consequently, we obtain $w_1 \rightarrow^*_{\mathcal{R}} w_3$, $w_2 \rightarrow^*_{\mathcal{R}} w_3$ and thus $w_1\sigma' \downarrow_{\mathcal{R}} w_2\sigma'$. This means, that all conditions are indeed satisfied as was to be shown.

(B) Clearly we have $u' = u[p \leftarrow r\sigma] \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow} u[\pi \leftarrow s' \,|\, \pi \in \Pi_d][p \leftarrow r\sigma'] = u'_4$ by a parallel shared step contracting $s$ into $s'$. Then induction (due to a decrease of the measure in the second or third component) yields the existence of some $v_1$ with $u'_4 \rightarrow^* v_1 \,{}^*\!\!\leftarrow v$.

(C) We have to show that there exists a reduct $v_2$ of $u'_3$ which can also be obtained from $u'_4$, but in such a way that in the latter reduction sequence only redexes strictly smaller than $s$ (w.r.t. $>_1$) are contracted. For that purpose it suffices, by definition of $u'_2$, $u'_3$ and $u'_4$, that for the extracted divergence

$$u'_3/p \mathrel{\leftarrow\!\!\!\!+\!\!\!\!+} u'_2/p \rightarrow u'_4/p \;,$$

i.e., for

$$(l\sigma')[\pi'_c \leftarrow s' \,|\, p\pi'_c \in \Pi_c] \mathrel{\leftarrow\!\!\!\!+\!\!\!\!+} l\sigma' \rightarrow r\sigma' \;,$$

we can find a common reduct of $u'_3/p = (l\sigma')[\pi'_c \leftarrow s' \,|\, p\pi'_c \in \Pi_c]$ and $u'_4/p = r\sigma'$ which can be reached from the latter by contracting only redexes strictly smaller than $s$ (w.r.t. $>_1$). Since by construction the above divergence is a feasible innermost[5] instance of a shared parallel critical peak of $\mathcal{R}$ (determined by overlapping $l_1 \rightarrow r_1 \Longleftarrow P_1$ into $l \rightarrow r \Longleftarrow P$), the global joinability assumption for such shared parallel critical peaks indeed ensures this special form of joinability.

(D) Finally, we have to show that the divergence $v_2 \,{}^*\!\!\leftarrow u'_4 \rightarrow^* v_1$ can be closed by finding a common reduct $v_3$. This is possible by repeatedly[6] applying induction, justified by a decrease in the first component of the measure, since only redexes strictly smaller than $s$ (w.r.t. $>_1$) are contracted in the reduction sequence from $u'_4$ to $v_2$ (cf. (C)). Hence we are done.

□

Next we shall discuss crucial aspects of this central result and its proof, thus deriving various interesting generalizations and consequences.

First of all we observe that in the above proof the case of a (possibly critical) overlay between the rules $l \rightarrow r \Longleftarrow P$ and $l_1 \rightarrow r_1 \Longleftarrow P_1$, i.e., for $p \in \Pi$,

[5] Note that by assumption no proper subterm of $s = l\sigma'/\pi'_c$ (for all $\pi'_c$ with $p\pi'_c \in \Pi_c$) is reducible!

[6] Doing this completely formally requires an additional (locally nested) induction on the length of the derivation $u'_4 \rightarrow^* v_2$.

is implicitly contained in case (2). In that case we have $\Pi_d = \Pi_v = \emptyset$, $\Pi = \Pi_c = \{p\}$ and the diagram in Fig. 3 collapses according to the identifications $u = u_1' = u_2'$, $u_1 = u_3'$, $\sigma' = \sigma$. Furthermore, the shared parallel step $u_1' \mathrel{+\!\!\!|\!\!\!\rightarrow} u_1$ (as well as $u_2' \mathrel{+\!\!\!|\!\!\!\rightarrow} u_3'$) actually is an ordinary reduction at position $p$, i.e., the extracted divergence

$$u_3'/p = r_1\tau \;{}_\lambda\!\leftarrow l_1\tau = s = u_2'/p = l\sigma \rightarrow_\lambda r\sigma = u_4'/p$$

corresponds to a (feasible) instance of the critical overlay of $\mathcal{R}$ between the rules $l \rightarrow r \Longleftarrow P$ and $l_1 \rightarrow r_1 \Longleftarrow P_1$ . Consequently, if there exists a common reduct $v_4$ (which is equal to $v_2/p$) of $u_3'/p$ and $u_4'/p$, as it is guaranteed by the global assumption, then the imposed ordering restriction for the reduction sequence $u_4'/p \rightarrow^* v_2/p$, namely that all contracted redexes are strictly smaller than $s$ (w.r.t. $>_1$), is vacuously satisfied. This follows from $u_2'/p = s \rightarrow_\lambda u_4'/p$ and hence $u_2'/p = s >_1 u_4'/p$ by definition of $>_1$. In other words, this observation makes clear that for critical overlays of $\mathcal{R}$ joinability is sufficient without any further restriction.

