On Proving Termination by Innermost Termination

Bernhard Gramlich*

INRIA Lorraine, 615, rue du Jardin Botanique, BP 101
54602 Villers-lès-Nancy, France
gramlich@loria.fr

Abstract. We present a new approach for proving termination of rewrite systems by innermost termination. From the resulting abstract criterion we derive concrete conditions, based on critical peak properties, under which innermost termination implies termination (and confluence). Finally, we show how to apply the main results for providing new sufficient conditions for the modularity of termination.

1 Introduction and Overview

Properties of rewriting under strategies are generally not well explored, despite the fact that such restricted versions of rewriting are of great importance, for instance as model for describing semantics of computation formalisms. *Innermost* rewriting, i.e., rewriting where always minimal redexes are chosen for replacement, corresponds to a call-by-value semantics (via innermost evaluation), and *outermost* rewriting, where always maximal redexes are chosen, may be used to model call-by-name semantics ([17]). Hence, the study of basic properties, in particular the fundamental ones of termination and confluence, of such restricted versions of rewriting is of independent interest. Moreover, from a conceptual point of view it is perfectly natural to ask: How do basic properties of rewriting under some strategy relate to the corresponding properties of arbitrary rewriting. Results in this direction may be useful in order to facilitate the verification of proof tasks for arbitrary rewriting.

Here we shall mainly be concerned with the termination property of innermost and arbitrary rewriting. In certain cases, verifying innermost termination may be much easier than proving general termination, i.e., termination of unrestricted rewriting. Another perhaps even more important aspect is that innermost rewriting has a much better modularity behaviour than arbitrary rewriting w.r.t. to (various syntactically restricted, e.g., disjoint types of) combinations of systems ([6]).

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Clearly, termination of arbitrary rewriting implies *innermost termination*, i.e., termination of innermost rewriting, but not vice versa. A very simple example illustrating this gap is the system $\mathcal{R} = \{f(a) \to f(a), a \to b\}$ which is obviously non-terminating, but innermost terminating.

Thus, developing criteria which guarantee the equivalence of termination and innermost termination is very useful. In [6] we have shown that for non-overlapping systems as well as for the more general case of locally confluent overlay systems innermost termination is indeed equivalent to termination.[2] The proof crucially relies both on the overlay property and on a transformation construction (for a non-terminating derivation) where all (parallel) maximal terminating (and thus also confluent) subterms are normalized. In this paper, starting from the result for non-overlapping systems, we shall develop another approach for obtaining new results. Actually, one basic idea has already been used in the technical report version of [6], however, without further exploiting its potential. The point is that under certain conditions, instead of performing a parallel normalization of the maximal terminating subterms one may proceed more 'economically' in the transformation construction, namely, by a parallel one-step reduction of all terminating innermost redexes. In order to make this construction well-defined and work, two syntactical properties will turn out to be crucial. From the obtained abstract result we shall derive a couple of more concrete critical peak based criteria which suffice for the equivalence of termination and innermost termination. Finally, we show how the presented equivalence criteria for termination and innermost termination easily yield new modularity results for termination.

The rest of the paper is structured as follows. In Section 2 we review some needed basic terminology and known results. Then, in the main Section 3 we develop the above sketched new approach. And finally, in Section 4, we show how to obtain corresponding new modularity results for termination.

2 Basics and Known Results

We assume familiarity with the basic no(ta)tions, terminology and theory of term rewriting (cf. e.g. [3], [9], [17]). The set of terms over some given signature $\mathcal{F}$ and some (disjoint) denumerable set $\mathcal{V}$ of variables is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Positions (in terms) are ordered by the prefix ordering $\leq$ as usual. The 'empty' root position is denoted by λ. If $p \leq q$ we say that p is *above* q (or q is *below* p). For *subtracting* positions we use $\backslash$, i.e., we write $p \backslash q = q'$ if $p = qq'$. Two positions p and q are said to be *parallel* (or *independent, disjoint*), denoted by $p \parallel q$, if neither $p \leq q$ nor $q \leq p$. The set of positions of a term s is denoted by $Pos(s)$. The sets of variable positions and of non-variable, i.e., function symbol, positions of s are denoted by $\mathcal{V}Pos(s)$ and $\mathcal{F}Pos(s)$, respectively. The subterm

[2] Implicitly, for non-overlapping systems this result also follows from [4] where termination conditions in terms of *forward* and *overlap* closures are investigated (cf. also [2]).

of s at some position $p \in Pos(s)$ is denoted by s/p. The result of replacing in s the subterm at position $p \in Pos(s)$ by t is denoted by $s[p \leftarrow t]$.

For rewrite rules $\;\to r$ of a term rewriting system (TRS) $\mathcal{R}^{\mathcal{F}} = \mathcal{R}$ (over $\mathcal{T}(\mathcal{F}, \mathcal{V})$) we require that l is not a variable, and that all variables of r occur in l (this excludes only degenerate cases). For reduction steps with the rewrite relation $\to_{\mathcal{R}} = \to$ induced by $\mathcal{R}$ we sometimes add additional information as in $s \to_{p,\sigma,l\to r} t$ with the obvious meaning. Furthermore, we make free use of context notations like $s = C[\sigma(l)] \to_{l\to r} C[\sigma(r) = t$. The innermost reduction relation $\xrightarrow[i]{}$ (induced by $\mathcal{R}$) is given by: $s \xrightarrow[i]{} t$ if $s = C[\sigma(l)] \to_{l\to r} C[\sigma(r)] = t$ for some $l \to r \in \mathcal{R}$, some context $C[.]$ and some substitution σ such that no proper subterm of $\sigma(l)$ is reducible. The parallel reduction relation (induced by $\mathcal{R}$) is denoted by $\dashv\!\vdash\!\!\!\to$. If s reduces to t by a parallel contraction of redexes in s at some set P of mutually disjoint positions, we also write $s \dashv\!\vdash\!\!\!\to_P t$.

A TRS is non-overlapping (NO) if it has no critical pairs. It is an overlay system (OS) if critical overlaps between rules of $\mathcal{R}$ occur only at the root position. For the properties termination (equivalently: strong normalization), weak termination (equivalently: weak normalization), confluence and uniform confluence (i.e., $\leftarrow \circ \to \subseteq (\mathrm{id} \cup \to \circ \leftarrow)$, with id being the identity relation) we also use the abbreviations SN, WN, CR and WCR[1], respectively. Innermost termination, i.e., termination (or strong normalization) of the innermost reduction relation $\xrightarrow[i]{}$, is abbreviated by SIN. Furthermore, we also use local versions ('below' some term) of these properties like SN(s) (meaning that there is no infinite derivation issuing form s). Completeness (or, equivalently, convergence) means termination plus confluence.

The following sufficient conditions for the (local and global) equivalence of termination and innermost termination are known.

Theorem 1. **([6])**

(1) Any innermost terminating (term in a) non-overlapping TRS is terminating (and hence complete).
(2) Any innermost terminating (term in a) locally confluent overlay TRS is terminating (and hence complete).

3 New Conditions for the Equivalence of Termination and Innermost Termination

3.1 The Basic Approach

As already mentioned, the basic idea pursued in this paper for deriving (SIN $\Longrightarrow$ SN) is to use as transformation construction a parallel one-step reduction of all terminating innermost redexes in the terms considered (instead of a parallel normalization of all maximal terminating redexes). Two properties are essential for this intended construction to work, namely, that contracting some (terminating) innermost redex in a term (at some fixed position) yields a unique result, and that certain 'innermost-critical' steps in infinite derivations can be avoided.

Definition 2. Let $\mathcal{R}$ be a TRS. We say that *innermost reduction (in $\mathcal{R}$) is unique* (at some fixed position), (denoted by UIR($\mathcal{R}$) or simply UIR), if

$$\forall s, t_1, t_2, p \in Pos(s) \,:\, s \xrightarrow[i]{}_p t_1 \wedge s \xrightarrow[i]{}_p t_2 \implies t_1 = t_2 \,.$$

To state the second property, we first introduce a new definition and some compact notations.

Definition 3. Let $\mathcal{R}$ be a TRS. A reduction step $s \to t$ is called *innermost-uncritical*, denoted by $s \xrightarrow[iu]{} t$, if $s \to_{p,\sigma,l\to r} t$ (for some p, σ, $l \to r \in \mathcal{R}$) such that either s/p is an innermost redex (of s), or else all innermost redexes of s strictly below p correspond to variable overlaps w.r.t. l, i.e., for every innermost redex s/pp' of s, $p' > \lambda$, we have: $p' \geq q$ for some $q \in \mathcal{V}Pos(l)$ (if $s \xrightarrow[iu]{} t$ with some p, σ, $l \to r$ as above, we also write $s \xrightarrow[iu]{}_{p,\sigma,l\to r} t$). Otherwise, i.e., if $s \to t$ but not $s \xrightarrow[iu]{} t$, the step $s \to t$ is called *innermost-critical* and denoted by $s \xrightarrow[ic]{} t$.

Remark. By definition, $s \xrightarrow[ic]{} t$ means $s \to_{p,\sigma,l\to r} t$ (for some p, σ, $l \to r \in \mathcal{R}$) such that there exists an innermost redex $s/pp' = \tau l'$ (for some τ and $l' \to r' \in \mathcal{R}$) of s with $p' > \lambda$ and $p' \in \mathcal{F}Pos(l)$. Obviously, this entails the existence of a corresponding critical pair by overlapping $l' \to r'$ into $l \to r$. Moreover note that an innermost reduction step is by definition innermost-uncritical, and an innermost-critical step must be non-innermost, due to $\to = \xrightarrow[iu]{} \uplus \xrightarrow[ic]{}$.

Now the second property required subsequently reads as follows.

Definition 4. Let $\mathcal{R}$ be a TRS. We say that *innermost-critical reduction steps can be avoided* (denoted by AICR($\mathcal{R}$) or simply AICR), if

$$\forall s, t \,:\, [s \to t \implies [s \xrightarrow[iu]{} t \vee \exists s' : s \xrightarrow[i]{} s' \to^* t]\,] \,.$$

This property guarantees that innermost-critical steps in infinite reductions can be avoided.

Lemma 5. *Let $\mathcal{R}$ be a TRS satisfying property AICR, and let t be a term. If t is non-terminating, then there exists an infinite derivation $t =: t_0 \to t_1 \to t_2 \ldots$ initiating from t which does only contain innermost-uncritical steps, i.e., for all $n \geq 0$ we have $t_n \xrightarrow[iu]{} t_{n+1}$.*

Proof. Let t be non-terminating. We give a construction for obtaining an infinite derivation issuing from t with the desired property. Assume that we have already constructed an initial segment

$$t =: t_0 \to t_1 \to \ldots \to t_n$$

of length n of an infinite derivation, i.e. with t_n non-terminating such that every step $t_k \to t_{k+1}$, $0 \leq k < n$ is innermost-uncritical. Now, if there exists a non-terminating t_{n+1} such that $t_n \xrightarrow[iu]{} t_{n+1}$, we can simply extend the initial segment by one step of the desired form. Otherwise, any step $t_n \to t'_{n+1}$ with t'_{n+1} non-terminating must be innermost-critical (note that at least one such t'_{n+1} must exist, due to non-termination of t_n). By AICR we conclude that there exists some term t_{n+1} with $t_n \xrightarrow[i]{} t_{n+1} \to^* t'_{n+1}$. This means that we can properly extend the initial segment to

$$t =: t_0 \to t_1 \to \ldots \to t_n \to t_{n+1}$$

with t_{n+1} still non-terminating where all steps have the desired form (note that $\xrightarrow[i]{} \subseteq \xrightarrow[iu]{}$). By induction we finally conclude that there exists an infinite derivation issuing from t which contains only innermost-uncritical steps. □