Secondly, we observe that by definition condition (*) in Theorem 6 obviously holds if $\mathcal{R}$ has quasi-overlay joinable shared parallel critical peaks. Hence, any subsequent result depending on (*) also holds under the latter slightly more restrictive precondition.

As an easy consequence of Theorem 6 we now obtain the following local criterion for confluence of terminating terms in a CTRS.

**Theorem 7. (local criterion for confluence of terminating terms)** *Let $\mathcal{R}$ be a CTRS satisfying (*). Then any terminating term $s$ is confluent, i.e., if $SN(s)$ and $t_1 \;{}^*\!\!\leftarrow s \rightarrow^* t_2$ then $t_1 \downarrow t_2$.*

*Proof.* We simply apply Theorem 6 choosing $u = s$, $t = t_1$, $v = t_2$ and $\Pi = \{\lambda\}$ (observing that 'proper' $\mathcal{R}'$-reductions, i.e., using the $\mathcal{R}'$-rule $eq(x,x) \rightarrow true$ are not needed in $t_1 \downarrow_{\mathcal{R}'} t_2$). □

A corollary of the local criterion in Theorem 7 is the following global result.

**Theorem 8. (global criterion for confluence of terminating CTRSs)** *Any terminating CTRS satisfying (*) is confluent.*

**Corollary 9.** *Any terminating CTRS with quasi-overlay joinable shared parallel critical peaks is confluent.*

For instance, the system $\mathcal{R}' = \mathcal{R} \cup \{f(x,x) \rightarrow x \Longleftarrow x \downarrow b\}$ with $\mathcal{R}$ as in Example 1 is easily shown to be confluent by applying Theorem 8 (or Corollary 9) whereas none of the results of Theorem 4 is applicable.

If all critical peaks of some CTRS $\mathcal{R}$ are overlays, then, according to our observation above, the ordering restriction imposed on the joinability requirement in (*) is vacuously satisfied. Hence, as a special case of Theorem 8 we get the well-known overlay confluence criterion of Theorem 4(3). If a divergence is a feasible instance of a shared parallel critical peak, then the ordering requirement is vacuously satisfied if we have joinability from left to right, i.e., if $t_1 \rightarrow^* t_2$.

Moreover, we observe that, by definition of (*), feasible non-innermost instances of shared parallel critical peaks can be safely ignored. In this sense, the above Theorems 7, 8 constitute critical peak criteria for confluence of CTRSs (in a local and global version, respectively) in the spirit of well-known "critical peak criteria" for confluence of unconditional TRSs as for instance described in [9], [2] (these criteria allow to ignore certain critical pairs when testing for confluence).

Finally, we now show how to generalize the results obtained by making use of orderings. Up to now we have only argued relative to the ordering $(\rightarrow \cup \rhd_{st})^{+}$. Henceforth, we shall consider the case that termination of $\mathcal{R}$ is shown by some well-founded ordering $\succ$ on terms (i.e., $\rightarrow \subseteq \succ$) which is stable w.r.t. contexts. First we generalize the definition of the quasi-overlay joinability notions introduced in Definition 5 by parameterizing the fixed ordering used there.

**Definition 10. ($\succ$-quasi-overlay joinability)**
The quasi-overlay joinability notions introduced in Definition 5 are generalized into corresponding $\succ$-quasi-overlay joinability notions simply by replacing $\rightarrow$ with $\succ$, i.e., the original $> = (\rightarrow \cup \rhd_{st})^{+}$ now becomes $> = (\succ \cup \rhd_{st})^{+}$.

Note that according to this new definition the old notion of quasi-overlay joinability now means $\rightarrow$-quasi-overlay joinability. Next we define (**) by

| | |
|---|---|
| all feasible innermost instances of all shared parallel critical peaks of $\mathcal{R}$ are $\succ$-quasi-overlay joinable | **(**)** |

and obtain the following generalization of Theorem 8.