Next we introduce some notions for reduction steps contracting terminating and non-terminating redexes, respectively. If $s \to t$ by contracting a terminating redex in s (i.e., $s \to_p t$ for some $p \in Pos(s)$, with $\mathrm{SN}(s/p)$), we write $s \to_{\mathrm{sn}} t$. If $s \to t$ by contracting a non-terminating redex in s (i.e., $s \to_p t$ for some $p \in Pos(s)$, with $\neg\mathrm{SN}(s/p)$), we write $s \to_{\neg\mathrm{sn}} t$. Obviously, every reduction step can be written as $s \to_{\mathrm{sn}} t$ or $s \to_{\neg\mathrm{sn}} t$. In other words, we have $\to = \to_{\mathrm{sn}} \cup \to_{\neg\mathrm{sn}}$.[3]

Lemma 6. *Let $\mathcal{R}$ be a TRS. The relation $\to_{\mathrm{sn}}$ is terminating (on all terms). Moreover, any infinite $\to$-derivation must contain infinitely many $\to_{\neg\mathrm{sn}}$-steps which are not $\to_{\mathrm{sn}}$-steps.*

Proof. Straightforward by structural induction. □

Now we formally introduce the mentioned transformation function.

Definition 7. Let $\mathcal{R}$ be a TRS satisfying property UIR. Then the transformation function Ψ on terms is defined by a parallel one-step reduction of all terminating innermost redexes, i.e.:

$$\Psi(t) := C[t'_1, \ldots, t'_n], \text{ if } t = C[t_1, \ldots, t_n],\ t_k \xrightarrow[i]{}_\lambda t'_k\ (1 \le k \le n),$$

where the t_k, $1 \le k \le n$, are all the terminating innermost redexes of t (equivalently: $s \mathrel{+\!\!\!\!\!\!\mid\!\!\mid\!\!\!\to}_P \Psi(s)$ where P consists of the positions of all terminating innermost redexes of s).

Reduction steps in a given derivation D can be transformed into reduction steps of the transformed derivation $\Psi(D)$ as follows.

Lemma 8. *Let $\mathcal{R}$ be a TRS satisfying property UIR, and let s, t be terms with $s \xrightarrow[iu]{}_{p,\sigma,l\to r} t$. Then we have:*

(1) $SN(s/p) \Longrightarrow \Psi(s) \to^* \Psi(t)$.

[3] Note, however, that it is possible that s reduces to t both by contracting a terminating or a non-terminating redex in s. For $\mathcal{R} = \{a \to b, f(a) \to f(b), f(b) \to f(a)\}$, we have both $f(a) \to_{\mathrm{sn}} f(b)$ and $f(a) \to_{\neg\mathrm{sn}} f(b)$. Hence, in general we do not have $\to_{\mathrm{sn}} \cap \to_{\neg\mathrm{sn}} = \emptyset$. Nevertheless this could be enforced by slightly modifying the definition of $\to_{\neg\mathrm{sn}}$ as follows: $\to_{\neg\mathrm{sn}} = \to \setminus \to_{\mathrm{sn}}$.

(2) $\neg SN(s/p) \Longrightarrow \Psi(s) \to^{+} \Psi(t)$.

Proof. Due to property UIR, the transformation Ψ is well-defined. Let P be defined by $s \mathrel{+\!\!\!|\!\!\!\to}_P \Psi(s)$ as in Definition 7.

(1) We distinguish two cases. First assume that s/p is an innermost redex of s. Then we know by definition of Ψ, SN(s/p) and property UIR that $\Psi(s)/p = t/p$, and moreover, that $\Psi(s)$ is obtained from t via $t \mathrel{+\!\!\!|\!\!\!\to}_{P\setminus\{p\}} \Psi(s)$. Now, in order to obtain $\Psi(t)$ from $\Psi(s)$ one may have to contract additionally some newly introduced terminating innermost redexes in t. If $\sigma(r)$ is irreducible, new (terminating) innermost redexes can only be introduced above p in t, and only at positions disjoint from those of $P \setminus \{p\}$. If $\sigma(r)$ is reducible, all new innermost redexes in t must be below p (and all of these are indeed terminating, since the contracted redex s/p of s was terminating). Hence, in both cases, we obtain $\Psi(t)$ from $\Psi(s)$ by an additional parallel innermost reduction (at positions disjoint from those of P). Thus, we get $\Psi(s) \to^{*} \Psi(t)$ as desired. In the other case s/p is a terminating non-innermost redex. Since the step $s \to_{p,\sigma,l\to r} t$ is non-innermost innermost-uncritical, every innermost redex of s (there exists at least one!) strictly below p, s/pq ($q > \lambda$), must correspond to a variable overlap, i.e., $q \geq u$, for some $u \in \mathcal{V}Pos(l)$. This implies (together with UIR) that $\Psi(s)/p$ is still an instance of l, let's say $\Psi(s)/p = \sigma'(l)$, with $\sigma \mathrel{+\!\!\!|\!\!\!\to} \sigma'$. Hence we get $s \xrightarrow[i]{}^{+} \Psi(s) \to_{p,\sigma',l\to r} \Psi(s)[p \leftarrow \sigma'(r)] =: t'$ and $s \to_{p,\sigma,l\to r} t \mathrel{+\!\!\!|\!\!\!\to}_Q t'$, where $Q = P \setminus \{q \in P \,|\, q \geq p\} \cup P'$ with P' consisting of the positions of those (terminating) innermost redexes of t which are descendents of the terminating innermost redexes of s below p w.r.t. the step $s \to_{p,l\to r} t$. Again, in order to obtain $\Psi(t)$ from t', additionally some newly introduced terminating innermost redexes (above or below p) may have to be reduced. Thus we obtain $\Psi(s) \to t' \to^{*} \Psi(t)$, hence $\Psi(s) \to^{+} \Psi(t)$ as desired.

(2) Again we distinguish two cases. First assume that s/p is an innermost redex of s. Since s/p is non-terminating we have $p \notin P$ and $p \parallel P$. Hence the two steps commute: $s \underset{i}{\mathrel{+\!\!\!|\!\!\!\to}}_P \Psi(s) \xrightarrow[i]{}_p t'$, $s \xrightarrow[i]{}_p t \underset{i}{\mathrel{+\!\!\!|\!\!\!\to}}_P t'$. Moreover, $\Psi(t)$ is obtained from t' by additionally contracting those terminating innermost redexes which are introduced (in t and also in t') by the step $s \xrightarrow[i]{}_p t$ (and $\Psi(s) \xrightarrow[i]{}_p t'$, respectively) above or below p. Again the positions of the latter introduced (terminating) innermost redexes are disjoint from those of $P \setminus \{p\}$. Hence we have $s \underset{i}{\mathrel{+\!\!\!|\!\!\!\to}}_P \Psi(s) \xrightarrow[i]{}_p t' \mathrel{+\!\!\!|\!\!\!\to} \Psi(t)$ and thus $\Psi(s) \to^{+} \Psi(t)$. In the other case s/p is a non-terminating, non-innermost redex. Hence, every innermost redex of s (there exists at least one!) strictly below p, s/pq ($q > \lambda$), must correspond to a variable overlap, i.e., $q \geq u$, for some $u \in \mathcal{V}Pos(l)$. This implies (together with UIR) that $\Psi(s)/p$ is still an instance of l, let's say $\Psi(s)/p = \sigma'(l)$, with $\sigma \mathrel{+\!\!\!|\!\!\!\to} \sigma'$. Hence, as in the corresponding second subcase of **(1)**, we then obtain $\Psi(s) \to t' \to^{*} \Psi(t)$ and thus $\Psi(s) \to^{+} \Psi(t)$ as desired. □

Next we prove that under certain conditions the transformation Ψ preserves the possibility of infinite reductions.

Lemma 9. *Let $\mathcal{R}$ be a TRS satisfying properties UIR and AICR. Then, for any term t we have: $SN(\Psi(t)) \Longrightarrow SN(t)$.*

Proof. Assume $\neg\mathrm{SN}(t)$, i.e., there exists an infinite derivation D issuing from t. Due to property AICR and Lemma 5 we may assume that in

$$D : t =: t_0 \to t_1 \to t_2 \to \ldots$$

every step is innermost-uncritical. If $t_k \xrightarrow[iu]{}_{p,\sigma,l\to r} t_{k+1}$ with $\mathrm{SN}(t_k/p)$ (for some $p, \sigma, l \to r$), then, by Lemma 8(1), we know $\Psi(t_k) \to^* \Psi(t_{k+1})$. If $t_k \xrightarrow[iu]{}_{p,\sigma,l\to r} t_{k+1}$ with $\neg\mathrm{SN}(t_k/p)$ (for some $p, \sigma, l \to r$), we conclude by Lemma 8(2), that $\Psi(t_k) \to^+ \Psi(t_{k+1})$. Clearly, steps of the form are $t_k \to_p t_{k+1}$ with $\mathrm{SN}(t_k/p)$ are $\to_{\mathrm{sn}}$-steps, and steps of the form are $t_k \to_p t_{k+1}$ with $\neg\mathrm{SN}(t_k/p)$ are $\to_{\neg\mathrm{sn}}$-steps. Hence, by Lemma 6, D contains infinitely many $\to_{\neg\mathrm{sn}}$-steps of the latter form $t_k \xrightarrow[iu]{}_{p,\sigma,l\to r} t_{k+1}$ with $\neg\mathrm{SN}(t_k/p)$. We conclude that the transformed derivation

$$\Psi(D) : \Psi(t) = \Psi(t_0) \to^* \Psi(t_1) \to^* \Psi(t_2) \to^* \ldots$$

is infinite, too. Hence we are done. □

Now we are in a position to establish the main abstract result of this section.

Theorem 10. *Let $\mathcal{R}$ be a TRS satisfying properties UIR and AICR. Then, for any term t we have: SIN(t) $\iff$ SN(t).*

Proof. For a proof by contradiction (of SIN $\implies$ SN, the other direction is trivial), suppose that t admits an infinite derivation. Every infinite derivation starting from t must contain a non-innermost step, due to $\mathrm{SIN}(t)$. Now consider an infinite derivation D starting from t with the property that the first non-innermost step is essential: Contracting any innermost redex at that point would result in a term with the property SN. Let

$$D : t =: t_0 \to t_1 \to \ldots \to t_n \to t_{n+1} \to \ldots$$

where $t_n \to t_{n+1}$ is the first non-innermost step. Note that this step must be innermost-uncritical (i.e., $t_n \xrightarrow[iu]{} t_{n+1}$) because, otherwise, it could not be essential in the above sense, due to property AICR. By assumption, contracting an innermost redex in t_n yields a terminating term. This implies in particular that $\Psi(t_n)$, which is obtained from t_n by a (non-empty) parallel reduction step contracting all innermost redexes of t_n (all of which must be terminating), is terminating. But then, by Lemma 9 we conclude that t_n is terminating, too. Hence, we have a contradiction (to the infinity of D). □

3.2 Concrete Equivalence Criteria

The above criterion for the implication (SIN $\implies$ SN) is still rather abstract in nature. Subsequently we shall investigate more concrete syntactic conditions satisfying the properties UIR, uniqueness of innermost reduction (at some fixed position), and AICR, the possibility to avoid (non-innermost) innermost-critical steps in infinite reductions. This will result in concrete critical peak conditions for the considered TRS $\mathcal{R}$.

First we introduce some more useful terminology for reasoning about critical peaks (and pairs).