**Theorem 11. (generalized confluence criterion)**
*Let $\mathcal{R}$ be a CTRS such that $\rightarrow \subseteq \succ$ for some well-founded ordering $\succ$ on terms which is stable w.r.t. contexts. If $\mathcal{R}$ satisfies (**) then it is confluent.*

*Proof.* It suffices to prove the local version, i.e., to show confluence below some arbitrary term $s$ which, by assumption, must be terminating. The proof is virtually the same as for Theorem 7, namely via the corresponding version of Theorem 6. The only difference is that instead of $>_1 = (\rightarrow \cup \rhd_{st})^{+}{}_{\text{below } s}$ we employ $>_1 := (\succ \cup \rhd_{st})^{+}{}_{\text{below } s}$. Since $\succ$ is stable w.r.t. contexts, $>_1$ is well-founded. $\square$

**Corollary 12.** *Let $\mathcal{R}$ be a CTRS such that $\rightarrow \subseteq \succ$ for some well-founded ordering $\succ$ on terms which is stable w.r.t. contexts. If $\mathcal{R}$ has $\succ$-quasi-overlay joinable shared parallel critical peaks then it is confluent.*

To see the benefits of Theorem 11, let us give a simple example.

*Example 2.* The CTRS

$$\mathcal{R} = \begin{cases} f(x,x) \rightarrow g(x,x) & \\ a \rightarrow b & \\ h(f(x,x)) \rightarrow a & \\ h(g(x,y)) \rightarrow b & \Longleftarrow \quad h(f(x,y)) \downarrow b \end{cases}$$

is terminating via the recursive path ordering $>$ induced by the following precedence $>_{\mathcal{F}}$: $f >_{\mathcal{F}} g >_{\mathcal{F}} a >_{\mathcal{F}} b$. There is only one shared parallel (inside) critical peak of $\mathcal{R}$, obtained by overlapping the first into the third rule at position 1: $h(g(x,x))\ {}_1{\leftarrow} h(f(x,x)) \to_\lambda a$ which is joinable as follows: $h(g(x,x)) \to b \leftarrow a$. We observe that for the redex contraction $a \to b$ here, the ordering requirement in (**) means to verify that the inner redex $f(x,x)$ in the overlap term $h(f(x,x))$ is strictly greater (w.r.t. $>_1 = (> \cup \rhd_{st})^+$) than the redex $a$ contracted to establish joinability. This does indeed hold due to $f(x,x) > a$. Applying Theorem 11 (or Corollary 12) yields confluence of $\mathcal{R}$. Observe that Theorem 8 is not applicable here, since $f(x,x)\ (\to \cup \rhd_{st})^+\ a$ does not hold. We note that $\mathcal{R}$ is neither an overlay CTRS, nor left-linear, nor decreasing. The latter follows since any decreasing ordering $\succ$ for $\mathcal{R}$ would have to satisfy $h(g(x,x)) \succ h(f(x,x)) \succ h(g(x,x))$, due to the last and the first rule, which contradicts irreflexivity. Hence, all other known confluence criteria, in particular those of Theorem 4, are not applicable.

In view of the conditions (*) and (**) both of which rely on joinability properties of shared parallel critical peaks, one may ask whether the corresponding properties for all ordinary critical peaks would already suffice for the results proved above. Actually, this is not the case, even for the case of terminating systems with left-to-right joinable critical peaks. This is witnessed by Example 1 above.

Finally let us remark that it is not difficult to give examples of terminating CTRSs that are confluent but where confluence cannot be shown by any of the presented criteria.

## 5 Conclusion and Related Work

Summarizing, we have succeeded in developing a new approach for verifying confluence of terminating (terms in) conditional rewrite systems. It is based on the notion of shared parallel critical peaks, and on an appropriate generalization of the notion of 'overlay joinability'. In fact, originally our approach was motivated by the result of [5] (cf. Theorem 4(3) above) in the following sense: We wanted to understand the crucial properties needed for the approach of [5] to work. Two main aspects turned out to be crucial for the presented generalization:

- A localized version of the overlay confluence criterion of [5] (Theorem 4(3)), which only requires a local termination assumption: This result was already obtained in [6], using an induction ordering (similar to the one in [13]) different from the one used in the proof of the corresponding [5, Lemma 2].
- The observation that shared parallel critical peaks constitute the natural candidates for capturing the relevant critical divergences to be taken into account.

Finally, let us briefly mention some related work. In [12], besides several other extensions of known results (also on non-terminating CTRSs), it is shown that the essential ideas and results of our approach also carry over to the more general (and technically more involved) setting of *constructor-based positive/negative CTRSs*. That the notion of *parallel critical peaks*, which is the "unshared" version of *shared parallel critical peaks*, may also be useful is shown in [7] where a sufficient parallel critical peak based condition for confluence of possibly non-terminating, unconditional left-linear TRSs is presented.

Further interesting confluence criteria for CTRSs which have not yet been mentioned can be found in [1], [3] and [11] (the latter two dealing with non-terminating systems).

**Acknowledgements:** We would like to thank Paul Taylor for his diagrams and the referees for some useful comments.

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