Definition 11. Let $\mathcal{R}$ be a TRS. A critical peak $s\ {}_p\!\leftarrow u \rightarrow_\lambda t$ (and its corresponding critical pair $\langle s = t \rangle$) is *joinable* if $s \downarrow t$. It is *trivial* if $s = t$. It is said to be *left-to-right joinable* if $s \rightarrow^* t$. It is *overlay joinable* or *outside joinable* if $p = \lambda$ and $s \downarrow t$. If $p > \lambda$ it is an *inside* critical peak, if $p = \lambda$ it is an *outside* critical peak or *critical overlay*. If $p = \lambda$ and $s/p = s$ is an innermost redex, we speak of an *innermost critical overlay*. $\mathcal{R}$ *has joinable critical peaks (pairs)* (JCP) if all its critical peaks (pairs) are joinable. $\mathcal{R}$ is *weakly non-overlapping* (WNO) if it has only trivial critical peaks. $\mathcal{R}$ *has left-to-right joinable critical peaks* (LRJCP) if all its critical peaks are left-to-right joinable. $\mathcal{R}$ *has strongly left-to-right joinable critical peaks* (SLRJCP) if all its critical peaks are left-to-right joinable and, moreover, all its outside critical peaks are even trivial. $\mathcal{R}$ *has weakly left-to-right joinable critical peaks* (WLRJCP) if all its inside critical peaks are left-to-right joinable and all its outside critical peaks are joinable. $\mathcal{R}$ *has overlay joinable critical peaks* (OJCP) if every critical peak of $\mathcal{R}$ is overlay joinable (note: OJCP $\iff$ OS $\wedge$ JCP). $\mathcal{R}$ *has weakly overlay joinable critical peaks* (WOJCP) if all its inside critical peaks are trivial, and all its outside critical peaks are joinable.

Next we give a critical peak condition which is sufficient for the crucial properties UIR and AICR needed above.

Definition 12. Let $\mathcal{R}$ be a TRS. We say that $\mathcal{R}$ *satisfies CPC* (CPC($\mathcal{R}$) or CPC for short) if the following holds: For every every critical peak $t_1\ {}_p\!\leftarrow s \rightarrow_\lambda t_2$ of $\mathcal{R}$ we have:[4]

(1) if $p = \lambda$ and both steps are innermost (i.e., s is an innermost redex), then $t_1 = t_2$, and
(2) if $p > \lambda$ and the inside step is innermost (i.e., $s \xrightarrow[i]{}_p t_1$), then $t_1 \rightarrow^* t_2$.

Lemma 13. *For any TRS, CPC implies both UIR and AICR.*

Proof. Assume CPC. Then, UIR follows easily by definition and CPC (1). Proving AICR, the possibility of avoiding innermost-critical steps, amounts to showing

$$\forall s, t\ :\ [s \rightarrow t \implies [s \xrightarrow[iu]{} t \vee \exists s' : s \xrightarrow[i]{} s' \rightarrow^* t]\,]\,.$$

Now, w.l.o.g. we may assume $s \rightarrow t$, but not $s \xrightarrow[iu]{} t$, hence $s \xrightarrow[ic]{} t$. This means (cf. Remark 3.1) $s \rightarrow_{p,\sigma,l\rightarrow r} t$ (for some p, σ, $l \rightarrow r$) such that there exists an innermost redex in $s/p = \sigma(l)$ at some position $q \in \mathcal{F}Pos(l)$, $q > \lambda$, with $s/pq = \sigma(l)/q = \tau(l')$ for some rule $l' \rightarrow r' \in \mathcal{R}$. But this implies that the divergence

$$s/p[q \leftarrow \tau(r')]\ {}_q\!\leftarrow s/p = \sigma(l) \rightarrow_\lambda \sigma(r) = t/p$$

[4] In other words, these requirements mean: every innermost critical overlay must be trivial, and every inside critical peak where the inside step is innermost must be left-to-right joinable.

is an instance of a critical peak of the form in (2) above. Therefore we get $s/p[q \leftarrow \tau(r')] \to^* \sigma(r)$ and hence

$$s \xrightarrow[i]{}_{pq} s[pq \leftarrow \tau(r')] \to^* s[p \leftarrow \sigma(r)] = t$$

as desired. □

Remark. Note that the reverse implication in Lemma 13 does not hold. For example, the TRS

$$\mathcal{R} = \begin{cases} f(a,a) \to f(a,a) \\ \quad\;\; a \to b \\ f(b,x) \to f(x,x) \end{cases}$$

satisfies both UIR and AICR, but not CPC. The point is that for the inside critical peak $f(a,b)\ {}_{2}\!\leftarrow f(a,a) \to_\lambda f(a,a)$ we do not have $f(a,b) \to^* f(a,a)$. However, the step $f(a,a) \to f(a,a)$ which is innermost-critical has two innermost-critical redexes. And, indeed, choosing the left innermost-critical redex a corresponds to the other critical peak $f(b,a)\ {}_{1}\!\leftarrow f(a,a) \to_\lambda f(a,a)$ which is left-to-right-joinable, yielding $f(a,a) \xrightarrow[i]{} f(b,a) \to f(a,a)$.

Combining Theorem 10 and Lemma 13 we obtain the following (local and global) results.

Theorem 14. *Any innermost terminating (term in a) TRS $\mathcal{R}$ satisfying the critical pair condition CPC is terminating.*

In particular, we get the following concrete critical pair criteria, which extend the equivalence result (SIN $\iff$ SN) for non-overlapping systems (cf. Theorem 1(1)) to more general classes of TRSs.

Theorem 15. *An innermost terminating (term in a) TRS $\mathcal{R}$ is terminating if $\mathcal{R}$*

(1) is weakly non-overlapping (WNO), or
(2) has strongly left-to-right joinable critical peaks (SLRJCP), or
(3) has left-to-right joinable critical peaks (LRJCP).

Proof. (1) is a special case of (2). And (2) holds by Theorem 14 and the fact that according to the definition of SLRJCP any violation of the critical pair condition CPC (in Lemma 13) is impossible. (3) is not a direct consequence of Theorem 14 and Lemma 13, but holds for similar reasons. Namely, LRJCP still implies AICR as is easily seen. However, UIR need not hold any more. Nevertheless, a careful inspection of Definitions 3, 4, 7, Lemmas 8, 9 and Theorem 10 reveals that UIR is only needed in the slightly weakened version

$$\forall s, t_1, t_2, p \in Pos(s) \ : \ s \xrightarrow[i]{}_p t_1 \ \wedge \ s \xrightarrow[i]{}_p t_2 \ \wedge \ \mathrm{SN}(s/p) \implies t_1 = t_2 \,.$$

And indeed, an innermost divergence $t_1\ {}_\lambda\!\leftarrow s \to_\lambda t_2$ with $\mathrm{SN}(s)$ which is non-trivial, i.e., with $t_1 \neq t_2$, cannot be an instance of a non-trivial innermost critical overlay. Because then, by left-to-right-joinability of all innermost critical overlays (according to LRJCP), we could conclude both $t_1 \to^* t_2$ and $t_2 \to^* t_1$, thus contradicting $\mathrm{SN}(s)$.[5] $\square$

Interestingly, and in contrast to the situation for non-overlapping and for locally confluent overlay systems, TRSs satisfying the critical pair condition CPC need not be (locally) confluent.

Example 1. The TRS

$$\mathcal{R} = \begin{cases} f(x) \to x \\ f(a) \to b \\ f(a) \to c \\ \quad a \to f(a) \end{cases}$$

is easily verified to satisfy CPC (in particular, we note that CPC ignores the critical overlays between the first three rules since they comprise non-innermost divergences). However, $\mathcal{R}$ is not locally confluent, as shown by the non-joinable critical overlay $b \;{}_{\lambda}\!\leftarrow f(a) \to_{\lambda} c$.

We observe that in this example $\mathcal{R}$ is obviously not innermost terminating. And, fortunately, it turns out that SIN combined with CPC does indeed imply not only termination, but also confluence, hence completeness.

Theorem 16. *Any innermost terminating TRS satisfying the critical peak condition CPC is complete, i.e., terminating and confluent.*

Proof. Let $\mathcal{R}$ be a TRS with $\mathrm{SIN}(\mathcal{R})$ and $\mathrm{CPC}(\mathcal{R})$. By Theorem 14 we know that $\mathcal{R}$ must be terminating. Thus, it remains to show confluence. This can be done directly by induction (for the stronger local version of the result): Using $\to^+$ as well-founded ordering, one proves

$$\forall s : [\,[\forall t : s \to^+ t \Longrightarrow \mathrm{CR}(t)] \implies \mathrm{CR}(s)\,]$$

by induction via a (not too difficult, but tedious) case analysis concerning the shape of the initial one-step divergence issuing from s in an arbitrary divergence $t_1 \;{}^{+}\!\leftarrow s \to^+ t_2$, and exploiting $\mathrm{CPC}(\mathcal{R})$. However, we shall not present this proof in detail but simply reduce it to a special case of some known result. More precisely, we apply the following *critical pair criterion* for confluence of terminating TRSs due to [8]:

> If $\mathcal{R}$ is a terminating TRS such every critical pair which corresponds to a *prime* critical peak is joinable, then $\mathcal{R}$ is confluent.

Here, a critical peak $t_1 \;{}_{p}\!\leftarrow s \to_{\lambda} t_2$ of $\mathcal{R}$ is *prime* if it is not *composite*. And it is *composite* if the inner redex s/p has a proper reducible subterm. Indeed, it is straightforward to verify that CPC ignores only composite critical peaks, and the ones considered are clearly joinable according to CPC. Hence, applying this result we get confluence of $\mathcal{R}$ for free. $\square$

[5] In fact, any TRS $\mathcal{R}$ which has a non-trivial, left-to-right joinable critical overlay is necessarily non-terminating. For this reason, such systems are not so interesting from a termination point of view.

Remark. Interestingly, it seems difficult to prove the implication (CPC $\wedge$ SIN $\Longrightarrow$ CR) directly, without making use of SN (which is allowed according to Theorem 14).

Example 1 above showed that, given CPC, we cannot simply drop innermost termination when trying to infer confluence or termination (or both of them). In view of (the proof of) Theorem 16 one might be tempted to conjecture that the combination of innermost termination and the critical pair criterion of [8], i.e., joinability of all prime critical peaks, suffices for guaranteeing confluence and termination or at least one of them. However, this is also not the case.

Example 2. The TRS

$$\mathcal{R} = \begin{cases} f(b) \to f(c) \\ f(c) \to f(b) \\ \quad b \to a \\ \quad c \to d \end{cases}$$

is clearly innermost terminating and locally confluent, but non-terminating and non-confluent, and both critical peaks to be considered are obviously prime.

Let us mention that Theorem 15 captures only some special cases of Theorem 14. According to the latter result, for inferring termination from innermost termination it suffices to show that every innermost critical overlay is trivial, and every inside critical pair where the inside step is innermost is left-to-right joinable. In particular, non-trivial critical overlays may exist provided they comprise a non-innermost divergence.

Example 3. The TRS

$$\mathcal{R} = \begin{cases} f(a) \to b \\ f(a) \to f(c) \\ \quad a \to d \\ f(d) \to b \\ f(c) \to b \\ \quad d \to c \end{cases}$$

innermost terminating and satisfies CPC, hence must be complete by Theorem 14. But note that there exists a non-trivial critical overlay (between the first two rules) which is ignored by CPC since it is not a divergence by innermost reduction.

As exhibited above, under uniqueness of innermost reduction (UIR, cf. Definition 2), left-to-right joinability of all critical pairs (LRJCP, cf. Definition 11) has turned out to be sufficient for the implication (SIN $\Longrightarrow$ SN) to hold (Theorem 15 (3)). Hence, in view of the fact that overlay joinability of all critical pairs (OJCP) also suffices for (SIN $\Longrightarrow$ SN) (Theorem 1(2)), it is quite natural to ask whether these two criteria can be combined. In other words, we have the following

Open Problem. Does termination follow from innermost termination if all critical overlays are joinable and all inside critical peaks are even left-to-right joinable? (or, more concisely: WLRJCP $\wedge$ SIN $\Longrightarrow$ SN?)

Unfortunately, we have neither been able to prove this criterion nor to find a counterexample. The careful reader may recognize that the problem lies with allowing both non-trivial innermost critical overlays and inside critical peaks. In fact, even for the special case that we require any inside critical peaks to be trivial, i.e., for the conjecture (WOJCP $\wedge$ SIN $\Longrightarrow$ SN), we could not provide a solution. Intuitively, the reason for our failure is due to the problem, that the two different proof techniques we applied somehow seem to be incompatible. More precisely, the latter proof technique presented here crucially relies on uniqueness of innermost reduction, which is destroyed by allowing non-trivial innermost critical overlays. And the other proof technique of [6] (for Theorem 1(2)) makes essential use of the (pure) overlay property which conflicts with allowing inside critical peaks.

Finally, we present some further results, relying on the properties UIR and AICR, which generalize the corresponding results of [6] for non-overlapping TRSs.

Theorem 17. *For any TRS $\mathcal{R}$ the following properties hold:*

(1) $UIR \Longrightarrow WCR^1(\xrightarrow[i]{}) \Longrightarrow CR(\xrightarrow[i]{})$.
(2) $UIR \Longrightarrow [\,WIN(s) \iff SIN(s)\,]$.
(3) $UIR \wedge AICR \Longrightarrow [\,WIN(s) \iff SIN(s) \iff SN(s)\,]$.
(4) $UIR \wedge AICR \Longrightarrow [\,s \xrightarrow[i]{} t \wedge SN(t) \Longrightarrow SN(s)\,]$.

Proof. (1) is straightforward by definition of UIR and using the well-known fact that uniform confluence (WCR^1) implies confluence (CR). (2) follows from (1), observing the equivalences (WIN $\iff$ $\mathrm{WN}(\xrightarrow[i]{})$) and (SIN $\iff$ $\mathrm{SN}(\xrightarrow[i]{})$) (which also hold 'below' some term s), and applying the following old result from Newmman ([14]): $\mathrm{WCR}^1 \wedge \mathrm{WN} \Longrightarrow \mathrm{SN}$. Finally, (3) is a consequence of (2) and Theorem 10, and (4) is a corollary of (3). □

4 Applications in Modularity

The study of the modularity behaviour of rewrite systems (w.r.t. to the preservation of important properties) under various types of combinations has become a very active and fruitful area of research (cf. [12], [15], [7] for surveys). A property P of rewrite systems is said to be *modular* (for disjoint systems) if, for all (disjoint) systems $\mathcal{R}_1^{\mathcal{F}_1}$, $\mathcal{R}_2^{\mathcal{F}_2}$ and $\mathcal{R}^{\mathcal{F}}$ with $\mathcal{R}^{\mathcal{F}} = (\mathcal{R}_1 \uplus \mathcal{R}_2)^{\mathcal{F}_1 \uplus \mathcal{F}_2}$: $P(\mathcal{R}_1^{\mathcal{F}_1}) \wedge P(\mathcal{R}_2^{\mathcal{F}_2}) \iff P(\mathcal{R}^{\mathcal{F}})$. The 'if'-part here is most often trivial, so, the interesting question is whether the combination inherits the property from the constituent systems. Under disjoint unions, confluence is well-known to be modular, whereas termination is not modular ([19]).

The known positive results for modularity of termination are, roughly speaking, based on three different approaches concerning the essential ideas and proof structures (cf. [7]):

- the **general approach via an abstract structure theorem** where the basic idea is to reduce non-termination in the union to non-termination of a slightly modified generic version of one of the systems ([5] [16]),
- the **modular approach via modularity of innermost termination** where sufficient criteria for the equivalence of innermost termination (SIN) and general termination (SN) are combined with the (not so difficult to establish) modularity of SIN ([6][6]), and
- the **syntactic approach via left-linearity** which in essence is based on commutation and uniqueness properties for rewriting in left-linear systems ([20], [18]).

Due to lack of space we cannot give a more detailed account of the (numerous) papers on the subject and the above classification, but shall only concentrate on the **modular approach via modularity of innermost termination** (for the case of disjoint unions). The 'modular' idea here is quite simple: Since innermost termination is modular (which is easily proved by structural induction, cf. [6]), it suffices to ensure that, for the disjoint union, innermost termination implies termination. According to Theorem 1, this is the case, if both constituent systems are (besides terminating) non-overlapping, or locally confluent overlay systems.[7] Alternately, one can also use Theorems 14-16 to derive corresponding modularity results.

Lemma 18. *The properties UIR, CPC, WNO and SLRJCP are modular for disjoint unions of TRSs.*[8]

Proof. Straightforward by the respective definitions and the facts that in the disjoint union of two TRSs the set of critical peaks is the disjoint[9] union of the sets of critical peaks in the disjoint component TRSs, and that joinability of a critical peak belonging to one system does not depend on the other system. □

Theorem 19. *Termination and completeness are modular for disjoint unions of TRSs satisfying CPC.*

[6] Cf. also [13] where (*composable*) combinations of *constructor systems*, which are in particular overlay systems, are considered.

[7] Observe that the the properties NO (being non-overlapping), OS (being an overlay system) and local confluence (which is eqivalent to JCP, joinability of all critical pairs) are obviously modular!

[8] In fact, a couple of further related syntactical (e.g., critical peak) properties are also easily seen to be modular.

[9] This does not hold necessarily for critical pairs, due to degenerate cases where a critical pair consists of two variables. However, such critical pairs do not constitute a substantial problem either, since they are either trivial or not joinable (due to our assumption of excluding variables as left hand sides).

Proof. Straightforward by combining Theorem 16, Lemma 18 (i.e., the modularity of CPC) and the modularity of innermost termination. □

In view of the implications (WNO $\Longrightarrow$ CPC) and (SLRJCP $\Longrightarrow$ CPC), and Lemma 18 this entails the following.

Corollary 20. *Termination and completeness are modular for disjoint unions of weakly non-overlapping TRSs (WNO) and of TRSs with strongly left-to-right-joinable critical peaks (SLRJCP).*

Let us give an example for illustrating the applicability of Theorem 19 (and of Corollary 20).

Example 4. The disjoint TRSs

$$\mathcal{R}_1 = \begin{cases} f(a,b,x) \to f(x,x,x) \\ a \to c \\ f(c,b,x) \to f(x,x,x) \end{cases}$$

and

$$\mathcal{R}_2 = \begin{cases} K(x,y,y) \to x \\ K(y,y,x) \to x \end{cases}$$

are both terminating and confluent, and moreover have strongly left-to-right joinable critical peaks (SLRJCP) as is easily verified. Hence, Corollary 20 yields completeness of their disjoint union. We note that none of the previous modularity results is applicable here. In particular, $\mathcal{R}_1$ is not an overlay system, and $\mathcal{R}_2$ is not left-linear.

We remark that the same approach as above also yields corresponding modularity results for the more general cases of *constructor-sharing* TRSs ([11]) and of *composable* TRSs ([13], [15]). This is straightforward, since innermost termination (SIN) as well the other syntactical properties involved (CPC, WNO, SLRJCP) are modular for such combinations, too. Moreover, it seems plausible that the new criteria for the equivalence (SIN $\iff$ SN) can also be used for deriving new inheritence results for termination of *hierarchical* combinations of TRSs (as in [10], [1]).

References

1. N. Dershowitz. Hierarchical termination. In N. Dershowitz and N. Lindenstrauss, editors, *Proc. of 4th Int. Workshop on Conditional and Typed Rewriting Systems, Jerusalem, Israel (1994)*, *Lecture Notes in Computer Science* 968, pages 89–105. Springer-Verlag, 1995.
2. N. Dershowitz and C. Hoot. Natural termination. *Theoretical Computer Science*, 142(2):179–207, May 1995.
3. N. Dershowitz and J.-P. Jouannaud. Rewrite systems. In J. van Leeuwen, editor, *Formal models and semantics, Handbook of Theoretical Computer Science*, volume B, chapter 6, pages 243–320. Elsevier - The MIT Press, 1990.

4. O. Geupel. Overlap closures and termination of term rewriting systems. Technical Report MIP-8922, Fakultät für Informatik, Universität Passau, July 1989.
5. B. Gramlich. Generalized sufficient conditions for modular termination of rewriting. *Applicable Algebra in Engineering, Communication and Computing*, 5:131–158, 1994.
6. B. Gramlich. Abstract relations between restricted termination and confluence properties of rewrite systems. *Fundamenta Informaticae*, 24:3–23, 1995. Special issue on term rewriting systems, ed. D. A. Plaisted. (A preliminary version of this paper appeared as 'Relating innermost, weak, uniform and modular termination of term rewriting systems' in Proc. *LPAR'92*, *LNAI* 624, pages 285–296, see also SEKI-Report SR-93-09, FB Informatik, Univ. Kaiserslautern).
7. B. Gramlich. *Termination and Confluence Properties of Structured Rewrite Systems*. PhD thesis, Fachbereich Informatik, Universität Kaiserslautern, Jan. 1996.
8. D. Kapur, D. Musser, and P. Narendran. Only prime superpositions need be considered in the Knuth-Bendix completion procedure. *Journal of Symbolic Computation*, 1988(6):19–36, 1988.
9. J. W. Klop. Term rewriting systems. In S. Abramsky, D. Gabbay, and T. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 2, chapter 1, pages 2–117. Clarendon Press, Oxford, 1992.
10. M. Krishna Rao. Modular proofs for completeness of hierarchical term rewriting systems. *Theoretical Computer Science*, 151(2):487–512, Nov. 1995.
11. M. Kurihara and A. Ohuchi. Modularity of simple termination of term rewriting systems with shared constructors. *Theoretical Computer Science*, 103:273–282, 1992.
12. A. Middeldorp. *Modular Properties of Term Rewriting Systems*. PhD thesis, Free University, Amsterdam, 1990.
13. A. Middeldorp and Y. Toyama. Completeness of combinations of constructor systems. *Journal of Symbolic Computation*, 15:331–348, Sept. 1993.
14. M. Newman. On theories with a combinatorial definition of equivalence. *Annals of Mathematics*, 43(2):223–242, 1942.
15. E. Ohlebusch. *Modular Properties of Composable Term Rewriting Systems*. PhD thesis, Universität Bielefeld, 1994. Report 94-01.
16. E. Ohlebusch. On the modularity of termination of term rewriting systems. *Theoretical Computer Science*, 136:333–360, 1994.
17. D. A. Plaisted. Equational reasoning and term rewriting systems. In D. Gabbay, C. Hogger, and J. Robinson, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming – Logical Foundations*, volume 1 of *Handbooks of Logic in Computer Science and of Logic in Artificial Intelligence and Logic Programming*, pages 273–364. Clarendon Press, Oxford, 1994.
18. M. Schmidt-Schauß, M. Marchiori, and S. Panitz. Modular termination of r-consistent and left-linear, term rewriting systems. *Theoretical Computer Science*, 149(2):361–374, Oct. 1995.
19. Y. Toyama. On the Church-Rosser property for the direct sum of term rewriting systems. *Journal of the ACM*, 34(1):128–143, 1987.
20. Y. Toyama, J. Klop, and H. Barendregt. Termination for the direct sum of left-linear term rewriting systems. In N. Dershowitz, editor, *Proc. 3rd Int. Conf. on Rewriting Techniques and Applications*, *Lecture Notes in Computer Science* 355, pages 477–491. Springer-Verlag, 1989.

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